

The Modulus-Based Levenberg-Marquardt Method for Solving Linear Complementarity Problem

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Abstract. As applying the Levenberg-Marquardt method to the reformulation of linear complementarity problem, a modulus-based Levenberg-Marquardt method with non-monotone line search is established and the global convergence result is presented. Numerical experiments show that the proposed method is efficient and outperforms the modulus-based matrix splitting iteration method.

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1. Introduction

Let \mathbb{R}^n and $\mathbb{R}^{n \times n}$ be the n -dimensional real vector space and the n -by- n real matrix space, respectively. In this paper, we consider the linear complementarity problem, abbreviated as $\text{LCP}(q, M)$, for finding a pair of real vectors w and $z \in \mathbb{R}^n$ such that

$$w := Mz + q \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T w = 0, \quad (1.1)$$

where $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ is a given large, sparse and real matrix, and

$$q = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$$

is a given real vector. Here, the notation \geq denotes the componentwise defined partial ordering between two vectors and the superscript T denotes the transpose of a vector.

The linear complementarity problem was introduced by Lemke in 1964, but it was Cottle and Dantzig [1] who formally defined the linear complementarity problem and called it the fundamental problem. The $\text{LCP}(q, M)$ of the form (1.1) often arises in many scientific computing and engineering applications, e.g., the linear and quadratic programming, the economies with institutional restrictions upon prices, the optimal stopping in Markov chain, and the free boundary problems; see [2, 3, 5] for details.

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For the solution of the large and sparse $LCP(q, M)$, the pivot algorithms based on simplex type processes require too many pivots, destroy sparsity, have exponential computational complexity and suffer from round-off errors [10]. Therefore, iterative methods, such as projected relaxation method [11], were constructed and widely discussed. Mangasarian [12] and Ahn [13] established the convergence theory of the projected iterative method when the matrix is either symmetric or nonsymmetric.

By equivalently reformulating the $LCP(q, M)$ as an implicit fixed-point equation, Van Bokhoven [6] presented a modulus iteration method, which was defined as the solution of linear equations at each iteration. Moreover, Bai [7] presented a class of modulus-based matrix splitting iteration methods which not only provided a general framework for the modified modulus method [8] and nonstationary extrapolated modulus algorithms [9], but also yielded a series of modulus-based relaxation methods which outperform the projected relaxation method as well as the modified modulus method in computing efficiency. With respect to matrix splitting method and modulus-based method, we can also refer to [18, 20–23, 25, 26, 28–33] and the references therein.

As we all know, the implicit fixed-point equation which is equivalent to the $LCP(q, M)$ is a absolute value equation. Iqbel et al. [14] proposed Levenberg-Marquardt method for solving absolute value equations, which is the combination of steepest descent and the Gauss-Newton methods. They proved the global convergence of new method when using the Goldstein line search. Li and Fukushima [15] presented a non-monotone line search for nonlinear equations, that is

$$\|F(x_k + \alpha d_k)\|^2 \leq (1 + \eta_k)\|F(x_k)\|^2 - \sigma_1 \alpha^2 \|d_k\|^2 - \sigma_2 \alpha^2 \|F(x_k)\|^2, \quad (1.2)$$

where $F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function, σ_1 and σ_2 are positive constants and the positive sequence $\{\eta_k\}$ satisfies

$$\sum_{k=0}^{\infty} \eta_k < \infty. \quad (1.3)$$

It is noticeable that as $\alpha \rightarrow 0^+$, the left hand side of (1.2) goes to $\|F(x_k)\|^2$, while the right hand side tends to the positive constant $(1 + \eta_k)\|F(x_k)\|^2$. Thus, (1.2) is satisfied for all sufficiently small $\alpha > 0$. Hence, one can obtain α_k by means of a backtracking process. This non-monotone line search can guarantee the global convergence of the Levenberg-Marquardt method [19].

Inspired by the above mentioned, we present the Levenberg-Marquardt method with a non-monotone line search for the $LCP(q, M)$.

The outline of this paper is as follows. We give some basic notations, definitions and lemmas in Section 2 and establish the modulus-based Levenberg-Marquardt method for linear complementarity problem in Section 3. In Section 4, the global convergence of the modulus-based Levenberg-Marquardt method is proved. In Section 5, the numerical experiments are presented to show the effectiveness of our method. In the final section we give the concluding remarks.

2. Preliminaries

We briefly introduce some necessary notations, definitions and lemmas. For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, the infinity norm of x is defined as $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, the Euclidean norm of $\|x\|$ is defined as $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$ and $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$ denotes the absolute value of x . For any $A \in \mathbb{R}^{n \times n}$, $\|A\|$ denotes the spectral norm defined by $\|A\| = \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, where $\|x\|$ is the Euclidean norm.

Definition 2.1. ([4, Definition 1.4]) *A vector-valued function $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called Lipschitz continuous if there exists a positive real constant c such that for all vectors $x, y \in D$*

$$\|F(x) - F(y)\| \leq c\|x - y\|.$$

We say F is locally Lipschitz continuous if for every $x \in D$ there exists a neighborhood $U = S(x, \delta)$ of x such that F restricted to U is Lipschitz continuous.

Let $M = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix $M \in \mathbb{R}^{n \times n}$, Ω be a positive diagonal matrix, γ be a positive constant. By utilizing the matrix splitting and the idea of acceleration, the LCP(q, M) can be equivalently transformed into the system of implicit fixed-point equations [16]

$$(M_1 + \Omega)x = N_1x + (\Omega - M_2)|x| + N_2|x| - \gamma q. \quad (2.1)$$

Moreover, with specific choices of the matrix splitting and iteration parameters, (2.1) can yield a series of accelerated modulus-based matrix splitting iteration methods. For example, let $M = D - L - U$ with D , $-L$ and $-U$ being the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of M , and

$$M_1 = \frac{1}{\alpha}(D - \beta L), \quad N_1 = \frac{1}{\alpha}[(1 - \alpha)D + (\alpha - \beta)L + \alpha U], \quad M_2 = D - U \quad \text{and} \quad N_2 = L,$$

where α and β are prescribed relaxation parameters. Then (2.1) reduces to the accelerated modulus-based accelerated overrelaxation (AMAOR) iteration method

$$\begin{aligned} & (D + \alpha\Omega - \beta L)x^{k+1} \\ & = [(1 - \alpha)D + (\alpha - \beta)L + \alpha U]x^k + \alpha(\Omega - D + U)|x^k| + \alpha L|x^{k+1}| - \alpha\gamma q. \end{aligned}$$

It also gives the accelerated modulus-based successive overrelaxation (AMSOR) iteration method, the accelerated modulus-based Gauss-Seidel (AMGS) iteration method and the accelerated modulus-based Jacobi (AMJ) iteration method when $\alpha = \beta$, $\alpha = \beta = 1$ and $\alpha = 1, \beta = 0$, respectively.

It is clear that the system of implicit fixed-point equations (2.1) is equivalent to the system of absolute value equations

$$(M + \Omega)x + (M - \Omega)|x| + \gamma q = 0. \quad (2.2)$$

Particularly, let $\Omega = I$, $\gamma = 1$, we have the following lemma.

Lemma 2.1. *For the LCP(q, M), the following statements hold true:*

(1) *If (w, z) is a solution of the LCP(q, M), then $x = (z - w)/2$ satisfies the system of absolute value equations*

$$(M + I)x + (M - I)|x| + q = 0. \quad (2.3)$$

(2) *If x satisfies the system of absolute value equations (2.3), then $z = |x| + x$ and $w = |x| - x$ is a solution of the LCP(q, M).*

Let $F(x)$ be given by

$$F(x) = (M + I)x + (M - I)|x| + q. \quad (2.4)$$

Then solving LCP(q, M) is equivalent to solving the system of absolute value equations $F(x) = 0$, where F is a function from \mathbb{R}^n into \mathbb{R}^n as defined in (2.4). It is noticed that there is no method that gives a solution which converges very rapidly compared to existing methods because of the non-differentiability of the function F . Hence, Foutayeni et al. [27] constructed a sequence of smooth functions $F_r \in C^\infty$ which are uniformly convergent to the function F and showed that an approximation solution of LCP(q, M) is obtained by solving $F_r(x) = 0$ for r is large enough. Here, the sequence of smooth functions $F_r : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$F_r(x) = (M + I)x + (M - I)(x^2 + e^{-r})^{\frac{1}{2}} + q, \quad (2.5)$$

where $r \in \mathbb{N}$ and

$$(x^2 + e^{-r})^{\frac{1}{2}} := \left((x_1^2 + e^{-r})^{\frac{1}{2}}, (x_2^2 + e^{-r})^{\frac{1}{2}}, \dots, (x_n^2 + e^{-r})^{\frac{1}{2}} \right)^T \in \mathbb{R}^n. \quad (2.6)$$

Specifically, Foutayeni et al. [27] derived the following results.

Lemma 2.2. ([27]) *The sequence of smooth functions $\{F_r\}_{r \geq 1}$ converges uniformly to F on \mathbb{R}^n when $r \rightarrow +\infty$.*

Lemma 2.3. ([27]) *If x_r^* is a solution of the equation $F_r(x) = 0$, then x_r^* is an approximation solution of the equation $F(x) = 0$ for r is large enough.*

In the following analysis, our goal is to build a method for solving $F_r(x) = 0$ for r is large enough.

3. Proposed method

In this section, we suggest Levenberg-Marquardt method with non-monotone line search for the nonlinear equations (2.5) which is the reformulation of the LCP(q, M). Firstly, we take

$$\Psi_r(x) = \frac{1}{2} \|F_r(x)\|^2 \quad (3.1)$$

as the merit function of (2.5). When solving (2.5) by Levenberg-Marquardt method, we obtain the Jacobian matrix of $F_r(x)$ is

$$J^{(r)}(x) = F_r'(x) = (M + I) + (M - I)D_{x,r}, \quad (3.2)$$

where

$$D_{x,r} = \text{diag}(d_1, d_2, \dots, d_n), \quad d_i = \frac{x_i}{\sqrt{x_i^2 + e^{-r}}}. \quad (3.3)$$

Together with the definition of $F_r(x)$, we have the following lemma.

Lemma 3.1.

- (1) $F_r(x)$ is Lipschitz continuous.
- (2) $J^{(r)}(x) = F'_r(x)$ is Lipschitz continuous.
- (3) $J^{(r)}(x) = F'_r(x)$ is bounded.

Proof. (1) Let us use the result of mean value theorem [4, Theorem 1.5] on vector function F_r . Then for all x, y , we have

$$\|F_r(y) - F_r(x)\| \leq \sup_{0 \leq t \leq 1} \|F'_r(x + t(y-x))\| \|y-x\|. \quad (3.4)$$

Let $z = x + t(y-x)$. By using (3.2) and (3.3), we have

$$\begin{aligned} \sup_{0 \leq t \leq 1} \|F'_r(z)\| &= \sup_{0 \leq t \leq 1} \|(M+I) + (M-I)D_{z,r}\| \\ &= \sup_{0 \leq t \leq 1} \left\| (M+I) + (M-I) \text{diag} \left(\frac{z_1}{\sqrt{z_1^2 + e^{-r}}}, \dots, \frac{z_n}{\sqrt{z_n^2 + e^{-r}}} \right) \right\| \\ &\leq \sup_{0 \leq t \leq 1} \left\{ \|M+I\| + \|M-I\| \max_{1 \leq i \leq n} \left| \frac{z_i}{\sqrt{z_i^2 + e^{-r}}} \right| \right\} \leq \|M+I\| + \|M-I\|. \end{aligned} \quad (3.5)$$

Together (3.4) with (3.5) yields

$$\begin{aligned} \|F_r(y) - F_r(x)\| &\leq \sup_{0 \leq t \leq 1} \|F'_r(x + t(y-x))\| \|y-x\| \\ &= \sup_{0 \leq t \leq 1} \|F'_r(z)\| \|y-x\| \leq (\|M+I\| + \|M-I\|) \|y-x\|. \end{aligned} \quad (3.6)$$

Hence, from Definition 2.1, $F_r(x)$ is Lipschitz continuous.

(2) From (3.2), by some calculations, we have

$$\begin{aligned} \|F'_r(y) - F'_r(x)\| &= \left\| [(M+I) + (M-I)D_{y,r}] - [(M+I) + (M-I)D_{x,r}] \right\| \\ &= \|(M-I)(D_{y,r} - D_{x,r})\| \leq \|M-I\| \|D_{y,r} - D_{x,r}\| \\ &= \|M-I\| \max_{1 \leq i \leq n} \left| \frac{y_i}{\sqrt{y_i^2 + e^{-r}}} - \frac{x_i}{\sqrt{x_i^2 + e^{-r}}} \right| \\ &= \|M-I\| \max_{1 \leq i \leq n} \left| \frac{e^{-r}}{(z_i^2 + e^{-r})^{3/2}} (y_i - x_i) \right|, \quad z_i = x_i + t(y_i - x_i), \quad t \in (0, 1) \\ &\leq e^{\frac{r}{2}} \|M-I\| \max_{1 \leq i \leq n} |y_i - x_i| = e^{\frac{r}{2}} \|M-I\| \|y-x\|_\infty \leq e^{\frac{r}{2}} \|M-I\| \|y-x\|, \end{aligned} \quad (3.7)$$

where the fifth equality use the Lagrange mean value theorem of real value function $g(\xi) = \frac{\xi}{\sqrt{\xi^2 + e^{-r}}}$. It then follows from Definition 2.1 that $F'_r(x)$ is Lipschitz continuous.

(3) The result can be obtained by the analysis of (1). The proof is completed. \square

Now, we can outline our new Levenberg-Marquardt method with non-monotone line search for solving linear complementarity problem as follows:

Algorithm 3.1 Levenberg-Marquardt method with non-monotone line search.

Input : A matrix $M \in \mathbb{R}^{n \times n}$, a vector $q \in \mathbb{R}^n$, an accuracy parameter $\varepsilon > 0$, a positive parameter $r > 0$ (default $r = 100$).

Begin :

Step 0. Give an arbitrary point x_0 in \mathbb{R}^n , $\mu > 0$, $\sigma_1, \sigma_2 > 0$, $\omega, \rho \in (0, 1)$ and the sequence $\{\eta_k\}$ satisfying in (1.3). Set $k := 0$.

Step 1. Compute $F_k = F_r(x_k)$ and $J_k = J^{(r)}(x_k)$.

Step 2. If $\|J_k^T F_k\| \leq \varepsilon$, stop. Otherwise, set $\lambda_k = \mu \|F_k\|^{\delta_k}$ where

$$\delta_k = \begin{cases} \frac{1}{\|F_k\|}, & \text{if } \|F_k\| \geq 1, \\ 1, & \text{otherwise.} \end{cases} \quad (3.8)$$

Step 3. Solve the linear system

$$(J_k^T J_k + \lambda_k I)d = -J_k^T F_k \quad (3.9)$$

to compute d_k .

Step 4. If

$$\|F_r(x_k + d_k)\| \leq \omega d_k, \quad (3.10)$$

then, take $\alpha_k = 1$ and go to Step 6. Otherwise, go to Step 5.

Step 5. Compute $\alpha_k = \max\{1, \rho^1, \rho^2, \dots\}$ with $\alpha_k = \rho^i$ satisfying

$$\|F_r(x_k + \alpha_k d_k)\|^2 \leq (1 + \eta_k) \|F_r(x_k)\|^2 - \sigma_1 \alpha_k^2 \|d_k\|^2 - \sigma_2 \alpha_k^2 \|F_r(x_k)\|^2, \quad (3.11)$$

where the positive sequence $\{\eta_k\}$ satisfies (1.3).

Step 6. Set $x_{k+1} = x_k + \alpha_k d_k$. Set $k := k + 1$ and go to Step 1.

end

Remark 3.1. As $\alpha \rightarrow 0^+$, the left hand side of (3.11) goes to $\|F_k\|^2$ while the right hand side tends to the positive value $(1 + \eta_k) \|F_k\|^2$, thus (3.11) is satisfied for sufficiently small $\alpha > 0$. This shows that Algorithm 3.1 is efficient.

4. Convergence analysis

In this section, we will show that Algorithms 3.1 is global convergence. Firstly, we define

$$\Omega = \{x \mid \|F_r(x)\| \leq e^{\eta/2} \|F_0\|\}, \quad (4.1)$$

where η is a positive constant such that

$$\sum_{k=0}^{\infty} \eta_k \leq \eta < \infty. \quad (4.2)$$

According to Lemma 3.1, $F_r(x)$ is Lipschitz continuous and its Jacobian matrix $J^{(r)}(x)$ is Lipschitz continuous and bounded, so $F_r(x)$ and $J^{(r)}(x)$ have the same properties on Ω , i.e., there exists a positive constant L such that for all $x, y \in \Omega$

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad (4.3a)$$

$$\|J^{(r)}(x) - J^{(r)}(y)\| \leq L\|x - y\|, \quad (4.3b)$$

$$\|J^{(r)}(x)\| \leq L. \quad (4.3c)$$

Now we state the following two lemmas that show the sequence $\{x_k\}$ generated by Algorithm 3.1 belongs to Ω and the sequence $\{\|F_k\|\}$ converges.

Lemma 4.1. ([17]) *Let $\{a_k\}$ and $\{r_k\}$ be positive sequences satisfying $a_{k+1} \leq (1 + r_k)a_k + r_k$, $\forall k = 0, 1, \dots$ and $\sum_{k=0}^{\infty} r_k < \infty$. Then $\{a_k\}$ converges.*

Lemma 4.2. *Let the sequence $\{x_k\}$ be generated by Algorithm 3.1. Then*

(1) *the sequence $\{\|F_k\|\}$ converges and $x_k \in \Omega$ for all $k \geq 0$.*

(2) *the sequence $\{\|F_k\|\}$ is bounded, that is, there exists a constant $\mathcal{M} > 0$ such that*

$$\|F_k\| \leq \mathcal{M}, \quad \forall k \geq 0. \quad (4.4)$$

Proof. From (3.10) and (3.11), we have

$$\|F_{k+1}\|^2 \leq (1 + \eta_k)\|F_k\|^2.$$

Due to $\{\eta_k\}$ is a positive sequence, we have

$$\|F_{k+1}\|^2 \leq (1 + \eta_k)\|F_k\|^2 + \eta_k.$$

Lemma 4.1 implies that $\{\|F_k\|^2\}$ and so $\{\|F_k\|\}$ are convergent. Moreover, from the above inequality, we deduce that

$$\|F_{k+1}\| \leq (1 + \eta_k)^{1/2} \|F_k\| \leq \dots \leq \prod_{i=0}^k (1 + \eta_i)^{1/2} \|F_0\|.$$

Thus

$$\begin{aligned}\|F_{k+1}\| &\leq \left(\frac{1}{k+1} \sum_{i=0}^k (1+\eta_i)\right)^{\frac{k+1}{2}} \|F_0\| \leq \left(1 + \sum_{i=0}^k \eta_i/(k+1)\right)^{\frac{k+1}{2}} \|F_0\| \\ &\leq \left(1 + \frac{\eta}{k+1}\right)^{\frac{k+1}{2}} \|F_0\| \leq e^{\eta/2} \|F_0\|,\end{aligned}$$

where the second inequality use arithmetic-geometric means inequality, the third inequality use the basis fact $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$ and the relation (4.2). This inequality means $x_k \in \Omega$ for all k . The proof of (1) is completed. Part (1) and the definition of Ω implies that the sequence $\{\|F_k\|\}$ is bounded. The proof is completed. \square

Lemma 4.3. *Let the sequence $\{x_k\}$ be generated by Algorithm 3.1. If (3.10) holds for infinite k , then $\{\|F_k\|\}$ converges to zero. In other words, if there exists a positive constant c such that $\|F_k\| \geq c$ holds for sufficiently large k , then (3.10) holds for finite k .*

Proof. Denote the index sets

$$I_j = \{k \leq j \mid (3.10) \text{ holds}\}, \quad H_j = \{0, 1, \dots, j\} \setminus I_j, \quad j = 1, 2, \dots.$$

If (3.10) holds for infinite k , then as $j \rightarrow \infty$, $\text{card}(I_j) \rightarrow \infty$, where $\text{card}(I_j)$ is the number elements of I_j . From (3.10) and (3.11), we have

$$\begin{aligned}\|F_{k+1}\| &\leq \left(\prod_{i \in H_k} (1+\eta_i)^{1/2} \prod_{i \in I_k} \eta\right) \|F_0\| \\ &= \left(\prod_{i \in H_k} (1+\eta_i)^{1/2}\right) \rho^{\text{card}(I_k)} \|F_0\| \\ &\leq e^{\frac{\eta}{2}} \rho^{\text{card}(I_k)} \|F_0\| \rightarrow 0, \text{ as } k \rightarrow \infty.\end{aligned}$$

So $\|F_k\| \rightarrow 0$. The proof is completed. \square

Theorem 4.1. *Algorithm 3.1 either terminates in a finite number of steps or satisfies*

$$\liminf_{k \rightarrow \infty} \|J_k^T F_k\| = 0. \quad (4.5)$$

Proof. By contradiction, suppose there exist $\tau > 0$ and an integer \bar{k} such that

$$\|J_k^T F_k\| \geq \tau, \quad \forall k \geq \bar{k}. \quad (4.6)$$

This together with (4.3c) implies that

$$\|F_k\| \geq L^{-1} \tau \quad (4.7)$$

holds for sufficiently large k . So, by Lemma 4.3, the inequality (3.10) holds for finite k . On the other hand, from (3.11), we have

$$\|F_{k+1}\|^2 \leq (1 + \eta_k) \|F_k\|^2 - \sigma_2 \alpha_k^2 \|F_k\|^2,$$

then

$$\sigma_2 \alpha_k^2 \|F_k\|^2 \leq \|F_k\|^2 - \|F_{k+1}\|^2 + \eta_k \|F_k\|^2.$$

Thus, according to Lemma 4.2 (2), results that

$$\sigma_2 \sum_{k=0}^m \alpha_k^2 \|F_k\|^2 \leq \|F_0\|^2 - \|F_{m+1}\|^2 + \sum_{k=0}^m \eta_k \|F_k\|^2 \leq \|F_0\|^2 + M \sum_{k=0}^m \eta_k,$$

which implies

$$\sum_{k=0}^{\infty} \alpha_k^2 \|F_k\|^2 < \infty.$$

Then $\lim_{k \rightarrow \infty} \alpha_k \|F_k\| = 0$. This relation together with (4.7) yields

$$\lim_{k \rightarrow \infty} \alpha_k = 0. \quad (4.8)$$

Now, let $J_k = U_k \Sigma_k V_k^T$ be the singular value decomposition (SVD) of J_k , where U_k, V_k are two orthogonal matrices and $\Sigma_k = \text{diag}(\sigma_{k,1}, \sigma_{k,2}, \dots, \sigma_{k,n})$ with $\sigma_{k,1} \geq \sigma_{k,2} \geq \dots \geq \sigma_{k,n} \geq 0$. Then, we have

$$\begin{aligned} \|(J_k^T J_k + \lambda_k I)^{-1}\| &= \|V_k (\Sigma_k^2 + \lambda_k I)^{-1} V_k^T\| \\ &= \|(\Sigma_k^2 + \lambda_k I)^{-1}\| = \max_{i \in \{1, 2, \dots, n\}} (\sigma_{k,i}^2 + \lambda_k)^{-1} \leq \lambda_k^{-1}. \end{aligned} \quad (4.9)$$

This inequality together with (3.9), (4.3c) and (4.4) implies that

$$\begin{aligned} \|d_k\| &= \|(J_k^T J_k + \lambda_k I)^{-1} J_k^T F_k\| \leq \|(J_k^T J_k + \lambda_k I)^{-1}\| \|J_k\| \|F_k\| \\ &\leq L \lambda_k^{-1} \|F_k\| = \frac{L}{\mu} \|F_k\|^{1-\delta_k}. \end{aligned}$$

If $\|F_k\| < 1$, then $\delta_k = 1$, hence

$$\|d_k\| \leq \frac{L}{\mu}. \quad (4.10)$$

If $\|F_k\| \geq 1$, then $\delta_k = \frac{1}{\|F_k\|}$, hence

$$\|d_k\| \leq \frac{L}{\mu} \|F_k\|^{1-\frac{1}{\|F_k\|}} \leq \frac{L}{\mu} \|F_k\| \leq \frac{L}{\mu} \mathcal{M}. \quad (4.11)$$

Let $\mathcal{M}_1 = \max\{1, \mathcal{M}\}$, together (4.10) with (4.11) yields

$$\|d_k\| \leq \frac{L}{\mu} \mathcal{M}_1. \quad (4.12)$$

If $\liminf_{k \rightarrow \infty} \|d_k\| = 0$, then we have from (3.9) and (4.3c) that

$$\liminf_{k \rightarrow \infty} \|J_k^T F_k\| = \liminf_{k \rightarrow \infty} \|(J_k^T J_k + \lambda_k I) d_k\| = 0, \quad (4.13)$$

which obtains a contradiction to (4.6). Hence there exists a constant $\nu > 0$, such that

$$\liminf_{k \rightarrow \infty} \|d_k\| > \nu. \quad (4.14)$$

From line search (3.11), we have

$$\begin{aligned} & \|F(x_k + \bar{\alpha}_k d_k)\|^2 - \|F_k\|^2 \\ & > -\bar{\alpha}_k^2(\sigma_1 \|d_k\|^2 + \sigma_2 \|F_k\|^2) + \eta_k \|F_k\|^2 > -\bar{\alpha}_k^2(\sigma_1 \|d_k\|^2 + \sigma_2 \|F_k\|^2), \end{aligned}$$

where $\bar{\alpha}_k = \frac{\alpha_k}{\rho}$. Combine this inequality with (4.3) yields

$$\begin{aligned} & \bar{\alpha}_k^2(\sigma_1 \|d_k\|^2 + \sigma_2 \|F_k\|^2) > -(\|F(x_k + \bar{\alpha}_k d_k)\|^2 - \|F_k\|^2) \\ & = -2F_k^T [F(x_k + \bar{\alpha}_k d_k) - F_k] - \|F(x_k + \bar{\alpha}_k d_k) - F_k\|^2 \\ & \geq -2F_k^T [F(x_k + \bar{\alpha}_k d_k) - F_k] - L^2 \bar{\alpha}_k^2 \|d_k\|^2. \end{aligned} \quad (4.15)$$

On the other hand, by the mean-value theorem [4, Theorem 1.6], we have

$$\begin{aligned} & F_k^T [F(x_k + \bar{\alpha}_k d_k) - F_k] = \bar{\alpha}_k F_k^T J_k d_k + F_k^T \int_0^1 (J(x_k + t\bar{\alpha}_k d_k) - J_k) \bar{\alpha}_k d_k dt \\ & \leq \bar{\alpha}_k F_k^T J_k d_k + \frac{1}{2} L \mathcal{M} \bar{\alpha}_k^2 \|d_k\|^2 = -\bar{\alpha}_k d_k^T (J_k^T J_k + \lambda_k I) d_k + \frac{1}{2} L \mathcal{M} \bar{\alpha}_k^2 \|d_k\|^2, \end{aligned} \quad (4.16)$$

which, together with (4.15), yields

$$\bar{\alpha}_k(\sigma_1 \|d_k\|^2 + \sigma_2 \|F_k\|^2) > 2d_k^T (J_k^T J_k + \lambda_k I) d_k - L \mathcal{M} \bar{\alpha}_k \|d_k\|^2 - L^2 \bar{\alpha}_k \|d_k\|^2.$$

Hence

$$\bar{\alpha}_k [(\sigma_1 + L \mathcal{M} + L^2) \|d_k\|^2 + \sigma_2 \|F_k\|^2] > 2d_k^T (J_k^T J_k + \lambda_k I) d_k \geq 2\lambda_k \|d_k\|^2, \quad (4.17)$$

where the last inequality is due to the semi-positive definite of $J_k^T J_k$. So, from the inequality (4.17), we have

$$\bar{\alpha}_k > \frac{2\lambda_k \|d_k\|^2}{(\sigma_1 + L \mathcal{M} + L^2) \|d_k\|^2 + \sigma_2 \|F_k\|^2}. \quad (4.18)$$

Consequently, we can deduce from (4.4), (4.12), (4.14) and (4.18) that $\{\alpha_k\}$ is bounded away from zero, which contradicts with (4.8) and the proof is completed. \square

5. Numerical experiments

In this section, we represent some numerical examples to demonstrate the effectiveness of our algorithm from the aspects of iteration steps (denoted by 'Iter'), elapsed CPU time in

seconds (denoted by 'CPU') and the norm of absolute residual vectors (denoted by 'Res'). Here, 'Res' is defined as

$$\text{Res}(z^k) := \left\| \min(Mz^k + q, z^k) \right\|_2,$$

where z^k is the k th approximate solution to the linear complementarity problem (1.1), and the minimum is taken componentwise.

All of the tests were run on the Intel (R) Core (TM), where the CPU is 2.40 GHz and the memory is 8.0 GB, the programming language was MATLAB R2015a. The stopping criteria for all methods are $\text{Res}(z^k) \leq 10^{-5}$ or k reaches the maximal number of iteration, e.g., 5000. Moreover, for Algorithm 3.1, we set the parameter $r = 100$, $\mu = 0.5$, $\sigma_1 = \sigma_2 = 0.55$, $\omega = 0.5$, $\rho = 0.8$, $\eta_k = 0.5^k$.

We compare our method with 'AMSOR' method presented in [16] as follows:

$$\begin{aligned} & (D + \alpha\Omega - \alpha L)x^{k+1} \\ = & [(1 - \alpha)D + \alpha U]x^k + \alpha(\Omega - D + U)|x^k| + \alpha L|x^{k+1}| - \alpha\gamma q. \end{aligned}$$

In numerical experiments, take $\gamma = 1$ and $\Omega = 5D$, and have 'AMSOR' converges, we also take different α for comparison.

Example 5.1. ([16]) Let m be a prescribed positive integer and $n = m^2$. Consider the LCP(q, M), in which $M \in \mathbb{R}^{n \times n}$ is given by $M = \widehat{M} + \nu I$ and $q \in \mathbb{R}^n$ is given by $q = -\left(\frac{1}{\alpha}D - L\right)z^*$, where

$$\widehat{M} = \begin{pmatrix} B & -I & O & \cdots & O & O \\ -I & B & -I & \cdots & O & O \\ O & -I & B & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -I \\ O & O & \cdots & \cdots & -I & B \end{pmatrix}, \quad z^* = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ \vdots \end{pmatrix},$$

where $B = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m}$, $I \in \mathbb{R}^{m \times m}$ is a unit matrix, $O \in \mathbb{R}^{m \times m}$ is the zero matrix, ν is a constant and α is the positive parameter used in 'AMSOR'. Obviously, M is a symmetric positive definite matrix. In this example, we take $\nu = 4$.

In Table 1, the iteration steps, the CPU time and the residual norms for the Levenberg-Marquardt method (Algorithm 3.1) and the accelerated modulus-based matrix splitting iteration method (AMSOR) for Example 5.1 are listed.

From Table 1, we can find that Algorithm 3.1 has higher precision for different choices n . Algorithm 3.1 needs less CPU time and iteration number. Especially, the iteration number of Algorithm 3.1 is far less than that of 'AMSOR' method.

Table 1: Numerical comparison of the testing methods for Example 5.1.

| m | | AMSOR([16]) | | | | Algorithm 3.1 | | | |
|----|------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| | | ($\alpha = 0.8$) | ($\alpha = 0.9$) | ($\alpha = 1.1$) | ($\alpha = 1.2$) | ($\alpha = 0.8$) | ($\alpha = 0.9$) | ($\alpha = 1.1$) | ($\alpha = 1.2$) |
| 10 | Iter | 90 | 74 | 57 | 54 | 3 | 3 | 3 | 3 |
| | CPU | 0.0577 | 0.0564 | 0.0434 | 0.0293 | 0.0280 | 0.0299 | 0.0291 | 0.0282 |
| | Res | 9.8518e-06 | 9.8531e-06 | 9.5125e-06 | 7.6435e-06 | 2.6153e-08 | 8.4000e-09 | 1.7749e-09 | 5.3528e-10 |
| 20 | Iter | 98 | 81 | 61 | 57 | 3 | 3 | 3 | 3 |
| | CPU | 0.2652 | 0.2961 | 0.2122 | 0.1594 | 0.1403 | 0.1387 | 0.1433 | 0.1319 |
| | Res | 9.2125e-06 | 9.3818e-06 | 9.6478e-06 | 9.9704e-06 | 6.4034e-07 | 1.9456e-07 | 2.5338e-08 | 8.0523e-09 |
| 30 | Iter | 101 | 84 | 63 | 59 | 4 | 3 | 3 | 3 |
| | CPU | 1.1397 | 1.0281 | 0.7868 | 0.7513 | 0.9609 | 0.7050 | 0.7298 | 0.7342 |
| | Res | 9.8448e-06 | 9.3716e-06 | 9.7976e-06 | 9.8046e-06 | 7.9535e-14 | 1.0505e-06 | 1.7578e-07 | 5.0147e-08 |
| 40 | Iter | 104 | 86 | 65 | 61 | 4 | 4 | 3 | 3 |
| | CPU | 4.4286 | 3.6525 | 2.7851 | 2.4528 | 3.7787 | 3.3502 | 2.6663 | 2.5685 |
| | Res | 8.6993e-06 | 9.2668e-06 | 8.5050e-06 | 8.1384e-06 | 1.7006e-13 | 9.0236e-14 | 4.9956e-07 | 1.7792e-07 |
| 50 | Iter | 105 | 88 | 66 | 62 | 4 | 4 | 3 | 3 |
| | CPU | 12.0005 | 10.0970 | 7.6646 | 7.1613 | 10.8706 | 10.0492 | 7.8830 | 7.7892 |
| | Res | 9.6343e-06 | 8.3616e-06 | 8.6148e-06 | 8.0671e-06 | 4.1890e-13 | 1.6673e-13 | 1.0589e-06 | 3.6835e-07 |

Example 5.2. ([24]) Consider the LCP(q, M), $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ are given below:

$$M = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 & 0 \\ 1 & 4 & -2 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -2 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ -4 \\ -4 \\ \vdots \\ -4 \\ -4 \end{pmatrix} \in \mathbb{R}^n.$$

It is easy to see that M is a nonsymmetric tridiagonal H_+ -matrix.

In table 2, the iteration steps, the CPU time and the residual norms for the Levenberg-Marquardt method (Algorithm 3.1) and the accelerated modulus-based matrix splitting iteration method (AMSOR) for Example 5.2 are listed.

From Table 2, we can find that Algorithm 3.1 has higher precision for different choices n . When $\alpha = 0.8$ and $\alpha = 0.9$, the CPU time and the iteration number of Algorithm 3.1 are far less than that of ‘AMSOR’ method. When $\alpha = 1.1$ and $\alpha = 1.2$, with the increasing of matrix dimension n , Algorithm 3.1 needs more CPU time. However, Algorithm 3.1 outperforms ‘AMSOR’ method in terms of the iteration number and the precision.

6. Conclusions

In this paper, the modulus-based Levenberg-Marquardt method is proposed and applied to the linear complementarity problem. The proposed method is well defined, the new algorithm is globally convergent by utilizing the non-monotone line search. Numerical

Table 2: Numerical comparison of the testing methods for Example 5.2.

| Dim(n) | | AMSOR([16]) ($\alpha = 0.8$) | AMSOR([16]) ($\alpha = 0.9$) | AMSOR([16]) ($\alpha = 1.1$) | AMSOR([16]) ($\alpha = 1.2$) | Algorithm 3.1 |
|--------|------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|---------------|
| 100 | Iter | 70 | 62 | 50 | 46 | 3 |
| | CPU | 0.0430 | 0.0393 | 0.0383 | 0.0405 | 0.0261 |
| | Res | 9.1563e-06 | 8.9179e-06 | 9.3475e-06 | 8.1329e-06 | 2.6713e-08 |
| 400 | Iter | 74 | 65 | 53 | 48 | 3 |
| | CPU | 0.2354 | 0.1972 | 0.1611 | 0.1635 | 0.1128 |
| | Res | 8.3568e-06 | 9.2419e-06 | 8.2115e-06 | 9.0419e-06 | 3.4113e-07 |
| 900 | Iter | 76 | 67 | 54 | 49 | 3 |
| | CPU | 0.9845 | 0.8334 | 0.7088 | 0.6231 | 0.6050 |
| | Res | 8.3474e-06 | 8.7428e-06 | 9.2991e-06 | 9.9547e-06 | 1.5609e-06 |
| 1500 | Iter | 77 | 68 | 55 | 50 | 4 |
| | CPU | 2.7097 | 2.4697 | 2.0325 | 1.8528 | 2.5815 |
| | Res | 8.7810e-06 | 8.9502e-06 | 9.0071e-06 | 9.3739e-06 | 2.3943e-13 |
| 2000 | Iter | 78 | 69 | 56 | 51 | 4 |
| | CPU | 5.3375 | 5.2777 | 4.0042 | 3.4825 | 5.1278 |
| | Res | 8.2470e-06 | 8.1804e-06 | 7.7889e-06 | 7.8805e-06 | 5.5586e-13 |

results indicate that the modulus-based Levenberg-Marquardt method with non-monotone line search is effective and robust for solving linear complementarity problem. Moreover, the modulus-based Levenberg-Marquardt method outperforms ‘AMSOR’ method in terms of the iteration number, the precision and the CPU time.

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