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A Numerical Study of Multiple Solutions for Laminar Flows in a Porous and Moving Channel

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Abstract. In this paper, based on the finite element formulation, we focus on multiple solutions and their evolution with time for a laminar flow in a permeable channel with expanding or contracting walls. Both Newtonian fluid and micropolar fluid are considered. For the Newtonian fluid model, we find that the profile of the unique solution in the case of injection remains the same for long time, which indicates that the solution may be stable. On the other hand, in the case of large suction, the profile of multiple solutions changes in time, which suggests that the multiple solutions may be unstable. Similar behaviors and conclusions are observed for the micropolar fluid model under different boundary parameters.

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1. Introduction

Studies of laminar flows in a porous channel have received considerable attention in recent years due to their wide range of applications in a number of engineering and biological models, for examples, the transport of biological fluids through vessels, the modeling of blood and air circulation in a respiratory system.

In the study of the refrigeration of steam and the separation process of U_{235} (a radioactive isotope of uranium element whose neutron number is 143), or in a gas diffusion process, Berman [1] obtained an asymptotic solution for small Reynolds numbers using a regular perturbation method assuming a steady, incompressible and laminar flow through a two dimensional porous channel with stationary walls in 1953. In the study, Berman also assumed that normal velocity is independent of the stream-wise coordinate and a fully developed flow. One of the key ideas in the proof is to transform the Navier-Stokes equations

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to a system of nonlinear ordinary differential equations that contains only a permeation Reynolds number by introducing the stream function.

In the following decades, many researchers have extended and generalized Berman's work, particularly in the following three directions.

First of all, numerous studies of channel flows through porous and rigid walls have been conducted. For both small and large Reynolds numbers *Re*, for example, Terrill [2] obtained the solutions using a perturbation method. Using an integral equations approach, Proudman [3] also investigated the solution for large Reynolds number cases. Later on, again using a singular perturbation method, Yuan [4] obtained solutions for the case of a large injection rate. Using the average method, Morduchow [5] obtained an analytical solution in the entire injection range.

Secondly, in seeking further generalization, Dauenhauer and Majdalani [6] considered the case of a laminar flow in a porous channel with expanding or contracting walls in 2003. By introducing an expansion ratio α and assuming it is a constant, the authors reduced the governing equations to an ordinary differential equation only involving the Reynolds number and the expansion ratio. The Runge-Kutta method coupled with a shooting method is used to obtained asymptotic solutions over a modest range of *Re* and α. Furthermore, not only the authors carried on the mechanism analysis to the related physical quantities of their model, but also obtained the relations between *Re* and α.

Thirdly, the existence of multiple solution is revealed in the course of discussions on Berman's work. Raithby [7] first discovered two numerical solutions in the research of the heat transfer in a rectangular channel at the entrance. In 1976, Robinson [8] carried out a numerical research about the multiplicity of the solutions for the flow in a porous channel and drawn the following conclusions. When $Re < 12.165$, the corresponding ordinary differential equation by the similarity transformation has only one numerical solution; but when $12.165 \leq Re < \infty$, the ODE has three solutions, which are labeled type I, type II, type III. Meanwhile, Raithby [7] also obtained two asymptotic solutions through a singular perturbation method. As for the stability analysis on a laminar flow, Sobey and Drazin [9] were the pioneers to investigate the stability and bifurcation of a symmetric flow in a two-dimensional rectangular regions in 1986. According to Sobey and Drazin, there is a unique solution which is stable for small Reynolds numbers. The unique solution will become unstable as the Reynolds reaches a critical value. Since then, many other interesting studies on this subject have been appeared in the literature, for examples, by Zaturska et al. [10], Waton et al. [11], Durlofsky and Brady [12], Cox and King [13].

Recently, using a homotopy method, Xu et al. [14] investigated multiple solutions of the Navier-Stokes equations for the Newtonian fluid model in a permeable channel with orthogonally moving walls in the context of a suction process assuming constant Reynolds number and the expansion ratio α. Two or three solutions are obtained under some values of *Re* and α. The authors also considered a more general case that α is a function of time and concluded that the solution reaches the steady state faster in the case of a larger suction rate $\alpha(t)$ compared that with a constant α. Following Xu et al. [14]'s work, Si et al. [15] investigated the same problem and obtained dual solutions for large suction rate by using a singular perturbation method. Based on a finite element method, Xu et al. [16]