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A Full Multigrid Method for Distributed Control Problems Constrained by Stokes Equations

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Abstract. A full multigrid method with coarsening by a factor-of-three to distributed control problems constrained by Stokes equations is presented. An optimal control problem with cost functional of velocity and/or pressure tracking-type is considered with Dirichlet boundary conditions. The optimality system that results from a Lagrange multiplier framework, form a linear system connecting the state, adjoint, and control variables. We investigate multigrid methods with finite difference discretization on staggered grids. A coarsening by a factor-of-three is used on staggered grids that results nested hierarchy of staggered grids and simplified the inter-grid transfer operators. A distributive-Gauss-Seidel smoothing scheme is employed to update the state- and adjoint-variables and a gradient update step is used to update the control variables. Numerical experiments are presented to demonstrate the effectiveness and efficiency of the proposed multigrid framework to tracking-type optimal control problems.

AMS subject classifications: 35Q93, 65N55, 49K20

Key words: PDE-constrained optimization, stokes equations, multigrid, staggered grids, finite difference.

1. Introduction

For the last decade, there is a growing interest among computation community to devise fast and efficient solution methods for large-scale distributed optimal control problems constrained by partial differential equations (PDEs). Optimal control problems constrained by the Stokes system form a stepping stone in the natural progression from the -now classical-Poisson-constrained test problem to problems constrained by more specialized and complex PDE systems modeling fluid flow such as Navier-Stokes, non-Newtonian Stokes, or the shallow water equations. Optimal control problems constrained by such PDE

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models play important roles in real-world applications, such as modeling of ice sheets or data assimilation for ocean flows and weather models.

How to effectively solve large scale algebraic systems arising from the discretization of PDEs is a fundamental question in scientific and engineering computing. For the positive definite linear systems corresponding to elliptic boundary value problems, multigrid methods are proven to be one of the most efficient algorithms [4, 13, 23]. However, it is much more challenging for saddle-point systems [2]. In the last few decades, much attention has been given to numerical solution of Stokes and/or Navier-Stokes systems, e.g., a distributive Gauss-Seidel relaxation based on the least squares commutator is devised for the saddle-point systems arising from the discretized Stokes equations in [22]. In [1], a distributed relaxation method for the incompressible Stokes problem has been advertised. A large selection of solution methods for linear systems in saddle point form is presented with an emphasis on iterative methods for large and sparse problems in [2].

As a motivation of this work, suppose that we have a flow that satisfies the Stokes equations in some domain with some given boundary condition, and then we have some mechanism (e.g., a magnetic field application) to change the forcing variable of the PDE. Suppose we have given functions, the so-called desired states. Then the question is how do we choose the forcing term, while satisfying the Stokes equations? One way of formulating such problem is by minimizing a cost functional of tracking-type with constrained PDE as Stokes equations, is presented in this article.

This paper aims at to construct an efficient multigrid scheme without any preconditioners, on staggered grids to solve (velocity and/or pressure tracking-type) distributed optimal control problem constrained by the Stokes equations. A significant amount of work has been devoted to develop multigrid methods for optimal control problems in the recent years, for example, see a review article [3] and the references therein. However, less attention has been given specifically to optimal control problems constrained by the Stokes system. In [9], multigrid preconditioners to accelerate the solution process of a distributed optimal control problem constrained by the Stokes equations are constructed. Recently, [19] proposed a robust all-at-once multigrid method for the Stokes control problem. This article extends the work [7], where a formulation and multigird solution for Cauchy-Riemann optimal control problems has been presented to control problems constrained by Stokes equations. Adopting a coarsening by factor-of-three strategy on staggered grids to these distributed control problems has the potential advantage of simplifying the inter-grid transfer operators, coarsening more quickly, and ultimately reducing the number of levels and parallel computations.

The paper is organized as follows. In the next Section 2, an optimal control problem with a cost functional of tracking-type is considered and its solution is characterized as an optimality system, in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. Discretization with finite difference on staggered grids, on non-uniform meshes, is presented in Section 3. In Section 4, a full multigrid scheme is discussed for solving the optimality system. It is shown that a coarsening by a factor-of-three of the mesh sizes is advantageous, easy to implement inter-grid transfer operators, and a nested hierarchy of staggered grids is obtained. A distributive-Gauss-Seidel relaxation scheme with inter-grid transfer operators is explained