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Selection of Bases in Operational Calculus and its Applications

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Abstract. In this article, we present a new method for selecting a base that corresponds to the modified left shift operator in operational calculus. The method is illustrated by Emden-Fowler equation and differential equations with variable coefficients. The method, combined with Pade approximant, is also applied to solve a differential-difference equation which was solved by the Adomian decomposition method. Since the new method does not involve integrals, it is more efficient than the one in the literature.

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Key words: Operational calculus, differential-difference equation, series solution, selection of base.

1. Introduction

In 2010, Gabriel Bengochea and Luis Verde-Star proposed a method called operational calculus for modified shift operators on an abstract space of formal Laurent series to solve linear equations [1]. By introducing a linear algebraic setting, they solved many types of linear functional equations. Integrals are not involved in this method. What's more, some nonlinear problems and other types of problems, such as Emden-Fowler equation [6], difference equations and fractional differential equations [7] can also be solved by this operational calculus.

In operational calculus, an important thing is to choose a modified left shift operator L and the corresponding base $\{p_k\}$. Bengochea and Verde-Star [1] have given some bases in dealing with some kinds of problems such as differential equations with constant coefficients, differential equations with variable coefficients, difference equations and fractional

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differential equations. Moreover, they have also presented a general rule for choosing the modified left shift operators as well as the corresponding bases [1]. Some extensions of operational calculus are [2,3].

Given a modified left shift operator, a new method is proposed for choosing the corresponding base in this paper. Emden-Fowler equation and differential equations with variable coefficients are taken as examples to illustrate this method. In addition, the new method and Pade approximant (called POCM) are used to solve a differential-difference equation which has been solved by Adomian decomposition method (please refer to [4,5] for this method). This solution is almost the same as the one in the literature. However, the new approach doesn't involve integrals. So it is more efficient than the previous method.

The paper is organized as follows. In Section 2, an approach is proposed for selecting bases in operational calculus, and Emden-Fowler equation and differential equations with variable coefficients are used to illustrate and verify the approach. In Section 3, operational calculus and Pade approximant are applied to solve some differential-difference equations, and some comparisons are made. Finally, some concluding remarks are given in the last section.

2. Selection of bases in operational calculus

2.1. Selection rule of bases

The key part in operational calculus is the selection of the modified left shift operator and the corresponding base. According to [1], they must satisfy the following conditions:

$$\begin{cases} Lp_k = p_{k-1}, \quad \text{if} \quad k \neq 0, \\ Lp_0 = 0. \end{cases} \tag{2.1}$$

A theorem will be given in this paper to guide the way to select the base that corresponds to the modified left shift operator in operational calculus. Before giving the theorem, we give a definition first.

Definition 2.1. *Let A be the set of functions defined on the field of complex numbers $\mathbb{C}$. L is an operator defined on A. If there exists an operator L^{-1} such that for any function $f \in A$, there exists a constant c_f such that*

$$L^{-1}Lf(z) = f(z) - c_f, \tag{2.2}$$

then L^{-1} is called the weak left inverse operator of L, and c_f is called the left irreversible constant of f about L. If there exists an operator L^{-1} such that for any function $f \in A$, there exists a constant c_f such that

$$LL^{-1}f(z) = f(z) - c_f, \tag{2.3}$$

then L^{-1} is called the weak right inverse operator of L, and c_f is called the right irreversible constant of f about L. If L^{-1} is both the weak left inverse operator and the weak right inverse operator of L, we say L^{-1} is the weak inverse operator of L.