Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 1, pp. 160-168
doi: 10.4208/nmtma.OA-2017-0017 February 2018

Local Convergence for a Fifth Order Traub-Steffensen-Chebyshev-Like Composition Free of Derivatives in Banach Space

Ioannis K. Argyros[1,*] and Santhosh George[2]

[1] *Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA*

[2] *Department of Mathematical and Computational Sciences, NIT Karnataka, India 575025*

Received 13 September 2016; Accepted (in revised version) 24 March 2017

Abstract. We present the local convergence analysis of a fifth order Traub-Steffensen-Chebyshev-like composition for solving nonlinear equations in Banach spaces. In earlier studies, hypotheses on the Fréchet derivative up to the fifth order of the operator under consideration is used to prove the convergence order of the method although only divided differences of order one appear in the method. That restricts the applicability of the method. In this paper, we extended the applicability of the fifth order Traub-Steffensen-Chebyshev-like composition without using hypotheses on the derivatives of the operator involved. Our convergence conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

AMS subject classifications: 65D10, 49M15, 74G20, 41A25

Key words: Traub-Steffensen-Chebyshev-like composition, restricted convergence domain, radius of convergence, local convergence.

1. Introduction

We are concerned with the problem of approximating a solution x^* of the equation

$$F(x)=0, \tag{1.1}$$

where $F: D \subseteq \mathscr{B}_1 \longrightarrow \mathscr{B}_2$ is a Fréchet differentiable operator between Banach spaces $\mathscr{B}_1, \mathscr{B}_2$. Most of the solution methods for solving (1.1) are iterative and for iterative methods order of convergence is an important issue. Convergence analysis of higher order iterative methods require assumptions on the higher order Fréchet derivatives of the operator F. That restricts the applicability of these methods. In this study, we consider the local

*Corresponding author. *Email addresses:* `iargyros@cameron.edu` (I. K. Argyros), `sgeorge@nitk.ac.in` (S. George)

convergence for fifth order Traub-Steffensen-Chebyshev-like composition free of derivatives in Banach space studied by Sharma and Kumar in [16]. The method is defined by

$$y_n = x_n - A_n^{-1}F(x_n), \tag{1.2a}$$
$$z_n = y_n - A_n^{-1}F(y_n), \tag{1.2b}$$
$$x_{n+1} = z_n - B_nA_n^{-1}F(z_n), \tag{1.2c}$$

where

$$A_n = [w_n, x_n; F],$$
$$B_n = 2I - A_n^{-1}[z_n, y_n; F],$$
$$w_n = x_n + \beta F(x_n),$$

and $[., .; F]$ is a divided difference of order one on D^2.

Throughout this paper $L(\mathscr{B}_2, \mathscr{B}_1)$ denotes the set of bounded linear operators between $\mathscr{B}_1$ and $\mathscr{B}_2$ and $B(z,\rho), \bar{B}(z,\rho)$ stand, respectively for the open and closed balls in $\mathscr{B}_1$ with center $z \in \mathscr{B}_1$ and of radius $\rho > 0$.

The motivations for the construction of this method are that is derivative free, of convergence order five and efficient. Notice that in [16] favorable comparisons with other methods using similar information have been provided to show the advantages of the proposed method. The aim of this paper is not to present those comparisons but to extend the applicabilty of method (1.2) in the more general setting of a Banach space. We refer the reader to [16] for more detailed advantages, motivations and comparisons.

Convergence analysis in [16] is based on Taylor expansions and assumptions on the Fréchet derivative F up to the order five when $\mathscr{B}_1 = \mathscr{B}_2 = \mathbb{R}^i$. That limits the applicability of this method. As a motivational example, let us define function F on $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$F(x) = \begin{cases} x^5 \sin\frac{1}{x} + x^2 + x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, $x^* = 0$, is a solution of $F(x) = 0$. We have that

$$F'(x) = 5x^4 \sin\frac{1}{x} - x^3 \sin\frac{1}{x} + 2x + 1, \quad F'(0) = 1,$$
$$F''(x) = 20x^3 \sin\frac{1}{x} - 5x^2 \cos\frac{1}{x} - 3x^2 \cos\frac{1}{x} - x \sin\frac{1}{x} + 2,$$
$$F'''(x) = 60x^2 \sin\frac{1}{x} - 36x \cos\frac{1}{x} + \sin\frac{1}{x} - \frac{1}{x} \cos\frac{1}{x}.$$

Then, obviously function F does not have bounded third derivative in I. In this study we use only assumptions on the first Fréchet derivative of the operator F in our convergence analysis, so that the method (1.2) can be applied to solve equations but the earlier results cannot be applied [1–18] (see Example 3.2). Moreover, we avoid the usage of high order derivatives, since we rely on the computational and approximate computational order of