Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 2, pp. 354-382
doi: 10.4208/nmtma.OA-2017-0032 May 2018

Convergence and Stability Analysis of Exponential General Linear Methods for Delay Differential Equations

Jingjun Zhao[1], Yu Li[1,2] and Yang Xu[1,*]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*
[2] *College of Science, Heilongjiang Bayi Agricultural University, Daqing 163319, China*

Received 6 March 2017; Accepted (in revised version) 14 October 2017

Abstract. In this paper, we study the convergence and stability properties of explicit exponential general linear methods for delay differential equations. We prove that, under some assumptions, for delay differential equations in Banach spaces, these numerical methods converge essentially with the order $\min\{P, Q+1\}$, where P and Q denote the order and stage order of the methods for ordinary differential equations, respectively. By using an interpolation procedure for the delay term, we analyze the linear and nonlinear stability of exponential general linear methods for two classes of delay differential equations. The sufficient conditions on the stability of exponential general linear methods for the test delay differential equations are provided. Several numerical experiments are given to demonstrate the conclusions.

AMS subject classifications: 65L20

Key words: Delay differential equation, exponential general linear method, convergence, stability.

1. Introduction

Let X be a real or complex Banach space with the norm $\|\cdot\|$, $D \subset X$ be another densely embedded Banach space, $\tau \geq 0$ be a positive constant delay, and $g : [0,T] \times D \times D \to X$ be a given continuous mapping. In this paper, we consider the initial value problem of the delay differential equations (DDEs) of the form

$$\begin{cases} y'(t) = Ly(t) + g(t, y(t), y(t-\tau)), & t \in (0,T], \\ y(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $L : D \to X$ is a linear operator, and $\phi : [-\tau, 0] \to D$ is a initial continuous mapping. It is well known that the equations with delay which contain some of the past states

*Corresponding author. *Email addresses:* `hit_zjj@hit.edu.cn` (J. J. Zhao), `yangx@hit.edu.cn` (Y. Xu)

of the systems are more realistic to describe phenomena in nature, and they have comprehensive application background. Especially, the initial value problem of the form (1.1) which arises in the process of semi-discretization for the initial-boundary value problem of delay parabolic equation occurs in many branches of science and engineering fields such as physiology, epidemiology and electrical circuit simulation, see [2, 23, 33].

Various numerical methods have been studied for solving DDEs in $\mathbb{R}^n$, $\mathbb{C}^n$ and Hilbert space, for example, Runge-Kutta methods, linear multistep methods and general linear methods, see [3] and the references therein. Most of the research has focused on the convergence and stability properties of the numerical methods for DDEs. The convergence of the numerical methods such as Runge-Kutta methods and general linear methods for DDEs has been studied in [11, 15]. The stability of the numerical methods when adapted to DDEs has been investigated by using the system of linear DDEs

$$\begin{cases} y'(t) = Ly(t) + My(t-\tau), & t > 0, \\ y(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where $L, M \in \mathbb{R}^{d\times d}$. A significant number of important results of stability about this test equation have been found, for example, the stability properties of Runge-Kutta methods for Eq. (1.2) were studied in [14] which showed that any A-stable Runge-Kutta method is P-stable. For linear multistep methods, [24] investigated their stability behaviour for the linear neutral DDEs. The stability properties of general linear methods for delay differential equation can be found in [12], which investigated their linear stability properties by considering the neutral DDEs with multiple delays. Recently, the nonlinear stability of numerical methods for DDEs were discussed. For example, the concepts of RN-stability and GRN-stability were introduced in [25, 32], and the corresponding nonlinear stability properties of one-point collocation rules and Runge-Kutta methods were investigated. In [26–29], the stability results of some numerical methods, including θ-methods, Runge-Kutta methods and linear multistep methods, for nonlinear DDEs in Banach space were obtained.

On the other hand, exponential integrators have been applied to solve DDEs. In fact, exponential integrators are a class of numerical methods originally designed for the numerical solutions of semilinear problems, see [10]. They aim to solve the linear part of the problems exactly and then solve the remaining part numerically. There are three main classes of exponential integrators, which include exponential Runge-Kutta method (ERKs), exponential linear multistep methods (ELMMs) and exponential general linear methods (EGLMs). For the details of these methods, one can refer to [4, 9, 21] and so on. To our knowledge, for exponential integrators, only [30,31] studied the convergence and stability properties of ERKs for DDEs. Up to now, there is few result published on EGLMs for solving DDEs.

The purpose of this paper is to investigate the convergence and stability properties of EGLMs for DDEs. We give out the convergence results of EGLMs for nonlinear DDEs with constant delay in Banach space, analyze the stability of EGLMs for the test DDE (1.2), and also discuss the nonlinear stability of EGLMs. The outline of this paper is as follows. In