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A Modified Weak Galerkin Finite Element Method for the Poroelasticity Problems

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Abstract. A modified weak Galerkin finite element method is applied to the poroelasticity problems, in which, we use the piece-wise polynomial space to approximate the displacement and the pressure, and we utilize the weak derivative operators to replace the classical ones in the modified weak Galerkin algorithm. Based on the traditional weak Galerkin finite element method, the modified method reduces the total amount of computation by eliminating the degrees of freedom on the boundaries. The error estimates are given and the numerical results are reported to illustrate our theoretical results.

AMS subject classifications: 65N30, 65N15, 76S05

Key words: Modified weak Galerkin, finite element method, poroelasticity.

1. Introduction

The poroelasticity describes the coupled mechanics and flow in porous media, which is widely used in such engineering field, as environmental engineering [4, 5, 18], reservoir engineering [2,3,16], the earthquake engineering [7], biomechanics [17,19,21], and materials science [28]. Since the equations of poroelasticity are so complex, that is difficult to find the analytical solutions. For this reason, the numerical methods for the poroelasticity problems such as continuous Galerkin element method [13, 14], the least-squares mixed finite element method [1,6,12], the discontinuous Galerkin method [15], and weak Galerkin finite element methods [20] are extensively studied.

Weak Galerkin (WG) finite element method [9, 23, 24] refers to a generalization of the classical finite element method, in which, we use discrete weak differential operators to replace the classical ones. In addition the stabilizers are introduced to enforce the weak continuous property. In the WG methods, the approximate finite element are given by piecewise polynomials. As a result, it is more convenient to construct the high-order

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WG spaces, and the relaxation of the continuity enables the application of WG methods on polygonal meshes. The weak functions given in the classical WG method have two componenets which are defined both on the interior and the boundaries of the elements respectively. We consider a modified weak Galerkin (MWG) finite element method in this paper, where componenets are determined by the interior components. Consequently, the MWG methods have fewer degrees of freedom than the WG methods.

The WG method is first introduced in [23] for solving the second-order elliptic equation by using the Raviart-Thomas elements and the Brezzi-Douglas-Marini elements. Later in [9, 24], the stabilizers are introduced. With the help of stabilizers, the general piecewise polynomial spaces are applied to the methods on polygonal meshes. This WG discretization is used to solve many model problems, such as the Stokes equation [25, 26], the Brinkman equation [8, 29], the biharmonic equation [10, 30], the linear elasticity equation [22, 27], and the poroelasticity equation [20].

In [20], the authors propose a WG method for the poroelasticity problem by a mixed finite element method including the pressure, fluid flux, and displacement, in which the Raviart-Thomas-Nedelec space is used to satisfy the boundary condition. In this paper, we shall study a MWG method for the poroelasticity problem with the displacements and pressure. We consider the following model problem:

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\cdot(\nabla\mathbf{u})+\alpha\nabla p=\mathbf{F}, \tag{1.1a}$$

$$\frac{\partial}{\partial t}(c_0 p+\alpha\nabla\cdot\mathbf{u})-\nabla\cdot(k\nabla p)=Q \tag{1.1b}$$

with the boundary conditions

$$\mathbf{u}=\mathbf{u}_g, \quad \text{on } \partial\Omega, \tag{1.2a}$$

$$p=p_d, \quad \text{on } \partial\Omega, \tag{1.2b}$$

$$\mathbf{u}=\mathbf{u}_0, \quad \text{in } \Omega, \text{ when } t=0, \tag{1.2c}$$

$$p=p_0, \quad \text{in } \Omega, \text{ when } t=0, \tag{1.2d}$$

where Ω is a polygonal or a polyhedral domain in $\mathbb{R}^d (d=2,3)$, $\lambda>0$ is the dilation modulus, $\mu>0$ refers to the shear modulus, $c_0\geq 0$ denotes the combined porosity of the medium and compressibility of the fluid, $\alpha>0$ is the Biot-willis constant, and $k>0$ gives the hydraulic conductivity of the media. A weak formulation for (1.1a)-(1.2d) is given by

$$\begin{aligned}
&((\lambda+\mu)\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w})+(\mu\nabla\mathbf{u},\nabla\mathbf{w})-(\alpha p,\nabla\cdot\mathbf{w})=(\mathbf{F},\mathbf{w}), &&\forall\mathbf{w}\in[H_0^1(\Omega)]^d,\\
&(\alpha q,\nabla\cdot\mathbf{u}_t)+(c_0p_t,q)+(k\nabla p,\nabla q)=(Q,q), &&\forall q\in H_0^1(\Omega).
\end{aligned}$$

The corresponding MWG approximation algorithm is given in this paper, which, compared with the traditional WG method, could reduce the computing costs significantly.

The rest of this paper is organized as follows. In Section 2, we introduce the MWG algorithm which will be used in this paper. Section 3 is devoted to give the error equations and the error estimates for the semi-discrete form. The error analysis for the fully-discrete form is given in Section 4. Section 5 is presented to give the proof of an important lemma. Numerical tests are presented in Section 5.