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High Order Hierarchical Divergence-Free Constrained Transport $H(div)$ Finite Element Method for Magnetic Induction Equation

Wei Cai[1,*], Jun Hu[2] and Shangyou Zhang[3]

[1] *Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA*
[2] *School of Mathematical Sciences, Peking University, Beijing 100871, China*
[3] *Department of Mathematics, University of Delaware, Newark, DE 19716, USA*

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. In this paper, we propose to use the interior functions of an hierarchical basis for high order BDM_p elements to enforce the divergence-free condition of a magnetic field B approximated by the $H(div)$ BDM_p basis. The resulting constrained finite element method can be used to solve magnetic induction equation in MHD equations. The proposed procedure is based on the fact that the scalar $(p-1)$-th order polynomial space on each element can be decomposed as an orthogonal sum of the subspace defined by the divergence of the interior functions of the p-th order BDM_p basis and the constant function. Therefore, the interior functions can be used to remove element-wise all higher order terms except the constant in the divergence error of the finite element solution of the B-field. The constant terms from each element can be then easily corrected using a first order $H(div)$ basis globally. Numerical results for a 3-D magnetic induction equation show the effectiveness of the proposed method in enforcing divergence-free condition of the magnetic field.

AMS subject classifications: 65M60, 76W05
Key words: $H(div)$ finite elements, hierarchical basis, MHD, divergence-free.

1. Introduction

Numerical modeling of magneto-hydrodynamic fluids has shown that the observance of the zero divergence of the magnetic field plays an important role in reproducing the correct physics in plasmas [3]. Various numerical techniques have been devised to ensure the computed magnetic field to be divergence-free [6]. In the early

*Corresponding author. *Email addresses:* `wcai@uncc.edu` (W. Cai), `hujun@math.pku.edu.cn` (J. Hu), `szhang@udel.edu` (S. Y. Zhang)

work of [3] a projection approach was used to correct the magnetic field to have a zero divergence. A more natural way to satisfy this constraint is through a class of the so-called constrained transport (CT) numerical methods based on the ideas in [5]. In most CT algorithms for the MHD, the surface averaged magnetic flux over the surface of a 3-D element is used to represent the magnetic field so normal continuity of the magnetic field can be assured while the volume averaged conserved quantities are used for mass, momentum, and energy variables.

In this paper, we will propose a high order transport finite element method using a recently developed high order hierarchical basis for the BDM_p element [4] for the magnetic induction equation in the MHD problems. The divergence condition on the B field is enforced through corrections with interior functions in the basis set such that the global divergence-free condition will be satisfied.

The paper is organized as follows. In Section 2, we will present the hierarchical $H(div)$ basis functions in various modes (edge, face, and interior). In Section 3, we will first characterize the divergence of the interior basis functions for the hierarchical $H(div)$ basis, then, we introduce a two-step procedure to remove non-zero divergence in the finite element solution. Numerical test of the proposed procedure will be carried out for a 3-D magnetic induction equation in Section 4. Finally, a conclusion is given in Section 5.

2. Basis functions for the tetrahedral element

In this section we present hierarchical shape functions proposed in [2] for the $H(div)$-conforming tetrahedral BDM_p element on the canonical reference 3-simplex. The shape functions are grouped into several categories based upon their geometrical entities on the reference 3-simplex [1]. The basis functions in each category are constructed so that they are also orthonormal within each category on the reference element.

Any point in the 3-simplex K^3 is uniquely located in terms of the local coordinate system (ξ,η,ζ). The vertexes are numbered as $\mathbf{v}_0(0,0,0)$, $\mathbf{v}_1(1,0,0)$, $\mathbf{v}_2(0,1,0)$, $\mathbf{v}_3(0,0,1)$. The barycentric coordinates are given as

$$\lambda_0 := 1-\xi-\eta-\zeta, \quad \lambda_1 := \xi, \quad \lambda_2 := \eta, \quad \lambda_3 := \zeta. \tag{2.1}$$

The directed tangent on a generic edge $\mathbf{e}_j = [j_1, j_2]$ is defined as

$$\tau^{\mathbf{e}_j} := \tau^{[j_1,j_2]} = \mathbf{v}_{j_2} - \mathbf{v}_{j_1}, \quad j_1 < j_2. \tag{2.2}$$

The edge is parameterized as

$$\gamma_{\mathbf{e}_j} := \lambda_{j_2} - \lambda_{j_1}, \quad j_1 < j_2. \tag{2.3}$$

A generic edge can be uniquely identified with

$$\mathbf{e}_j := [j_1, j_2], \quad j_1 = 0,1,2, \quad j_1 < j_2 \le 3, \quad j = j_1 + j_2 + \mathrm{sign}(j_1), \tag{2.4}$$