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The Legendre Galerkin-Chebyshev Collocation Method for Space Fractional Burgers-Like Equations

Yubo Yang[1,*] and Heping Ma[2]

[1] *Nanhu College, Jiaxing University, Jiaxing, Zhejiang 314001, China*
[2] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

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Abstract. In this paper, a Legendre Galerkin Chebyshev collocation method for the Burgers-like equations with fractional nonlinear term and diffusion term is developed. This method is based on the Legendre-Galerkin variational form, but the nonlinear term and the right-hand term are treated by Chebyshev-Gauss interpolation. Rigorous stability and convergence analysis are developed. Numerical examples are shown to demonstrate the accuracy, stability and effectiveness of this method.

AMS subject classifications: 65M70, 65M12, 65M15, 26A33, 35R11
Key words: Space fractional Burgers-like equations, Legendre Galerkin Chebyshev collocation method, stability and convergence analysis.

1. Introduction

In this paper, we consider the following space fractional Burgers-like equations [1]:

$$\begin{cases} \partial_t U(x,t) + {}_{-1}\mathscr{D}_x^{\mu_1} f(U(x,t)) - \nu\, {}_{-1}\mathscr{D}_x^{2\mu_2} U(x,t) = g(x,t), & (x,t)\in(-1,1)\times(0,T), \\ U(-1,t)=U(1,t)=0, & t\in(0,T), \\ U(x,0)=U_0(x), & x\in(-1,1), \end{cases} \tag{1.1}$$

where $\nu>0, 0<\mu_1<1, \frac{1}{2}<\mu_2<1$, $f\in C^2(-\infty,\infty)$ and ${}_{-1}\mathscr{D}_x^{\mu_1}, {}_{-1}\mathscr{D}_x^{2\mu_2}$ are the left Riemann-Liouville derivative operators of order μ_1 and $2\mu_2$ respectively. Usually, $\mu_1\le\mu_2$. For $n-1<\mu<n, n\in\mathbb{N}$, the left and right Riemann-Liouville derivative operators ${}_{-1}\mathscr{D}_x^{\mu}$ and ${}_x\mathscr{D}_1^{\mu}$ are defined as (cf. [28])

$$\begin{aligned} {}_{-1}\mathscr{D}_x^{\mu}u(x) &= \frac{\partial^n}{\partial x^n}\left[{}_{-1}\mathscr{D}_x^{-(n-\mu)}u(x)\right] = \frac{1}{\Gamma(n-\mu)}\frac{\partial^n}{\partial x^n}\int_{-1}^{x}(x-s)^{n-\mu-1}u(s)\mathrm{d}s, \\ {}_x\mathscr{D}_1^{\mu}u(x) &= (-1)^n\frac{\partial^n}{\partial x^n}\left[{}_x\mathscr{D}_1^{-(n-\mu)}u(x)\right] = \frac{(-1)^n}{\Gamma(n-\mu)}\frac{\partial^n}{\partial x^n}\int_{x}^{1}(s-x)^{n-\mu-1}u(s)\mathrm{d}s, \end{aligned}$$

*Corresponding author. *Email addresses:* `boydman_xm@mail.zjxu.edu.cn` (Y. B. Yang), `hpma@shu.edu.cn` (H. P. Ma)

respectively, where ${}_{-1}\mathscr{D}_x^{-\mu}$ and ${}_x\mathscr{D}_1^{-\mu}$ are the left and right Riemann-Liouville integral operators respectively, defined as

$$
\begin{aligned}
{}_{-1}\mathscr{D}_x^{-\mu}u(x) &= \frac{1}{\Gamma(\mu)}\int_{-1}^{x}(x-s)^{\mu-1}u(s)\mathrm{d}s, \quad \mu>0,\\
{}_x\mathscr{D}_1^{-\mu}u(x) &= \frac{1}{\Gamma(\mu)}\int_{x}^{1}(s-x)^{\mu-1}u(s)\mathrm{d}s, \quad \mu>0.
\end{aligned}
$$

A large variety of physically motivated fractional Burgers equations can be found in literature (see, e.g. [2–4]), including applications to hydrodynamics, statistical mechanics, physiology and molecular biology. The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe [3]. The fractional derivative results from the memory effect of the wall friction through the boundary layer [3,5,6]. The same form can be found in other systems such as shallow-water waves [7] and waves in bubbly liquids [8]. We refer to [3,9–12] for an incomplete list of references on the applications of the fractional Burgers equation. The approximate solutions of time and/or space Burgers equations are obtained by several methods, such as Adomian decomposition method [4,13,14], homotopy analysis method [15,16], variational iteration method [17,18] and parametric spline functions method [19].

Besides, several numerical methods are used to solve the fractional Burgers equation, such as finite difference (FD) method [3,13,20–23], and spectral method [1,3,24]. Sugimoto [3] solved the space-fractional Burgers equation by FD method and Fourier spectral method in the numerical experiments, respectively. Fractional Burgers equation with fractional nonlinear term and diffusion term is introduced by Zayernouri et al. [1], in order to test the spectral collocation method based on the Jacobi polyfractonomials. The time-fractional Burgers equation is taken as a numerical example by Li [13] and Zhao [21], and solved by FD method. Esen et al. [23] developed a full discrete scheme for the time-fractional Burgers equation, based on FD method in time and Haar wavelet method in space. A space-time Legendre spectral collocation method is proposed to solve space-time fractional Burgers equation in [24]. No error analysis was explored in the above researches. Recently, a FD method is used to solve the time-fractional Burgers equation defined by a new generalized time fractional derivative [20], and stability analysis is provided. In [22], Li et al. proposed a linear implicit finite difference scheme for solving the time-fractional Burgers equation, and a convergence rate of $\mathscr{O}(\tau+h^2)$ is established, where τ and h are the temporal and spatial step sizes respectively.

The aim of this study is to develop a Legendre Galerkin Chebyshev collocation method for the Burgers-like equations with fractional nonlinear term and diffusion term. Due to the existence of the space fractional derivative operator, local methods such as FD method and finite element method (FEM) loss a big advantage that they enjoy for usual PDEs. On the other hand, the main disadvantage of global methods such as spectral method is no longer an issue for fractional PDEs [25,26]. Therefore, spectral method should be better suited for spatial discretization of the problem (1.1). In this paper, the Legendre Galerkin