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Diagonalized Chebyshev Rational Spectral Methods for Second-Order Elliptic Problems on Unbounded Domains

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Abstract. Diagonalized Chebyshev rational spectral methods for solving second-order elliptic problems on the half/whole line are proposed. Some Sobolev bi-orthogonal rational basis functions are constructed which lead to the diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier-like Chebyshev rational series. Numerical results demonstrate the effectiveness of the suggested approaches.

AMS subject classifications: 65N35, 41A20, 33C45, 35J15

Key words: Chebyshev rational spectral methods, Sobolev bi-orthogonal functions, second-order elliptic equations, numerical results.

1. Introduction

Many science and engineering problems are set on unbounded domains, such as fluid flows in an infinite strip, nonlinear wave equations in quantum mechanics and so on. How to accurately and efficiently solve such problems is a very important and difficult subject, since the unboundedness of the domain causes considerable theoretical and practical challenges. A variety of numerical approaches have been proposed and investigated for dealing with such problems. For spectral methods, we usually restrict calculations to some bounded subdomains and impose certain artificial boundary conditions. It is easy to be performed, but it lowers the accuracy sometimes. Alternatively, we may approximate the problems on unbounded domains directly by using some orthogonal polynomials/functions on unbounded domains, such as the Hermite spectral method and the Laguerre method, see, e.g., [1,2,5,8,9,12,15,16,18–20,24]. However, since the Laguerre and Hermite Gauss points are too concentrated near zero, the approximation results are usually not ideal, especially where the points are far away from zero. Another effective spectral method to approximate differential equations on unbounded domains is to use algebraically mapped

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Legendre, Chebyshev or Jacobi functions, i.e., the so-called Legendre, Chebyshev or Jacobi rational spectral methods [3, 4, 6, 10, 11, 22, 23]. Compared with the first two methods, we prefer the last one, since the distribution of the Gauss points is much reasonable. Accordingly, the numerical results would be better, especially for slow decay solutions.

As is well known, the utilization of Chebyshev rational functions usually leads to a sparse algebraic system (e.g., a nine-diagonal matrix for second-order problems on the whole line), the condition numbers increase as $\mathcal{O}(N^2)$ for second-order problems. However, in many cases, people still want to get a set of Fourier-like basis functions (see [7, 17]), which are orthogonal or bi-orthogonal with respect to certain Sobolev inner product involving derivatives, such that the corresponding algebraic system is diagonal (see [21]).

Recently, Liu et al. [13, 14] constructed the Fourier-like Sobolev orthogonal basis functions based on generalized Laguerre functions, and applied them to second and fourth order elliptic equations on the half line. Motivated by [13, 14, 21], the main purpose of this paper is to construct the Fourier-like Chebyshev rational basis functions, which are bi-orthogonal with respect to certain Sobolev inner product. On this basis, we further propose the diagonalized Chebyshev rational spectral methods for second-order elliptic problems on the whole/half line.

The main advantages of the suggested algorithms include: (i). The exact solutions and the approximate solutions can be represented as infinite and truncated Fourier-like Chebyshev rational series, respectively; (ii) The condition numbers for the resulting algebraic systems are equal to 1.

This paper is organized as follows. In Section 2, we introduce the Chebyshev rational functions on the whole/half line and their basic properties. In Section 3, we construct two kinds of Sobolev bi-orthogonal Chebyshev rational functions corresponding to the second-order elliptic equations on the whole/half line, and propose the diagonalized Chebyshev rational spectral methods. Some numerical results are presented in Section 4 to demonstrate the effectiveness and accuracy.

2. Notations and preliminaries

Let Λ be a certain interval and $\omega(x)$ be a weight function in the usual sense. For integer $r \geq 0$, we define the weighted Sobolev spaces $H^r_\omega(\Lambda)$ as usual, with the inner product $(u,v)_{r,\omega}$, the semi-norm $|v|_{r,\omega}$ and the norm $\|v\|_{r,\omega}$, respectively. We omit the subscript r or $\omega(x)$ whenever $r=0$ or $\omega(x)\equiv 1$. For simplicity, we denote $\partial_x^k v = d^k v/dx^k$, $v'' = d^2v/dx^2$ and $v' = dv/dx$.

2.1. Chebyshev polynomials

We first recall the Chebyshev polynomials. Let $I=(-1,1)$ and $T_k(y)$ be the Chebyshev polynomial of degree k, which is the eigenfunction of the singular Strum-Liouville problem (cf. [20]):

$$\sqrt{1-y^2}\partial_y\left(\sqrt{1-y^2}\partial_y T_k(y)\right) + k^2 T_k(y) = 0. \tag{2.1}$$