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Evaluation Finite Moment Log-Stable Option Pricing by a Spectral Method

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Abstract. The classical Black-Scholes pricing model is based on standard geometric Brownian motion, and the log-returns of this model are independent and Gaussian. However, most of the recent researches on the statistical properties of the log-returns make this hypothesis not always consistent. One of the ongoing issues of mathematical finance today is to design an efficient numerical algorithm for the pricing model, which might be modified from the standard Black-Scholes diffusion equation and would have favorable empirical results.

Of those financial models that have been already proposed, the most interesting include the Finite Moment Log-Stable (FMLS) process model and its fractional partial integral-differential equation. In this paper, we consider to use Gauss-Jacobi spectral method on a two-dimensional computation domain in order to discretize the FMLS fractional partial integral-differential equation, and further illustrate the flexibility and accuracy of the method by comparing the first order finite difference scheme for the pricing examples of European and American-styled options. Our results suggest that the global character of the Gauss-Jacobi method makes them well-suited to fractional partial integral-differential equations and can naturally take the global behavior of the solution into account and thus do not lead to an extra computational cost when moving from a second-order to a fractional-order diffusion model.

AMS subject classifications: 35R11, 65M70

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1. Introduction

Black and Scholes [1] were the first pioneers to evaluate the price of options by using mathematical instruments. They derived the famous Black-Scholes partial differential equation in 1973, which describes the prices of the European-styled options over time.

Shortly after that, Merton [2] expanded the mathematical understanding of the pricing model and coined the term Black-Scholes-Merton options pricing model. This well-known

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pricing model, which is based upon the standard geometric Brownian motion, is of the hypothesis that the natural logarithm for stock price at time t, S_t, follows a random walk or diffusion, with a deterministic drift μ [3]:

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t. \tag{1.1}$$

Here, the parameter $\sigma > 0$ is the volatility of the stock returns, $\mu > 0$ is the average compounded growth of the stock price S_t, and dB_t is the increment of Brownian motion that under the normal distribution.

On the basis of the traditional Black-Scholes-Merton (TBSM), the price of a European-styled option, $V(S,t)$, which is written on the traded asset S_t, may also be demonstrated as an advection-diffusion type equation by making the change of the variable $\ln S_t = y_t$, together with the supposition that the accession of the underlying Brownian motion should under the *equivalent martingale measure*, or equally, the risk-neutral probability measure:

$$\frac{\partial V(y,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(y,t)}{\partial y^2} + \left(r - q - \frac{1}{2}\sigma^2\right) \frac{\partial V(y,t)}{\partial y} = rV(y,t), \tag{1.2}$$

where r is the risk-free interest rate, q is the continuous dividends yield [4], and the log-returns should be independent and Gaussian.

Much efforts have been made to develop mathematical models that can better describe the prices of financial assets recently. One possible idea is to adopt a Lévy process instead of a Brownian motion to estimate the prices of the underlying assets, as a Lévy process allows the appearance of jumps with independent and stationary increments and discontinuous paths in the underlying random walks. Koponen [5], Boyarchenko and Levendorskiî [6] proposed a modified Lévy Stable process, i.e. the regular Lévy Process of Exponential type, which was first suggested in [7], to model the dynamics of securities. This modification introduced a damping effect in the tails of the LS distribution to ensure finite moments and gain mathematical tractability; this family of models is known in mathematical finance literature as the KoBoL family. Carr *et al.* [8] follow this idea, and propose a more realistic model to price the equity. This is done by considering the prices of underlying assets following a jump process or a Lévy process, and has quickly become one of the most widely used models for equity prices.

The plan of this paper is as follows: In Section 2, we introduce the general definitions and results of the Lévy process, option pricing models, the contents relevant to fractional calculus, and derive the target equation, which is the Finite Moment Log-Stable fractional partial differential equation (FPDE). Section 3 provides the numerical ideas of Gauss-Jacobi quadrature for solving partial integral-differential equations, then gives the explicit solutions and further smooths the singularity initial conditions. In Section 4, the numerical analysis for European- and American-styled option pricing problems are given, and finally, Section 5 concludes.