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Semi-Group Stability of Finite Difference Schemes in Corner Domains

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Abstract. In this article we are interested in the semi-group stability for finite difference schemes approximations of hyperbolic systems of equations in corner domains. We give generalizations of the results of [10] and [9] from the half space geometry to the quarter space geometry. The most interesting fact is that the proofs of [10] and [9] can be adaptated with minor changes to apply in the quarter space geometry. This is due to the fact that both methods in [10] and [9] are based on energy methods and the construction of auxiliary problems with strictly dissipative boundary conditions which are known to be suitable for the strong well-posed for initial boundary value problems in the quarter space.

AMS subject classifications: 65M06, 65M12

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1. Introduction

In this article we are interested in finite difference schemes for linear hyperbolic problems in the quarter space. Such problems read:

$$\begin{cases} L(\partial)u := \partial_t u + A_1\partial_1 u + A_2\partial_2 u + \sum_{j=3}^d A_j\partial_j u = f, & \text{in } [0,\infty[\times\Omega\times\mathbb{R}^{d-2}, \\ B_1 u_{|x_1=0} = g_1, & \text{on } [0,\infty[\times\partial\Omega_1\times\mathbb{R}^{d-2}, \\ B_2 u_{|x_2=0} = g_2, & \text{on } [0,\infty[\times\partial\Omega_2\times\mathbb{R}^{d-2}, \\ u_{|t=0} = u_0, & \text{on } \Omega\times\mathbb{R}^{d-2}, \end{cases} \tag{1.1}$$

where Ω denotes the quarter space $\mathbb{R}_+^2$ and $\partial\Omega_1$ (resp. $\partial\Omega_2$) is the component of the boundary associated to $\{x_1=0\}$ (resp. $\{x_2=0\}$). In (1.1) the coefficients in the interior, the A_j are matrices in $\mathbf{M}_{N\times N}(\mathbb{R})$ whereas the coefficient on the boundary B_1 (resp. B_2) is an

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element of $\mathbf{M}_{p_1\times N}(\mathbb{R})$ (resp. $\mathbf{M}_{p_2\times N}(\mathbb{R})$) where p_1 (resp. p_2) denotes the number of strictly positive eigenvalues of A_1 (resp. A_2).

Finite difference schemes in the quarter space are just discretizations of (1.1) and have pratical motivations in scientific computations. Indeed, due to the impossibility of modeling the full space $\mathbb{R}^d$ during a numerical simulation, all the schemes implemented in a computer lie in a large rectangle. So that numerically boundary conditions have to be specified even for the numerical approximation of a Cauchy problem. Thus the theoretical study of such schemes set in a domain with corners also have more pratical views. We can be more specific about theses pratical views, in such a way that we describe, for example, the question of absorbing boundary conditions for wave propagation (see for example [12]- [16] and [11]). These conditions are non physical ones and aim to minimize, as much as possible, the "parasite" reflections which occur when the wave hits the artificial boundaries implemented in the simulation of the Cauchy problem. Consequently these conditions are choosen in such a way that the reflections against the boundaries modify or influence as little as possible the approximation in the interior of the box. A similar method is the study of perfectly matched layers (see for example [5]) which consists in boundary conditions which will only modify the approximation in a small neighborhood of the boundary.

In this article we are interested in the stability of difference schemes approximations defined in a space with corner. But before we turn to a more precise description of the notion of stability for schemes with corner, we recall some elements of comparison with the notion of strong well-posedness for continuous problems.

Strong well-posedness means existence and uniqueness of the solution of (1.1) and that this solution is as regular (in the L^2-norm) as the data of the problem. Such a control of the solution by the data is refered as an energy estimate for (1.1). As far as I know even for homogeneous initial conditions (that is to say $u_0 \equiv 0$) the strong well-posedness of (1.1), under suitable conditions, has not been established yet. The main contribution about this question is due to [20], in which the author obtains, thanks to the introduction of a new inversibility condition (we refer to [20] or to [[3], Chapter 5] for more details), an energy estimate for the L^2-norm of the solution. However the regularity of the source terms of (1.1) asked to control the L^2-norm of the solution is not explicit. As a consequence, there is a non explicit number of losses of derivatives in the energy estimate and we can not conclude to the strong well-posedness.

However in a particular framework, more precisely for strictly dissipative boundary conditions, that is to say boundary conditions which make the energy decrease, the strong well-posedness (with homogeneous initial data) is established (see [[3], Chapters 4 and 5]). We also refer to [15] for a result dealing with three dimensional corners in which, thanks to the strict dissipativity and under an ellipticity assumption on the spatial symbol of the hyperbolic operator, the authors obtain a result of strong well-posedness for corners problems with non-homogeneous initial conditions.

We give some more details about the mentioned previous energy estimates. By analogy with the natural energy estimate in the half space geometry [17], the expected energy