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Error Splitting Preservation for High Order Finite Difference Schemes in the Combination Technique

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Abstract. In this paper we introduce high dimensional tensor product interpolation for the combination technique. In order to compute the sparse grid solution, the discrete numerical subsolutions have to be extended by interpolation. If unsuitable interpolation techniques are used, the rate of convergence is deteriorated. We derive the necessary framework to preserve the error structure of high order finite difference solutions of elliptic partial differential equations within the combination technique framework. This strategy enables us to obtain high order sparse grid solutions on the full grid. As exemplifications for the case of order four we illustrate our theoretical results by two test examples with up to four dimensions.

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1. Introduction

In many of todays applications high dimensional problems arise. Especially in the field of computational finance partial differential equations (PDEs) with several dimensions have to be solved to evaluate the price of financial products. Since the number of grid points in a tensor based grid grows exponentially with the dimension, the so called *curse of dimensionality* shows its effects very quickly. *Sparse grids* [17] and the *Combination Technique* [10] have proven their great ability to reduce the computational effort. Let $n \in \mathbb{N}_0$ denote the level and $h_n = 2^{-n}$ a mesh parameter of a numerical approximation, then the sparse grid representation of a function in d dimensions at level n has $\mathcal{O}\left(h_n^{-1}\log_2(h_n^{-1})^{d-1}\right)$ grid points, while the approximation accuracy is of order $\mathcal{O}\left(h_n^2\log_2(h_n^{-1})^{d-1}\right)$ under certain smoothness requirements, see e.g., [5]. Compared to a tensor based full grid with $\mathcal{O}(h_n^{-d})$ grid points and an accuracy of $\mathcal{O}(h_n^2)$ the total number of nodes is significantly decreased, whereas the accuracy is only deteriorated by the factor

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$\mathscr{O}\left(\log_2(h_n^{-1})^{d-1}\right)$. Thus, the sparse grid approach only suffers from the curse of dimensionality in a much lower extent. Sparse grids have successfully been used in [7, 9, 11] to solve PDEs with several dimensions. In order to construct a solution on the sparse grid, the combination technique can be used. It is based on linearly combining a sequence of solutions via interpolation. Since each solution can be computed independently of the others, the method is embarrassingly parallel. Hence, it can be efficiently implemented on a cluster to accelerate the computation time [7].

In the literature second order finite difference schemes are employed to solve each of the resulting subproblems and the solutions are combined to the sparse grid solution via multilinear interpolation. As far as we know, there exists only one article by Leentvaar and Oosterlee [14], where fourth order stencils are used. But the question, which interpolation technique is suitable, remains open. From an intuitive point of view it is clear that linear interpolation cannot preserve the order of the highly accurate subsolutions. In this paper we want to present interpolation techniques which do not interfere with the error splitting within the combination technique. Since high dimensional problems shall be solved, we use a tensor product approach to extend the univariate interpolation to the multivariate case.

The outline is as follows: in Sections 2 and 3 we give a short overview of the combination technique and motivate the need for high-order interpolation techniques. In Section 4 we take a closer look at the two dimensional test case. Here we can omit a complex notation and give the reader an idea of how the approach works. Later in Section 5 the framework is extended to the d dimensional case. Finally, numerical results are presented in Section 6.

2. Combination technique in a nutshell

Here we want to give a short introduction to the combination technique. It is based on linearly combining a sequence of discrete solutions to a more accurate solution. In order to achieve a higher accuracy, the error structure of the discrete solutions is exploited in such a way that low order errors cancel out. This can most easily be demonstrated in a two dimensional example. Let us consider the Poisson equation on the unit square $\Omega = (0,1)^2$,

$$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y) \qquad \text{on } \Omega, \tag{2.1a}$$

$$u(x,y) = g(x,y) \qquad \text{on } \partial\Omega \tag{2.1b}$$

with its discrete solution u_h on the grid Ω_h with mesh widths $h = (h_x, h_y)$ respectively. The discrete solution is computed via a standard second order finite difference scheme. Bungartz et al. were the first ones in [4], who proved with help of Fourier series of discrete and semi-discrete solutions that the error of the discrete solution consists of second order errors from each of the directions and one mixed error

$$\begin{aligned} &u_h(x,y) \\ =&u(x,y) + w_1(x,y;h_x)h_x^2 + w_2(x,y;h_y)h_y^2 + w_{1,2}(x,y;h_x,h_y)h_x^2 h_y^2. \end{aligned} \tag{2.2}$$