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A New Convergent Explicit Tree-Grid Method for HJB Equations in One Space Dimension

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Abstract. In this work we introduce a new unconditionally convergent explicit Tree-Grid Method for solving stochastic control problems with one space and one time dimension or equivalently, the corresponding Hamilton-Jacobi-Bellman equation. We prove the convergence of the method and outline the relationships to other numerical methods. The case of vanishing diffusion is treated by introducing an artificial diffusion term. We illustrate the superiority of our method to the standardly used implicit finite difference method on two numerical examples from finance.

AMS subject classifications: 65M75, 65C40

Key words: Tree-Grid Method, Hamilton-Jacobi-Bellman equation, Stochastic control problem.

1. Introduction

In this work we are interested in solving stochastic control problems (SCP) numerically. Such problems can be represented by so called Hamilton-Jacobi-Bellman (HJB) equations, and arise in many applications in physics, economics, or finance. This article is divided into five main sections. In this first introductory section, we define the stochastic control problem and HJB equation and discuss the most widely used numerical methods. In the Section 2, we derive the new Tree-Grid Method–the main result of this paper. The convergence of this method is proven in Section 3. In Section 4, we test the performance of the method on two problems from finance. We compare the results with the ones from the standardly used implicit finite-difference method. Finally, Section 5 presents the conclusion.

1.1. Problem formulation

We are concerned with searching for the value function $V(s,t)$ of the following *general stochastic control problem* (SCP):

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$$V(s,t) = \max_{\theta(s,t)\in\bar{\Theta}} \mathbb{E}\left(\int_t^T \exp\left(\int_t^k r(S_l,l,\theta(S_l,l))dl\right) f(S_k,k,\theta(S_k,k))dk \right.$$
$$\left. + \exp\left(\int_t^T r(S_k,k,\theta(S_k,k))dk\right) V_T(S_T)\middle| S_t = s\right), \tag{1.1a}$$
$$dS_t = \mu\big(S_t,t,\theta(S_t,t)\big)dt + \sigma(S_t,t,\theta(S_t,t))dW_t, \tag{1.1b}$$
$$0 < t < T, \quad s \in \mathbb{R}, \tag{1.1c}$$

where s is state variable and t is time. Here, $\bar{\Theta}$ is space of all *suitable* control functions from $\mathbb{R}\times[0,T]$ to a set Θ. For our purpose, we will suppose Θ to be discrete. If this is not the case, we can easily achieve this property by its discretization. We also suppose that the functions r, f, μ, σ, V_T are chosen *suitably*. For example, we demand Lipschitz continuity of μ, σ:

$$\exists K > 0: \ \forall t \in [0,T], \quad \theta \in \Theta, \quad s_1, s_2 \in \mathbb{R}:$$
$$\big|\mu(s_1,t,\theta) - \mu(s_2,t,\theta)\big| + \big|\sigma(s_1,t,\theta) - \sigma(s_2,t,\theta)\big| \le K|s_1 - s_2|.$$

For a detailed analysis of suitability of coefficient and control functions we reffer to some classic stochastic control literature e.g. [14, 20]. Now following Bellman's principle, the *dynamic programming equation* holds:

$$V(s,t_j) = \max_{\theta(s,t)\in\bar{\Theta}_{t_j}} \mathbb{E}\left(\int_{t_j}^{t_{j+1}} \exp\left(\int_{t_j}^k r(S_l,l,\theta(S_l,l))dl\right) f(S_k,k,\theta(S_k,k))dk \right.$$
$$\left. + \exp\left(\int_{t_j}^{t_{j+1}} r(S_k,k,\theta(S_k,k))dk\right) V(S_{t_{j+1}},t_{j+1})\middle| S_{t_j} = s\right), \tag{1.2}$$

where $0 \le t_j < t_{j+1} \le T$ are some time-points and $\bar{\Theta}_{t_j}$ is a set of control functions from $\bar{\Theta}$ restricted to the $\mathbb{R}\times[t_j,t_{j+1})$ domain. Using this Eq. (1.2), it can be shown [20, 21], that solving the SCP (1.1a), (1.1b) is equivalent to solving the so-called Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial V}{\partial t} + \max_{\theta\in\Theta}\left(\frac{\sigma(\cdot)^2}{2}\frac{\partial^2 V}{\partial s^2} + \mu(\cdot)\frac{\partial V}{\partial s} + r(\cdot)V + f(\cdot)\right) = 0, \tag{1.3a}$$
$$V(s,T) = V_T(s), \tag{1.3b}$$
$$0 < t < T, \quad s \in \mathbb{R}, \tag{1.3c}$$

where $\sigma(\cdot)$, $\mu(\cdot)$, $r(\cdot)$, $f(\cdot)$ are functions of s,t,θ. We should note that the maximum operator in (1.1a) and (1.3a) can be replaced by minimum, (supremum, infimum) operator and the whole following analysis will hold analogously. Another possible generalization is the use of both infimum and supremum in so called stochastic differential games and corresponding Hamilton-Jacobi-Bellman-Isaac equation [16]. Use of even more general