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The Disc Theorem for the Schur Complement of Two Class Submatrices with γ-Diagonally Dominant Properties

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Abstract. The distribution for eigenvalues of Schur complement of matrices plays an important role in many mathematical problems. In this paper, we firstly present some criteria for H-matrix. Then as application, for two class matrices whose submatrices are γ-diagonally dominant and product γ-diagonally dominant, we show that the eigenvalues of the Schur complement are located in the Geršgorin discs and the Ostrowski discs of the original matrices under certain conditions.

AMS subject classifications: 15A24, 15A48

Key words: Schur complement, Geršgorin Theorem, Ostrowski Theorem, H-matrix.

1. Introduction and notations

In many fields such as control theory and computational mathematics, the theory of Schur complement plays an important role. A lot of work have been done on it. Based on the Geršgorin discs and Gassini ovals, Liu and Zhang firstly presented the notations of disc separations and considered the disc separations for diagonally dominant matrix and their Schur complement ([1]). Further, Liu obtained some estimates for dominant degree of the Schur complement and some bounds for the eigenvalues of the Schur complement by the entries the original matrix ([1–5]). For another, as the eigenvalue distribution problem on the Schur complement has important applications (see e.g., [2–4]), thus there are many researchers pay attention to it. Liu and Zhang considered the relation between the eigenvalues of the Schur complement and the submatrix for diagonally dominant matrix A with real diagonal elements in the paper ([1]). Cvetković and Nedović [6] generalized this result to the S-strictly diagonally dominant matrix.

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In [7], Liu and Huang obtained the number of eigenvalues with positive real part and with negative real part for the Schur complement of H-matrix with real diagonal elements. Later, Zhang *et al.* [8] generalized this result to the H-matrix with complex diagonal elements. Liu *et al.* presented some bounds for the eigenvalues of the Schur complement by the entries of original matrix ([2–5]). As stated in these papers above, if the eigenvalues of the Schur complement can be estimated by the elements of the original matrix, it easy to know whether a linear system could be transformed into two smaller one which can be solved by iteration. This kind of iteration, which has many advantages, is called the Schur-based iteration, as it converts the original system into two smaller ones by the Schur complement. Hence, investigating the distribution for eigenvalues of Schur complement is of great significance.

In the following, we recall some notations and definitions. Let $C^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,\dots,n\}$ and $A=(a_{ij})\in C^{n\times n}$, where $n\geq 2$ and let $N_1\cup N_2=N, N_1\cap N_2=\emptyset$. Denote

$$\alpha_i(A)=\sum_{j\in N_1,j\neq i}|a_{ij}|,\quad \beta_i(A)=\sum_{j\in N_2,j\neq i}|a_{ij}|,\quad P_i(A)=\alpha_i(A)+\beta_i(A);$$
$$\alpha_i'(A)=\sum_{j\in N_1,j\neq i}|a_{ji}|,\quad \beta_i'(A)=\sum_{j\in N_2,j\neq i}|a_{ji}|,\quad S_i(A)=\alpha_i'(A)+\beta_i'(A).$$

Take

$$N_r(A)=\Big\{i:i\in N,|a_{ii}|>P_i(A)\Big\};\qquad N_c(A)=\Big\{j:j\in N,|a_{jj}|>S_j(A)\Big\}.$$

The comparison matrix of A, which is denoted by $\mu(A)=(t_{ij})$, is defined as

$$t_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

It is known that A is a (row) diagonally dominant matrix (D_n) if for all $i=1,\dots,n$,

$$|a_{ii}|\geq P_i(A). \tag{1.1}$$

A is a γ-diagonally dominant matrix (D_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq \gamma P_i(A)+(1-\gamma)S_i(A),\quad \forall i\in N. \tag{1.2}$$

And A is called a product γ-diagonally dominant matrix (PD_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq [P_i(A)]^\gamma[S_i(A)]^{1-\gamma},\quad \forall i\in N. \tag{1.3}$$

If all inequalities in (1.1)-(1.3) hold, A is said to be strictly (row) diagonally dominant (SD_n), strictly γ-diagonally dominant (SD_n^γ), and strictly product γ-diagonally dominant (SPD_n^γ), respectively. If there exists a diagonal matrix D, with positive diagonal elements, such that AD is strictly diagonally dominant, strictly γ-diagonally dominant