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A Meshless Discrete Galerkin Method Based on the Free Shape Parameter Radial Basis Functions for Solving Hammerstein Integral Equations

Pouria Assari[1,*] and Mehdi Dehghan[2]

[1] *Department of Mathematics, Faculty of Sciences, Bu-Ali Sina University, Hamedan 65178, Iran*

[2] *Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran 15914, Iran*

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Abstract. In the current investigation, we present a numerical technique to solve Fredholm-Hammerstein integral equations of the second kind. The method utilizes the free shape parameter radial basis functions (RBFs) constructed on scattered points as a basis in the discrete Galerkin method to estimate the solution of integral equations. The accuracy and stability of the classical RBFs heavily depend on the selection of shape parameters. But on the other hand, the choice of suitable value for shape parameters is very difficult. Therefore to get rid of this problem, the free shape parameter RBFs are used in the new method which establish an effective and stable method to estimate an unknown function. We utilize the composite Gauss-Legendre integration rule and employ it to estimate the integrals appeared in the method. Since the scheme does not need any background meshes, it can be identified as a meshless method. The error analysis of the method is provided. The convergence accuracy of the new technique is tested over several Hammerstein integral equations and obtained results confirm the theoretical error estimates.

AMS subject classifications: 45A05, 41A25, 45E99

Key words: Hammerstein integral equation, discrete Galerkin method, free shape parameter RBF, meshless method, error analysis.

1. Introduction

Among the meshless methods, RBFs are effective techniques for interpolating an unknown function over a scattered set of points which have applied in the past few decades. The interpolation problems using traditional RBFs are usually ill-conditioned, i.e., the condition number of coefficient matrix is very large [18]. Therefore, a small perturbation in

*Corresponding author. *Email addresses:* `passari@basu.ac.ir` (P. Assari) and `mdehghan@aut.ac.ir` (M. Dehghan)

the initial data may produce large amount of perturbation in the solution. In most RBFs the stability and accuracy of solution depend heavily on the choice of a positive parameter called as shape parameter [50]. Many authors have investigated an optimal value for shape parameters which in general is a difficult work [26,27,30]. Therefore the implication of free shape parameter RBFs is very useful in compactions. The most famous free shape parameter RBFs are known in the literature as thin plate splines and powers which are interpreted as a generalization of univariate natural splines.

Many problems of mathematical physics, engineering and mechanics can be stated in the form of Hammerstein integral equations [24,39,40] which can be considered as follows:

$$u(x)-\lambda\int_a^b K(x,y)\Phi(y,u(y))\mathrm{d}y=f(x), \quad a\le x,\ \ y\le b,\ \ a,b\in\mathbb{R}, \tag{1.1}$$

where the kernel function $K(x,y)$ and the right hand side function $f(x)$ are given, the unknown function $u(x)$ must be determined, $\lambda\in\mathbb{R}$ is a non-zero constant and the known function Φ is continuous and nonlinear respect to the variable u. These types of integral equations also arise as a reformulation of boundary value problems with a certain nonlinear boundary condition [15,16,21].

Since the analytical solution of Hammerstein integral equations is mostly difficult, it is valuable to obtain their numerical solutions. Several methods based on the basic functions so-called projection methods including collocation and Galerkin methods have been considered for solving these types of integral equations. The discrete collocation method [12], the new collocation-type method [34,35], the iterated Galerkin method [32], the discrete Galerkin method [13] and the modified iterated projection method [28] have been applied to solve Hammerstein integral equations. Authors of [19] have described discrete Legendre spectral methods for solving these types of integral equations. The Nystrom methods with the existence of asymptotic error expansion [9,29] have been used to solve the Hammerstein Fredholm integral equations. Walsh-Hybrid functions [43] have been utilized to solve Hammerstein integral equation of the second kind. The Adomian decomposition method [48,49] has been investigated for the numerical solution of these types of integral equations. Haar wavelets [36], rationalized Haar wavelet [22], Legendre wavelets [1,31] have been studied for nonlinear Hammerstein integral equations.

In recent years, the meshless approximations have been applied for the numerical solution of various types of integral equations. The meshless discrete collocation schemes have been investigated based on the RBFs for solving linear and nonlinear integral equations on non-rectangular domains with sufficiently smooth kernels [2,3] and weakly singular kernels [7,8]. The RBFs have been applied for the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations [44]. The meshless product integration (MPI) method [6] has been proposed to solve one-dimensional linear weakly singular integral equations. The MLS methodology as a local meshless method has been used for solving linear and nonlinear two-dimensional integral equations on non-rectangular domains [5,41] and integro-differential equations [20]. The paper [4] has described a computational method for solving Fredholm integral equations with logarithmic kernels