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Nodal Discontinuous Galerkin Method for Time-Domain Lorentz Model Equations in Meta-Materials

Shanghui Jia[1], Changhui Yao[2,*] and Shuai Su[3]

[1] *School of Statistic and Mathematics, Central University of Finance and Economics, Beijing 100081, China*
[2] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, China*
[3] *Graduate School of China Academy of Engeering Physics, Beijing 100088, China*

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Abstract. In this paper, Nodal discontinuous Galerkin method is presented to approximate Time-domain Lorentz model equations in meta-materials. The upwind flux is chosen in spatial discrete scheme. Low-storage five-stage fourth-order explicit Runge-Kutta method is employed in time discrete scheme. An error estimate of accuracy $\mathcal{O}(\tau^4 + h^n)$ is proved under the L^2-norm, specially $\mathcal{O}(\tau^4 + h^{n+1})$ can be obtained. Numerical experiments for transverse electric (TE) case and transverse magnetic (TM) case are demonstrated to verify the stability and the efficiency of the method in low and higher wave frequency.

AMS subject classifications: 35L05, 76M10

Key words: Time-domain Lorentz model, meta-materials, Runge-Kutta method, nodal discontinuous Galerkin method.

1. Introduction

The discontinuous Galerkin (DG) method [1] has gained more popularity in solving various differential equations [2–6] in recent years for its great flexibility in mesh construction and its convenience in parallel implementation. During the past years, Many DG researches were explored for electromagnetic systems in the free space [7–14] and dispersive media [15–17] whose permittivity depended on the wave frequency. Very recently, the DG method was developed to solve Maxwell's equations in meta-materials [18, 19].

The meta-materials [20, 21] are artificially structured electromagnetic nano-materials with some exotic properties such as negative refraction index and amplification of evanescent waves. The advantages of meta-materials mainly come from their potential applications in diverse areas such as building a perfect lens, sub-wavelength imaging and cloaking. In the past decade, engineers and physicists had been engaged in many numerical simulations for Maxwell's equations in meta-materials. However, such simulations were almost

*Corresponding author. *Email addresses:* `shjia@lsec.cc.ac.cn` (S. H. Jia), `chyao@lsec.cc.ac.cn` (C. H. Yao), `sushuaiby@sina.cn` (S. A. Su)

established on either the classic finite-difference time-domain (FDTD) method or some commercial software packages with constraints and limitations. In recent years, some effort [18, 20–22, 24] in developing and analyzing some finite element methods(FEMs) for time-domain Maxwell's equations involving meta-materials were established.

In [21], the author proved stability, existence and uniqueness of the solution for the Lorentz model equations. In [31], the author provided the existence and uniqueness for a vector wave integro-differential equation. In [32], the author provided stability for the single pole Debye medium model equations.

In [17], A discontinuous Galerkin method for the numerical approximation of time-dependent Maxwell equations in three different dispersive media was introduced. Both the L^2-stability and error estimate of the DG method were discussed in detail. In [18], the author proposed a leap-frog discontinuous Galerkin method to solve the time-dependent Maxwell's equations in meta-materials. Conditional stability and error estimates were proved for the scheme. In [30], the author developed a nodal discontinuous Galerkin method for solving the time-dependent Maxwell's equations when meta-materials were involved. Both semi- and fully-discrete schemes were constructed. Numerical stability and error estimate were proved for both schemes.

In this paper, Nodal discontinuous Galerkin method is presented to approximate time-domain Lorentz model equations in meta-materials. The upwind flux is chosen in spatial discrete scheme. Low-storage five-stage fourth-order explicit Runge-Kutta method is employed in time discrete scheme. The energy is decreasing in time. An error estimate of accuracy $\mathcal{O}(\tau^4+h^n)$ is proved under the L^2-norm, specially $\mathcal{O}(\tau^4+h^{n+1})$ can be obtained. Numerical experiments for TE and TM cases are demonstrated to verify the stability and the efficiency of the method in low and higher wave frequency. The content of this paper is summarized as follows. In Section 2, we present the governing equations for meta-materials and for deformation. In Section 3, We develop a DG method and conduct theory analysis. In Section 4, We get the semi-discrete DG method and use the classic low-storage five-stage fourth-order explicit Runge-Kutta method [28] for time discretization. Then in Section 5, we implement numerical results for the meta-material model and analyze large wavenumber.

2. Time-domain Lorentz model

Time-domain Lorentz model is described by the following governing equations

$$\varepsilon_0\frac{\partial\widetilde{\mathbf{E}}}{\partial\tilde{t}}+\frac{\partial\widetilde{\mathbf{P}}}{\partial\tilde{t}}-\nabla\times\widetilde{\mathbf{H}}=0, \qquad (0,T]\times\Omega, \tag{2.1a}$$

$$\mu_0\frac{\partial\widetilde{\mathbf{H}}}{\partial\tilde{t}}+\frac{\partial\widetilde{\mathbf{M}}}{\partial\tilde{t}}+\nabla\times\widetilde{\mathbf{E}}=0, \qquad (0,T]\times\Omega, \tag{2.1b}$$

$$\frac{1}{\varepsilon_0\tilde{\omega}_{pe}^2}\frac{\partial^2\widetilde{\mathbf{P}}}{\partial\tilde{t}^2}+\frac{\tilde{\Gamma}_e}{\varepsilon_0\tilde{\omega}_{pe}^2}\frac{\partial\widetilde{\mathbf{P}}}{\partial\tilde{t}}+\frac{\tilde{\omega}_{e0}^2}{\varepsilon_0\tilde{\omega}_{pe}^2}\widetilde{\mathbf{P}}-\widetilde{\mathbf{E}}=0, \qquad (0,T]\times\Omega, \tag{2.1c}$$