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Superconvergence of the Finite Volume Method for Stokes Problems

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Abstract. This paper presents a superconvergence analysis of the finite volume method for Stokes problems using the $P_1 - P_1$ velocity-pressure element pair. Based on some superclose estimates on the interpolation function, we derive a superconvergence result of rate $\mathcal{O}(h^{\frac{3}{2}})$ for the post-processed velocity approximation in the H^1-norm and for the directly computed pressure approximation in the L_2-norm, respectively. Numerical experiments are provided to illustrate the theoretical analysis.

AMS subject classifications: 65N30, 65M60

Key words: Finite volume method, $P_1 - P_1$ element pair, superconvergence, Stokes problem.

1. Introduction

The finite volume method (FVM) is one of the most important numerical methods for solving partial differential equations (see, e.g., [5–7,9,15,16,18–20,22,23,28]). The main feature of FVM is that it inherits some physical conservation laws of original problems locally, which is very desirable in practical applications. The basic idea of FVM is to choose a standard finite element space as the trial space on a primal mesh and choose the piecewise constant space as the test space on a dual mesh. In this paper, we will study the superconvergence of FVM for Stokes problems using the $P_1 - P_1$ polynomial pair. The $P_1 - P_1$ velocity-pressure pair is the lowest equal-order element pair in all velocity-pressure element pairs for Stokes and Navier-Stokes problems. This element pair has the advantages of high efficiency in implementation and satisfied accuracy [3,4,12,14,17,18,24].

Superconvergence of numerical solutions has been an active research area in scientific computations since its practical importance in enhancing the accuracy of numerical solutions. In existing literatures, there have been a lot of works on superconvergence of finite element methods, see [8, 21, 25, 31] and the references therein. Compared with finite element methods, the study of superconvergence properties of FVM has been far behind. Recently, for the FVM solving Stokes and Navier-Stokes problems using the low order element pairs, some superconvergence results (see (1.1)) have been obtained in [1,11,13,19].

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We know that in general, using the $P_1 - P_1$ element pair, one only can obtain an $\mathcal{O}(h)$-order error estimate for velocity approximation in the H^1-norm and pressure approximation in the L_2-norm by the standard finite element method and FVM [12, 18, 24]. However, by using the L_2-projection post-processing technique [26], the following superconvergence estimate can be derived

$$\|\mathbf{u} - L_h\mathbf{u}_h\|_1 + \|p - L_h p_h\| \leq Ch^{\frac{4}{3}}(\|\mathbf{u}\|_3 + \|p\|_2), \tag{1.1}$$

where $L_h\mathbf{u}_h$ and $L_h p$ are the post-processed velocity and pressure solutions, respectively. Superconvergence result (1.1) is derived originally in [27] for finite element method solving Stokes problem, and then it also is obtained for FVM solving Stokes problem [1, 11] and Navier-Stokes problem [13, 19] with the stabilized $P_1 - P_1$ element pairs. As far as we know, for the FVM on triangle meshes, almost all known superconvergence results for Stokes and Navier-Stokes problems are based on the use of the L_2-projection method and estimate (1.1) is the best result under the assumption of the exact solution $(\mathbf{u}, p) \in [H^3(\Omega)]^2 \times H^2(\Omega)$.

The goal of this paper is to present a new superconvergence analysis for FVM solving Stokes problem with the stable lowest equal-order element pair (that is, the P_1 velocity element combining with the P_1 pressure macro-element). Our superconvergence analysis is based on the superclose estimate and the interpolation post-processing technique. For the strongly regular triangle meshes [31], we prove the following superconvergence result:

$$\|\mathbf{u} - Q_{2h}\mathbf{u}_h\|_1 + \|p - p_h\| \leq Ch^{\frac{3}{2}}(\|\mathbf{u}\|_3 + \|p\|_2), \tag{1.2}$$

where $\mathbf{u}_h$ and p_h are the finite volume velocity and pressure solutions, respectively, and $Q_{2h}\mathbf{u}_h$ is the interpolation post-precessed solution. Let us illustrate the significance of our work: (1) We obtain a higher order superconvergence estimate compared with the known result (1.1). (2) The interpolation post-processing method used in this paper costs less computation than the L_2-projection method which needs to solve the least-square discrete equations on a high order polynomial space. (3) We prove that the discrete pressure p_h itself (without post-processing) possesses the superconvergence of $\mathcal{O}(h^{\frac{3}{2}})$-order, the numerical evidence of such superconvergence has been observed in [2] for the stabilized finite element method and in [10] for the local discontinuous Galerkin method.

This paper is organized as follows. In Section 2, we introduce the stable $P_1 - P_1$ finite volume approximation to the Stokes problem. In Section 3, some superclose estimates are established for the interpolation function under the strongly regular mesh condition. Section 4 is devoted to the superconvergence estimates. In Section 5, numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notations $W^{m,p}(D)$ to indicate the usual Sobolev spaces on subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$, and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$. When $D = \Omega$, we omit the index D. The notations $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm, respectively, in the $L_2(\Omega)$ space. We will use letter C to represent a generic positive constant, independent of the mesh size h.