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A Uniformly Stable Nonconforming FEM Based on Weighted Interior Penalties for Darcy-Stokes-Brinkman Equations

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Abstract. A nonconforming rectangular finite element method is proposed to solve a fluid structure interaction problem characterized by the Darcy-Stokes-Brinkman Equation with discontinuous coefficients across the interface of different structures. A uniformly stable mixed finite element together with Nitsche-type matching conditions that automatically adapt to the coupling of different sub-problem combinations are utilized in the discrete algorithm. Compared with other finite element methods in the literature, the new method has some distinguished advantages and features. The Boland-Nicolaides trick is used in proving the inf-sup condition for the multi-domain discrete problem. Optimal error estimates are derived for the coupled problem by analyzing the approximation errors and the consistency errors. Numerical examples are also provided to confirm the theoretical results.

AMS subject classifications: 65N30, 76S05

Key words: Fluid structure interactions, Darcy-Stokes-Brinkman equations, Stokes equations, Darcy flow, discontinuous coefficient, nonconforming rectangular element, interior penalty, inf-sup condition, error estimates.

1. Introduction

There are many applications of the fluid structure interaction between a fluid flow and a porous media, a fluid flow and another fluid flow, or a porous media and another porous media with different physical parameters. In this paper, we consider such a fluid structure interaction problem that is modeled by the Darcy-Stokes-Brinkman equations,

$$\eta\mathbf{u} + \nabla\cdot\big(p\,\mathbf{I} - \nu\nabla\mathbf{u}\big) = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \qquad x\in\Omega\subset\mathbb{R}^2, \qquad a.e., \tag{1.1}$$

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where $\mathbf{u}(x)$ is the velocity, $p(x)$ is the pressure, and $\mathbf{f}(x)$ is an external force. We assume that $\nu(x)$ and $\eta(x)$ are nonnegative coefficients satisfying $\nu(x)+\eta(x)=\mu(x)$ with $0<m\le\mu(x)\le M$ a.e. on Ω. Note that if $\eta=0$, the equations become a Stokes flow while it is a Darcy flow if $\nu=0$, see Figure 1 for an illustration of different set-up of our interest in this paper. We use the non-slip boundary condition if the part of the boundary $\partial\Omega$ bordered with the flow with $\nu>0$, otherwise we use $\mathbf{u}\cdot\mathbf{n}=0$, where $\mathbf{n}$ is the normal direction.

Without loss of generality, we refer to normalized coefficients such that $m\le 1\le M$. In this case, the ratio $M/m\ge 1$ quantifies the spatial heterogeneity of the problem. We also denote $\mathbf{f}(x)$ as a vector-valued forcing term and $\mathbf{I}$ as the identity matrix. When both $\nu(x)$ and $\eta(x)$ are positive for any $x\in\Omega$, equation (1.1), complemented with boundary conditions, represents a standard problem called a generalized Stokes equation. In this paper, we are interested in the local singular limit case, i.e., when $\nu(x)\to 0$ or $\eta(x)\to 0$ in a sub-region of the domain. In the case of $\nu(x)=0$, a rigorous formulation of problem (1.1) requires us to differentiate between Stokes and Darcy subproblems and to introduce interface conditions. The aim of this work is to provide a finite element discretization scheme for the local singular limit cases. This will be achieved starting from the multi-domain formulation (2.1)-(2.7).

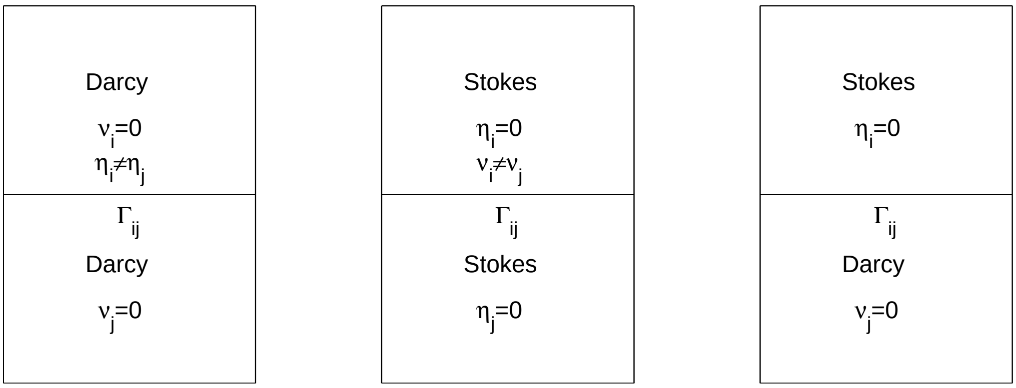

Figure 1: Diagrams of the fluid-structure interaction problem with three typical couplings.

There is rich literature on the coupling of viscous and inviscid sub-problems and applications [1, 2, 5, 9–13, 15–17, 19–21, 23–26, 30, 31]. It is desirable to develop a unified discretization framework for the problem. One difficulty is the treatment of interface conditions. Various approaches have been developed in the literature including Lagrange multipliers and mortar elements to satisfy the discrete interface conditions [5, 16, 20, 21, 25]. Generalizing the analysis in [9], D'angelo and Zunino [12] do with the coupling based on matching conditions due to the Nitsche method. This scheme is also particularly effective for the treatment of realistic applications, because interface conditions of practical interest, such as the ones proposed by Beavers and Joseph [3] and Saffman [27] for the coupling of free flows with porous media can naturally be embedded into the scheme. A finite difference approach that utilizes fast