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Linear Stability of Hyperbolic Moment Models for Boltzmann Equation

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

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Abstract. Grad's moment models for Boltzmann equation were recently regularized to globally hyperbolic systems and thus the regularized models attain local well-posedness for Cauchy data. The hyperbolic regularization is only related to the convection term in Boltzmann equation. We in this paper studied the regularized models with the presentation of collision terms. It is proved that the regularized models are linearly stable at the local equilibrium and satisfy Yong's first stability condition with commonly used approximate collision terms, and particularly with Boltzmann's binary collision model.

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Key words: Boltzmann equation, Grad's moment method, hyperbolic moment equation, linear stability.

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1. Introduction

Boltzmann equation [3] is the most important kinetic equation, governing the movement of a particle system, particularly the gas particles. Since the distribution function in the Boltzmann equation is in very high dimension, Grad [13] purposed the famous moment method for gas kinetic theory to reduce the kinetic equation into low-dimensional models. In more than half a century, Grad's moment equations were suffered by the lack of hyperbolicity [6,15]. Only very recently, in [4,5], the authors revealed the underlying reason that Grad's moment equations lost its hyperbolicity during the model reduction and purposed new reduced models of Boltzmann equation. The

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new models are referred to as globally Hyperbolic Moment Equations (HME) hereafter, which are symmetric quasi-linear systems [7] with global hyperbolicity.

As new models for fluid dynamics, one may prefer to carry out studies on some fundamental mathematical properties on HME before further numerical applications. Among these fundamental mathematical properties, linear stability is one of the most important points [2, 16, 19] for a system to be applied in numerical experiments. It should be noted that the linear stability is not automatically attained for models in fluid dynamics. For instance, famous Burnett equations and super-Burnett equations are discovered not linearly stable [2,18] and thus are ill-posed and rarely have practical applications.

Except for linear stability, Yong proposed a stability condition [21,22], called Yong's first stability condition, for nonlinear first-order hyperbolic systems with source term. With this stability condition, a formal asymptotic approximation of the initial-layer solution to the nonlinear problem has been constructed [22]. Furthermore, with some regularity assumption of the solution, the existence of classical solutions is guaranteed in the uniform time interval. The stability condition is essential for the nonlinear first-order hyperbolic system. And in [21,22], several classical models have been verified to satisfy the stability condition.

In this paper, we focus on the linear stability analysis of HME at local equilibrium and Yong's first stability condition. The collision term under consideration includes the commonly used approximate formations, such as BGK model [1], ES-BGK model [14], Shakhov model [17] and the original Boltzmann's collision term [3], particularly the binary collision term [9, 12]. We prove that both HME and Ordered globally Hyperbolic Moment Equations(OHME) are linearly stable at local equilibrium for all the four collision models and satisfy Yong's first stability condition.

We start with a brief review of HME and the collision term to be considered. The globally hyperbolic regularization enables us to write HME into an elegant quasi-linear form. It is essential to expand the distribution function at the local equilibrium, where the collision term vanishes. This property provides us some additional equalities which significantly simplify the linear stability analysis. For the binary collision model, the symmetry of the collision plays an important role, which indicates some induced symmetry in the Jacobian of the collision term. With some linear algebra, we proved that HME is linear stable at local equilibrium for all the four collision models. This proof is not trivial noticing that HME we are studying is for arbitrary order.

For Yong's first stability condition, the third inequality plays a major role. We verified this inequality by applying the results in the linear stability analysis, together with some linear algebraic technique. In such sense, Yong's first stability condition can be regarded as an enhanced version of linear stability for nonlinear balance laws.

OHME, first proposed in [10], is the hyperbolic version of ordered Grad's moment system, which includes the well-known Grad's 13 moment system. Since OHME can be derived from HME, the linear stability of OHME at the local equilibrium is deduced from that of HME, as well as Yong's first stability condition.

The rest of the paper is organized as following. Section 2 presents a brief intro-