Numer. Math. Theor. Meth. Appl. Vol. **10**, No. 2, pp. 331-350
doi: 10.4208/nmtma.2017.s07 May 2017

High Order Mass-Lumping Finite Elements on Simplexes

Tao Cui[1], Wei Leng[1], Deng Lin[1,2], Shichao Ma[1,2] and Linbo Zhang[1,2,*]

[1] *State Key Laboratory of Scientific and Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 East Zhongguancun Road, Beijing 100190, China*
[2] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

Received 19 October 2016; Accepted (in revised version) 17 January 2017

Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. This paper is concerned with the construction of high order mass-lumping finite elements on simplexes and a program for computing mass-lumping finite elements on triangles and tetrahedra. The polynomial spaces for mass-lumping finite elements, as proposed in the literature, are presented and discussed. In particular, the unisolvence problem of symmetric point-sets for the polynomial spaces used in mass-lumping elements is addressed, and an interesting property of the unisolvent symmetric point-sets is observed and discussed. Though its theoretical proof is still lacking, this property seems to be true in general, and it can greatly reduce the number of cases to consider in the computations of mass-lumping elements. A program for computing mass-lumping finite elements on triangles and tetrahedra, derived from the code for computing numerical quadrature rules presented in [7], is introduced. New mass-lumping finite elements on triangles found using this program with higher orders, namely 7, 8 and 9, than those available in the literature are reported.

AMS subject classifications: 65M60, 65D05, 65D32

Key words: Finite element, mass-lumping, simplex, numerical quadrature, multivariate polynomial interpolation, unisolvence.

1. Introduction

In this paper we refer by mass-lumping finite elements or simply mass-lumping elements, also known as spectral elements, to a special class of H^1-conforming Lagrangian finite element bases with (approximately) orthogonal basis functions [1, 2].

*Corresponding author. *Email addresses:* tcui@lsec.cc.ac.cn (T. Cui), wleng@lsec.cc.ac.cn (W. Leng), lind@lsec.cc.ac.cn (D. Lin), masc@lsec.cc.ac.cn (S. C. Ma), zlb@lsec.cc.ac.cn (L. B. Zhang)

Mass-lumping elements may have advantages over traditional elements when solving some types of problems. For example, for time dependent problems, such as time domain wave equations, when explicit time stepping is used, one obtains diagonal mass matrices with mass-lumping elements, thus avoiding solutions of large sparse linear systems of equations. Another example is eigenvalue problems arising from partial differential equations, the resulting discrete algebraic systems are standard eigenvalue problems when using mass-lumping elements, instead of generalized eigenvalue problems as obtained using traditional elements. For the case of one dimension, it is easy to construct mass-lumping elements using Gauss-Lobatto quadrature points. For two or higher dimensions and tensor-product elements such as quadrilaterals and hexahedra, mass-lumping elements can be constructed using tensor products of one dimensional elements. For two or higher dimensions and simplicial elements such as triangles (2D) and tetrahedra (3D), the construction of mass-lumping elements of orders higher than 1 becomes highly non trivial and there is still no systematic way of doing it. In the past two decades, numerical and symbolic algorithms have been designed to compute or search high order mass-lumping elements. So far, mass-lumping elements up to order 6 for triangles and order 3 for tetrahedra have been reported [1–5]. A 7-th order basis for triangles has been reported in [6], but its numerical quadrature precision is not high enough to ensure the approximation order of numerical solutions.

The purpose of this paper is to address problems related to the construction of mass-lumping elements on simplexes. More precisely, we give a systematic presentation of the class of polynomial spaces for constructing mass-lumping elements as proposed in the existing literature and discuss their properties as well as some related issues, including H^1-conformity and unisolvence of symmetric point-sets. An interesting property about unisolvent symmetric point-sets for the polynomial spaces used in the mass-lumping elements is revealed which, we believe, not only is useful in the computation of mass-lumping elements, but also has important implications in the more general field of polynomial interpolations. A program capable of computing high order mass-lumping elements on triangles and tetrahedra is introduced. It is derived from the program for computing symmetric quadrature rules on triangles and tetrahedra presented in [7]. With this program we were able to find out some new mass-lumping elements on triangles with higher orders than those reported in the literature. This work, as well as the work presented in [7], is a part of the project for developing the parallel adaptive finite element toolbox Parallel Hierarchical Grid (PHG) [8, 9].

The rest of the paper is organized as follows. Section 2 presents basic concepts, notations and properties used in this paper which are related to numerical quadrature and Lagrangian interpolation on simplexes. In Section 3, we discuss the problem of constructing mass-lumping elements, address some related issues including unisolvence of symmetric point-sets on simplexes, and present our numerical algorithm for computing mass-lumping elements on triangles and tetrahedra. In Section 4, we present the mass-lumping elements found using our program. The new elements, including elements of 7, 8 and 9-th order, are validated with numerical tests. Finally in Section 5, we conclude the paper with some remarks.