Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 3, pp. 569-603
doi: 10.4208/nmtma.2017-OA-0102 August 2018

Complete Eigenpairs of Hexagonal Seven-Point Laplacian in Fourier Vectors at Half-Integral Nodes

Daniel Lee*

[1] *Department of Applied mathematics, Tunghai University, Taichung 40704, Taiwan, Republic of China*

Received 27 August 2016; Accepted (in revised version) 19 November 2017

Abstract. Hexagonal grid methods are valuable in two-dimensional applications involving Laplacian. The methods are investigated on problems related to standard and anisotropic Laplacian using Fourier vectors in pure, mixed and combination types. Complete (positive) eigenvalues and eigenvectors are determined explicitly in terms of various bases in a unified structure. This work is the *smallest completion* of some previous works.

AMS subject classifications: 65N06, 65N12, 65N22, 15A18, 15A63

Key words: Compact scheme, eigenpairs, fourier vectors, hexagonal grid method, laplacian.

1. Introduction

Hexagonal (Hex) grid methods are of interest in many research studies. A partial list of relevant works includes : [4] (action potential in human heart modeling), [5–7] (numerical modeling in spherical coordinates), [15] (FFT method), [16] , [19] (in atmospheric and ocean models), and [26] (periodic boundary condition). In the article [14], Hex grid FD methods are derived in a finite volume (FV) approach involving standard Laplacian, and used in the simulation of electrical wave phenomena propagated in two-dimensional reversed-C type cardiac tissue, exhibiting both linear and spiral waves more efficiently than similar computation carried on rectangular FVs. We note these cited works all used standard Laplacian and mostly on one configuration of regular hexagon. We refer to [3] for seven-point method on a rectangular grid, also [25] for approximation of Laplacian on meshes of hexagons by a vectorized approach, which extends to three-dimensional applications, and [23] for analysis in non tensor-product applications. Aiming at a better understanding of two-dimensional anisotropic Laplacian ([20]) on a net of hexagons ([13]), we analyze at matrices level the discretizations based on hexagonal seven-point methods using Fourier bases of combination types. In case of a configuration consisting of (a subset of) Cartesian type regular hexagons, we denote by r the radius of a typical

*Corresponding author. *Email address:* danlee@thu.edu.tw (D. Lee)

Table 1: Lattices of type I and II regular hexagons.

Phase angle	Type I, $\varphi = 0$		Type II, $\varphi = -\pi/6$	
Center point	i^{even}	i^{odd}	j^{even}	j^{odd}
$cx(i,j)$	$(1.5i-0.5)r$		$2ih$	$(2i-1)h$
$cy(i,j)$	$2jh$	$(2j-1)h$	$(1.5j-0.5)r$	

Table 2: Local geometry at a hexagon : six vertices and neighbor centers with indices periodically extended when appropriate.

Phase angle	$\varphi \in \mathbb{R}$
Vertices	$V_k = \big(vx(*,k), vy(*,k)\big)\,,\;\; k = 1,\cdots,6$
$vx(*,k)$	$cx(*) + r\cos(\varphi + \frac{k\pi}{3})$
$vy(*,k)$	$cy(*) + r\sin(\varphi + \frac{k\pi}{3})$
Neighbor centers	$P_k = V_k + V_{k+1} - P_0, \quad k = 1,\cdots,6$

hexagon, $h(=\sqrt{3}r/2)$ the height, and $d(=2h)$ the *center-to-center* distance. At a typical center node, $P_0 = (x_0, y_0)$, the six neighbor (center) nodes are

$$P_j := (x_j, y_j) = (x_0, y_0) + d(\cos\xi_j, \sin\xi_j), \quad \xi_j = \varphi + \frac{\pi}{6} + \frac{j\pi}{3}, \quad 1 \le j \le 6.$$

Here the *phase angle*, φ, serves as the configurarion parameter. We name two particular instances type I ($\varphi = 0$) and type II ($\varphi = -\pi/6$). Hexagon centers in lattices of these two types are indexed as for an orthogonal Cartesian mesh as shown in Table 1. This helps the unification of software design in general cases. The geometry and neighborhood of a general Hex FV are specified in Table 2. We refer to Figs. 1 and 2.

For convenience, we abuse the notations and denote FV centers in a neighborhood (Fig. 2) by an *ordered list*,

$$\text{Type I}: \;\; \{P_j\}_{j=0}^{6} = \{P,\, P_{_N},\, P_{_{NW}},\, P_{_{SW}},\, P_{_S},\, P_{_{SE}},\, P_{_{NE}}\}, \tag{1.1a}$$

$$\text{Type II}: \;\; \{P_j\}_{j=0}^{6} = \{P,\, P_{_{NE}},\, P_{_{NW}},\, P_{_W},\, P_{_{SW}},\, P_{_{NW}},\, P_{_E}\}. \tag{1.1b}$$

We adopt similar abbreviations to primitive variables and stencils which will be introduced later. We note results on a Hex II grid can be derived from those on a Hex I grid, and vice versa.

Lemma 1.1. (Reflection principle for anisotropic Laplacian.) *The two configurations, type I and II regular hexagons centered at the origin together with the anisotropic Laplacian, are convertible from each other by applying reflection with respect to the main diagonal in the xy-plane, and therefore interchanging the two symbol lists (Fig. 2)*

$$\Big\{x, y, D_1, D_2, N, NW, SW, S, SE, NE\Big\} \;\; \text{and} \;\; \Big\{y, x, D_2, D_1, E, SE, SW, W, NW, NE\Big\}.$$