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Solving Constrained TV2L1-L2 MRI Signal Reconstruction via an Efficient Alternating Direction Method of Multipliers

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Abstract. High order total variation (TV^2) and ℓ_1 based (TV2L1) model has its advantage over the TVL1 for its ability in avoiding the staircase; and a constrained model has the advantage over its unconstrained counterpart for simplicity in estimating the parameters. In this paper, we consider solving the TV2L1 based magnetic resonance imaging (MRI) signal reconstruction problem by an efficient alternating direction method of multipliers. By sufficiently utilizing the problem's special structure, we manage to make all subproblems either possess closed-form solutions or can be solved via Fast Fourier Transforms, which makes the cost per iteration very low. Experimental results for MRI reconstruction are presented to illustrate the effectiveness of the new model and algorithm. Comparisons with its recent unconstrained counterpart are also reported.

AMS subject classifications: 90C25, 49M27, 68U10

Key words: Magnetic resonance imaging (MRI), high order total variation, alternating direction method of multipliers (ADMM), constrained model.

1. Introduction

Nowadays, magnetic resonance imaging (MRI) is crucial in diagnosis because of its noninvasive nature and glorious depiction of human organs and tissues. However, achieving high spatio-temporal resolutions is challenging in dynamic MRI due to the hardware limitations and the risk of peripheral nerve stimulation. Recently, compressive sensing

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(CS) theory [4] has been successfully applied to accelerate MRI scanning [34]. Partial data acquisition increases the spacing between read-out lines, thereby reducing scan time; however, this reduction in the number of recorded Fourier components leads to aliasing artifacts in images which must be removed by the image reconstruction process. Therefore, finding an approach for accurate reconstruction from highly undersampled k-space data is of great necessity for both quick MR image acquisition and clinical diagnosis.

Motivated by the compressive sensing theory, Lustig et al. [23] proposed their constrained model for the MR image reconstruction. In their work, this problem is formulated as follows:

$$\min_u \|u\|_{TV} + \tau\|\Phi^\top u\|_1, \quad s.t.\ \ \|Au - f\|_2 \leq \sigma, \tag{1.1}$$

where $\|u\|_{TV} = \int_\Omega |\nabla u|$ is the total variation [26] (TV), Φ is the wavelet transform (e.g. the Haar wavelet basis is used in our experiments). The superscript $^\top$ denotes transpose (conjugate) of matrices. $\|\Phi^\top u\|_1$ is the ℓ_1-norm of the representation of u under the wavelet transform Φ, $\tau > 0$ is a scalar which balances Φ sparsity with TV sparsity, $\sigma > 0$ is related to the noise level. Let $P \in \mathbb{R}^{p\times N}$ be a selection matrix containing p rows of the identity matrix of order N, $\mathscr{F}$ be the Fourier transform, and we have $A = P\mathscr{F}$.

Constrained problems are usually much more difficult to solve than the unconstrained ones. The unconstrained version [32] of model (1.1) reformulates the reconstruction as

$$\min_u \|u\|_{TV} + \tau\|\Phi^\top u\|_1 + \frac{\eta}{2}\|Au - f\|_2^2, \tag{1.2}$$

where the regularization parameter $\eta > 0$ is crucial to the reconstruction results: an improperly large weight for the data fidelity term results in serious residual artifacts, whereas an improperly small weight results in damaged edges and fine structures [10]. From the optimization theory, problems (1.1) and (1.2) are equivalent in the sense that solving one of the two will determine the parameter in the other such that both give the same solution.

A number of numerical methods have been proposed for solving the above unconstrained model (1.2) [9, 11, 30]. However, few works focus on solving the constrained problems (1.1) directly, while choosing a reasonable value of σ is usually much easier than find a suitable value of η. In order to overcome choosing of the regularization parameter, there is a good choice to solve the constrained model directly [1,2,13,19,25,36].

A significant advantage of TV regularization is that it preserves edges in the solution. However, the classical TV norm causes staircase effects in the smooth regions [8,22,24,28, 33]. To overcome the above drawback of the TV regularization, in 2003, a high order TV regularization norm $\|u\|_{TV^2} = \int_\Omega |\nabla^2 u|$ have been proposed by Tai et al in [24]. Inspired from TV^2 norm regularization, Xie et al. propose an unconstrained TV^2 based MR image reconstruction model [29] as:

$$\min_u \|u\|_{TV^2} + \alpha\|\Phi^\top u\|_1 + \frac{\eta}{2}\|Au - f\|_2^2, \tag{1.3}$$

where

$$\|u\|_{TV^2} = \|\nabla^2 u\| = \sqrt{(\nabla_{xx}u)^2 + (\nabla_{xy}u)^2 + (\nabla_{yx}u)^2 + (\nabla_{yy}u)^2}.$$