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An Unconditionally Stable and High-Order Convergent Difference Scheme for Stokes' First Problem for a Heated Generalized Second Grade Fluid with Fractional Derivative

Cuicui Ji and Zhizhong Sun*

Department of Mathematics, Southeast University, Nanjing 210096, P. R. China

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Abstract. This article is intended to fill in the blank of the numerical schemes with second-order convergence accuracy in time for nonlinear Stokes' first problem for a heated generalized second grade fluid with fractional derivative. A linearized difference scheme is proposed. The time fractional-order derivative is discretized by second-order shifted and weighted Grünwald-Letnikov difference operator. The convergence accuracy in space is improved by performing the average operator. The presented numerical method is unconditionally stable with the global convergence order of $\mathcal{O}(\tau^2+h^4)$ in maximum norm, where τ and h are the step sizes in time and space, respectively. Finally, numerical examples are carried out to verify the theoretical results, showing that our scheme is efficient indeed.

AMS subject classifications: 65M06, 65M12, 65M15, 35R11

Key words: Stokes' first problem, heat flow, fractional derivative, finite difference scheme, unconditional stability, convergence.

1. Introduction

In the last few decades, models of viscoelastic flow have attracted considerable interest due to its powerful potential to depict many processes in petroleum industries, chemical, biorheology, geophysical and so on. The Stokes' first problem for the flat plate as well as the Rayleigh-Stokes problem for an edge has been investigated by many researchers. For the details, readers can refer to the works [1–6].

It should be pointed out that fractional models have been found to be quite flexible in describing the properties of viscoelastic fluid, where the classical differential equations are modified by replacing time integer-order derivatives with time fractional-order derivatives. Nowadays, there exist many methods to solve these fractional models. The analytical methods mainly cover the Fourier sine transform and the fractional Laplace transform, for instance, [7–10]. Whereas, the exact solutions cannot work well in real applications,

*Corresponding author. *Email address:* `zzsun@seu.edu.cn` (Z. Z. Sun)

due to the complexity of the representation. Hence, the effective and realizable numerical methods have been resorted to compute the fractional models.

This paper numerically deals with the following nonlinear Stokes' first problem for a heated generalized second grade fluid with fractional derivative

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathscr{D}_t^{1-\beta}\left[\kappa_1 \frac{\partial^2 u(x,t)}{\partial x^2}\right] + \kappa_2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t),x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{1.1}$$

with initial condition

$$u(x,0) = 0, \quad 0 \le x \le L, \tag{1.2}$$

and boundary conditions

$$u(0,t) = \phi_1(t), \quad u(L,t) = \phi_2(t), \quad 0 < t \le T, \tag{1.3}$$

where $0 < \beta \le 1$, the constants $\kappa_1, \kappa_2 > 0$, $\phi_1(t), \phi_2(t)$ are sufficiently smooth functions and the symbol ${}_0\mathscr{D}_t^{1-\beta}$ is the Riemann-Liouville fractional derivative operator defined by

$${}_0\mathscr{D}_t^{1-\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\beta}} d\eta,$$

where $\Gamma(\cdot)$ is the gamma function. Additionally, we assume that (1.1)–(1.3) has a smooth solution $u(x,t)$ and the nonlinear source term $f(u,x,t)$ satisfies the local Lipschitz condition with respect to u in the neighborhood of the solution $u(x,t)$, that is to say, there exist two positive constants η_0 and L_f such that

$$|f(u(x,t)+\eta_1,x,t) - f(u(x,t)+\eta_2,x,t)| \le L_f|\eta_1 - \eta_2|, \quad 0 \le x \le L, \quad 0 \le t \le T, \tag{1.4}$$

where $|\eta_1| \le \eta_0$, $|\eta_2| \le \eta_0$, and L_f is called the Lipschitz constant.

Till now, some numerical methods have been presented to simulate Stokes' first problem for a heated generalized second grade fluid with fractional derivative (SFP-HGSGF-FD). Chen et al. [11] proposed one implicit and one explicit numerical approximation schemes for the SFP-HGSGF-FD with the fractional derivative discretized by first-order shifted Grünwald-Letnikov formula. They also analyzed the stability and convergence of both the schemes using a Fourier method. Wu [12] first integrated both sides of the SFP-HGSGF-FD to obtain an integro-differential equation, then an implicit numerical scheme was gained by numerical integral, which was unconditionally stable with the global convergence order of $\mathscr{O}(\tau + h^2)$ in L_2 norm. Lin and Jiang [13] gave the exact solution for the SFP-HGSGF-FD by the series expansion in the reproducing kernel space. Afterwards, the n-term approximation solution was obtained by truncating the series. Chen et al. [14] developed a numerical scheme with high spatial accuracy for a variable-order nonlinear SFP-HGSGF-FD, where the variable-order fractional derivative was discretized by the