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Analysis of Hexagonal Grid Finite Difference Methods for Anisotropic Laplacian Related Equations

Daniel Lee*

Department of Applied Mathematics, Tunghai University, Taichung 40704, Taiwan, Republic of China

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Abstract. Hexagonal grids are valuable in two-dimensional applications involving Laplacian. The methods and analysis are investigated in current work in both linear and non-linear problems related to anisotropic Laplacian. Ordinary and compact hexagonal grid finite difference methods are developed by elementary arguments, and then analyzed by perturbation for standard Laplacian. In the anisotropic case, analysis is done through reduction to the standard one by using Fourier vectors of mixed types. These hexagonal seven-point methods, with established theoretic stabilities and accuracies, are numerically confirmed in linear and semi-linear anisotropic Poisson problems, and can be applied also in time-dependent problems and in many applications in two-dimensional irregular domains.

AMS subject classifications: 65N06, 65N12, 65N22, 15A18, 15A63

Key words: Compact scheme, hexagonal grid method, matrix perturbation, fourier vectors.

1. Introduction

Hexagonal (Hex) grid methods are of interest in many research studies: [16] (direct method), [17] (has a formula without proof), [5] (seven-point method on rectangular grid with explicit form of eigen pairs), [27] (periodic boundary condition), [8–10] (numerical modeling in spherical coordinates), [7] (action potential in numerical heart modeling), [20] (showing advantages of Hex grids over commonly used square grids for use in atmospheric and ocean models). In the article [15], Hex grid FD methods are derived in a finite volume (FV) approach involving standard Laplacian, and used in the simulation of electrical wave phenomena propagated in two-dimensional reserved-C type cardiac tissue, exhibiting both linear and spiral waves more efficiently than similar computation carried on rectangular FVs. We note these cited works all used standard Laplacian and mostly on one configuration of regular hexagon.

*Corresponding author. *Email address:* `danlee@thu.edu.tw` (D. Lee)

Aiming at two-dimensional applications involving anisotropic Laplacian (e.g., [21]), we extend in the current work the hexagonal seven-point methods to solving partial differential equations involving anisotropic Laplacian. Discretizations based on both ordinary and compact hexagonal seven-point methods are developed and analyzed. Also considered are the associated Hex grid discretizations with three-color iterative update [2,13,24]. The applications include linear and semilinear anisotropic Poisson equations.

In case of a configuration consisting of (subset of) Cartesian type regular hexagons, we denote by r the radius of a typical hexagon, h $(=\sqrt{3}r/2)$ the height, and d $(=2h)$ the *center-to-center* distance. At a typical center node, $P_0=(x_0,y_0)$, the six neighbor (center) nodes are

$$P_j=(x_j,y_j)=(x_0,y_0)+d(\cos\xi_j,\sin\xi_j),\quad \xi_j=\varphi+\frac{\pi}{6}+\frac{j\pi}{3},\quad j=1,\cdots,6. \tag{1.1}$$

Here the *phase angle*, φ, serves as the configurarion parameter. We focus on two particular instances : type I ($\varphi=0$) and type II ($\varphi=-\pi/6$). Hexagon centers in lattices of these two types are indexed as for an orthogonal Cartesian mesh as shown in Table 1, while the geometry and neighborhood of a general Hex FV in Table 2. We refer to Figs. 1(a), 1(b), 2(a), and 2(b) for visualization, and propose a three-color (Figs. 1(c) and 1(d)) algorithm in §4.2. In applications, a two-dimensional irregular domain may be approximated by a sequence of (not necessarily Cartesian) nets of hexagons. Actually, previous work in numerical modeling of electrocardiogram in a reversed C-type domain relies on this (Algorithm 1 in [15]).

Table 1: Lattices of type I and II regular hexagons.

Phase angle	Type I, $\varphi=0$		Type II, $\varphi=-\pi/6$	
Center point	i^{even}	i^{odd}	j^{even}	j^{odd}
$cx(i,j)$	$(1.5i-0.5)r$		$2ih$	$(2i-1)h$
$cy(i,j)$	$2jh$	$(2j-1)h$	$(1.5j-0.5)r$	

Table 2: Local geometry at a hexagon : six vertices and six neighbor centers with indices periodically extended when appropriate.

Phase angle	$\varphi\in\mathbb{R}$
Vertices	$V_k=\big(vx(*,k),vy(*,k)\big)$, $k=1,2,\cdots,6$
$vx(*,k)$	$cx(*)+r\cos(\varphi+\frac{k\pi}{3})$
$vy(*,k)$	$cy(*)+r\sin(\varphi+\frac{k\pi}{3})$
Neighbor centers	$P_k=V_k+V_{k+1}-P_0,\qquad k=1,2,\cdots,6$

In analyzing the discrete anisotropic Laplacian, we note that, (i) spectral analysis of iterative methods on a net of hexagons seems not as easy as the analysis on square grids [12,25], because the set of finite trigonometric series is incomplete for the error analysis (even) on a single regular hexagon [18,19,26], and (ii) for Hex FVs of types I and II, con-