Numer. Math. Theor. Meth. Appl. Vol. **12**, No. 1, pp. 98-114
doi: 10.4208/nmtma.OA-2017-0129 February 2019

Weighted Integral of Infinitely Differentiable Multivariate Functions is Exponentially Convergent

Guiqiao Xu[1,*], Yongping Liu[2] and Jie Zhang[2]

[1] *Department of Mathematics, Tianjin Normal University, Tianjin, 300387, PRC*
[2] *Department of Mathematics, Beijing Normal University, Beijing, 100875, PRC*

Received 27 October 2017; Accepted (in revised version) 26 March 2018

Abstract. We study the problem of a weighted integral of infinitely differentiable multivariate functions defined on the unit cube with the L_∞-norm of partial derivative of all orders bounded by 1. We consider the algorithms that use finitely many function values as information (called standard information). On the one hand, we obtained that the interpolatory quadratures based on the extended Chebyshev nodes of the second kind have almost the same quadrature weights. On the other hand, by using the Smolyak algorithm with the above interpolatory quadratures, we proved that the weighted integral problem is of exponential convergence in the worst case setting.

AMS subject classifications: 41A63, 65Y20, 68Q25

Key words: Smolyak algorithm, infinitely differentiable function class, standard information, worst case setting.

1. Introduction

A multivariate numerical problem refers to a sequence of solution operators $S = \{S_d : F_d \to G_d\}_{d\in\mathbb{N}}$, where for each d, F_d is a class of functions with d variables and G_d is another space. Multivariate problems occur in many applications such as in computational finance, statistics and physics. To solve these solution operators, we often use information based algorithms that use finitely many information operations. Due to considering integral problem, in this paper, we only allow any function values to be an information operation.

Most of the work on multivariate computational problems has dealt with problems defined over classes of functions with finite smoothness. For such problem classes, the corresponding minimal error sequence often converges polynomially. However, there has been recent work in the worst case setting (see, e.g., [1-4]) on problems having infinite smoothness, including problems defined over spaces of analytic functions. For such problem classes, the convergence rate of the minimal error sequence will often be faster than

*Corresponding author. *Email addresses:* Xuguiqiao@aliyun.com (G. Q. Xu), ypliu@bnu.edu.cn (Y. P. Liu), zhangjie91528@163.com (J. Zhang)

polynomial (e.g., supper-polynomial or exponential). Noted that all the problems involved in above papers are over the weighted reproducing kernel Hilbert spaces, K. Suzuki [5] focused on the integral problem on a weighted L_1-normed space which consists of non-periodic smooth functions. It used the multivariate QMC rules on digital nets to prove that the corresponding minimal error sequence converges supper-polynomially. In this paper, we will consider a weighted integral problem on the following infinitely differentiable function class that was introduced in [6]:

$$F_d = \Big\{ f : [-1,1]^d \to \mathbb{R} \Big| \|f\|_{F_d} = \sup_{\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}_0^d} \|D^\alpha f\|_\infty < \infty \Big\}, \tag{1.1}$$

and we proved that the corresponding minimal error sequence converges exponentially. The reason that we use $[-1,1]$ instead of $[0,1]$ in [5, 6] is that we will use the Chebyshev nodes. We mainly used the Smolyak algorithm and the interpolatory quadratures based on the extended Chebyshev nodes. We would like to add that our proof is also suitable for non-weighted integral.

The paper is organized as follows. Section 2 contains some basic concepts and lemmas that will be needed in the proofs of our main results. In Section 3 we give the main results and their proofs.

2. Some concepts and lemmas

First, let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ respectively denote the sets of all positive integers, non-negative integers and real numbers.

Now we introduce the related concepts. Assume that each operator $S_d : F_d \to G_d$ is a continuous linear transformation, where F_d is a Banach space of d-variate real functions defined on $D_d \subset \mathbb{R}^d$ and G_d is another Banach space.

For each $d \in \mathbb{N}$, we consider the approximation of $S_d(f)$ for $f \in F_d$ by using *information-based algorithms* of the form

$$A_{n,d}(f) = \phi_{n,d}\left(L_1(f), \cdots, L_n(f)\right), \tag{2.1}$$

where $L_1, L_2, \cdots, L_n \in \Lambda^{\text{std}} = \{L | L(f) = f(t), \forall t \in D_d\}$ and $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary mapping. As a special case, we define $A_{0,d} = 0$.

The worst-case error of the algorithm $A_{n,d}$ is defined as

$$e(S_d, A_{n,d}, F_d, G_d) = \sup_{f\in F_d, \|f\|_{F_d}\le 1} \left\| S_d(f) - A_{n,d}(f) \right\|_{G_d}.$$

Furthermore, we define the nth minimal worst-case error as

$$e(n, S_d, F_d, G_d) = \inf_{A_{n,d}} e(S_d, A_{n,d}, F_d, G_d),$$

where the infimum is taken over all algorithms of the form (2.1).

Traditionally we consider the problems for which smoothness of functions is finite. In this case, the corresponding minimal error sequence is often of polynomial convergence rate. Recently, some authors considered some multivariate linear problems defined over classes of infinitely differentiable functions and find that the corresponding minimal error sequence is probably of exponential convergence rate rather than polynomial convergence rate. Now we introduce the notion of exponential convergence as follow.

We say that a multivariate problem S is of *exponential convergence* (EXP) if there exist a number $q \in (0,1)$ and functions $r, C, C_2 : \mathbb{N} \to (0,\infty)$ such that

$$e(n, S_d, F_d, G_d) \leq C(d) q^{\left(n/C_2(d)\right)^{r(d)}} \quad \text{for all } n, d \in \mathbb{N}. \tag{2.2}$$

For the integral problems, papers [1-3] proved that the multivariate integral problems on some weighted Korobov spaces are of exponential convergence, and [4] proved that the multivariate integral problems on some weighted Hermite spaces are also of exponential convergence. Noticed that all the problems are defined on some weighted reproducing kernel Hilbert spaces and the proofs are heavily depended on the properties of the corresponding reproducing kernels, K. Suzuki [5] introduced the non-Hilbert weighted spaces

$$F_{d,\mathbf{u}} = \left\{ f : [0,1]^d \to \mathbb{R} \,\middle|\, \|f\|_{F_{d,\mathbf{u}}} = \sup_{\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}_0^d} \frac{\|D^\alpha f\|_1}{\prod_{j=1}^d u_j^{\alpha_j}} < \infty \right\}$$

with $\mathbf{u} = \{u_j\}_{j\geq 1}$ a sequence of positive decreasing weights and

$$D^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_d}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_d} x_d} f, \qquad \forall\ \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d.$$

He considered the numerical approximation of integrals

$$INT_d(f) = \int_{[0,1]^d} f(\mathbf{x}) d\mathbf{x}, \qquad \forall f \in F_{d,\mathbf{u}} \tag{2.3}$$

by using quasi-Monte Carlo rules with digital nets and obtained that there exist positive $C(d)$ and $C_2(d)$ depending on d and $\mathbf{u}$ such that

$$e(n, INT_d, F_{d,\mathbf{u}}, \mathbb{R}) \leq C(d) e^{-C_2(d)\log^2 n} \quad \text{for all } n, d \in \mathbb{N}. \tag{2.4}$$

Noticed that $F_{d,\mathbf{u}}$ are also weighted spaces, in this paper we will consider the weighted integral problem on the non-weighted spaces F_d (see (1.1)), i.e.,

$$INT_d(f) = \frac{1}{\pi^d} \int_{[-1,1]^d} f(\mathbf{x}) \prod_{j=1}^d \frac{1}{\sqrt{1-x_j^2}} d\mathbf{x}, \qquad \forall f \in F_d, \tag{2.5}$$

where $\mathbf{x} = (x_1, \cdots, x_d)$. We will prove that the weighted integral problem $INT = \{INT_d\}_{d=1}^\infty$ is of exponential convergence (see Theorem 3.2).

Now we show why we consider weighted integral problem (2.5) rather than non-weighted integral problem. On the one hand, for $d=1$, the weighted integral with weight function $\frac{1}{\pi\sqrt{1-x^2}}$ is widely studied in the approximation theory, and the corresponding d-fold tensor product problem is (2.5). On the other hand, the quadrature weights in the Chenshaw -Curtis algorithm for the non-weighted integral problem are very complicated (see (2) in [7]), however we proved that the quadrature weights for the weighted integral problem (2.5) is almost the same (see (2.11)), i.e., the weighted integral (2.5) is easier to compute. The basic error estimates obtained in this paper are all equality. Moreover, by combining Chenshaw -Curtis algorithm given in [7] with the same proof used in this paper, we can easily obtain that the non-weighted integral problem on F_d is also of exponential convergence.

To obtain our results, we first give the quadrature formulae of weighted integral

$$I(f)=\frac{1}{\pi}\int_{-1}^{1}f(t)\frac{dt}{\sqrt{1-t^2}}\qquad \forall f\in C[-1,1] \tag{2.6}$$

on the extended Chebyshev nodes of the second kind.

Let

$$-1=x_{n+1}<x_n<\cdots<x_1<x_0=1 \tag{2.7}$$

be the zeros of $\left(1-x^2\right)V_n(x)$, where $V_n(x)$ is the nth Chebyshev polynomial of the second kind, i.e.,

$$V_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\qquad x=\cos\theta.$$

In this case the well-known Lagrange interpolation polynomial is given by (see [8])

$$Q_{n+2}(f,x)=\sum_{k=0}^{n+1}f(x_k)\phi_k(x), \tag{2.8}$$

where

$$\phi_k(x)=\frac{(-1)^{k+1}\left(1-x^2\right)V_n(x)}{(n+1)(x-x_k)},\qquad k=1,\cdots,n, \tag{2.9a}$$

$$\phi_0(x)=\frac{(1+x)V_n(x)}{2V_n(1)},\qquad \phi_{n+1}(x)=\frac{(1-x)V_n(x)}{2V_n(-1)}. \tag{2.9b}$$

Lemma 2.1. *The interpolatory quadrature of the weighted integral* (2.6) *on nodes* (2.7) *is*

$$T_{n+2}(f)=\frac{1}{\pi}\int_{-1}^{1}Q_{n+2}(f,t)\frac{dt}{\sqrt{1-t^2}}=\sum_{k=0}^{n+1}a_kf(x_k), \tag{2.10}$$

where

$$a_0=a_{n+1}=\frac{1}{2(n+1)},\qquad a_k=\frac{1}{n+1},\qquad k=1,\cdots,n. \tag{2.11}$$

Proof. Let $\mathbb{P}_n$ denote the set of algebraic polynomial of degree at most n. Then by the nature of the interpolation quadratures we know that for an arbitrary $p \in \mathbb{P}_{n+1}$, one has

$$\frac{1}{\pi}\int_{-1}^{1} p(t)\frac{dt}{\sqrt{1-t^2}} = \sum_{k=0}^{n+1} a_k p(x_k). \tag{2.12}$$

Let $p_0(x)=1, p_1(x)=x, p_2(x)=2x^2-1, p_s(x)=\left(1-x^2\right)V_{s-2}(x), s=3,\cdots,n+1$. Then by (2.12) we obtain

$$\sum_{k=0}^{n+1} a_k p_s(x_k) = \frac{1}{\pi}\int_{-1}^{1} p_s(t)\frac{dt}{\sqrt{1-t^2}} \qquad \text{for all } 0 \le s \le n+1. \tag{2.13}$$

Let $s=0,1,2$, respectively. Then by (2.13), $x_k = \cos\frac{k\pi}{n+1}$ and a simple computation it follows that

$$\sum_{k=0}^{n+1} a_k = \frac{1}{\pi}\int_{-1}^{1} 1\cdot\frac{dt}{\sqrt{1-t^2}} = 1, \tag{2.14a}$$

$$\sum_{k=0}^{n+1} a_k \cos\frac{k\pi}{n+1} = \sum_{k=0}^{n+1} a_k x_k = \frac{1}{\pi}\int_{-1}^{1} t\cdot\frac{dt}{\sqrt{1-t^2}} = 0, \tag{2.14b}$$

$$\sum_{k=0}^{n+1} a_k \cos\frac{2k\pi}{n+1} = \sum_{k=0}^{n+1} a_k(2x_k^2-1) = \frac{1}{\pi}\int_{-1}^{1} (2t^2-1)\cdot\frac{dt}{\sqrt{1-t^2}} = 0. \tag{2.14c}$$

For $s>2$, by virtue of the orthonormality of $\{V_n(x)\}_{n=1}^{\infty}$ with respect to the weight function $\sqrt{1-x^2}$ we obtain

$$\int_{-1}^{1} p_s(t)\cdot\frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} V_0(t)\cdot V_{s-2}(t)\cdot\sqrt{1-t^2}dt = 0. \tag{2.15}$$

Moreover, by a direct computation we obtain

$$\begin{aligned}
&\sum_{k=0}^{n+1} a_k p_s(x_k) = \sum_{k=0}^{n+1} a_k(1-x_k^2)V_{s-2}(x_k)\\
=&\sum_{k=0}^{n+1} a_k \sin\frac{k\pi}{n+1}\sin\frac{(s-1)k\pi}{n+1}\\
=&\frac{1}{2}\sum_{k=0}^{n+1} a_k\left(\cos\frac{(s-2)k\pi}{n+1} - \cos\frac{sk\pi}{n+1}\right).
\end{aligned} \tag{2.16}$$

From (2.13), (2.15) and (2.16) it follows that

$$\sum_{k=0}^{n+1} a_k\left(\cos\frac{(s-2)k\pi}{n+1} - \cos\frac{sk\pi}{n+1}\right) = 0, \qquad 3 \le s \le n+1. \tag{2.17}$$

From (2.14b), (2.14c), (2.17) and the induction we obtain

$$\sum_{k=0}^{n+1} a_k \cos\frac{sk\pi}{n+1} = \sum_{k=0}^{n+1} a_k \cos\frac{(s-2)k\pi}{n+1} = \cdots = 0, \qquad 3 \le s \le n+1. \tag{2.18}$$

By a direct computation we obtain

$$\sum_{k=0}^{n} \cos\frac{sk\pi}{n+1} = Re\left(\sum_{k=0}^{n} e^{\frac{sk\pi i}{n+1}}\right) = Re\left(\frac{1-e^{s\pi i}}{1-e^{s\pi i/(n+1)}}\right) = \begin{cases} 0, & s \text{ even}, \\ 1, & s \text{ odd}. \end{cases} \tag{2.19}$$

From (2.19) we checked that $a_0 = a_{n+1} = \frac{1}{2(n+1)}, a_k = \frac{1}{n+1}, k = 1,2,\cdots,n,$ satisfies the system of the linear equations given by (2.14a)-(2.14c) and (2.18) in the variables $a_0, a_1, \cdots, a_{n+1}$.

On the other hand, from [9, p. 204] we know that for an arbitrary $m \in \mathbb{N}$, one has

$$\cos m\theta = 2^{m-1}\cos^m\theta + p_{m-1}(\cos\theta), \tag{2.20}$$

where $p_{m-1} \in \mathbb{P}_{m-1}$. On using (2.20) and the computation of the Vandermonde determinant we obtain that the value of the coefficient determinant of the system of the linear equations given by (2.14a)-(2.14c) and (2.18) is

$$\begin{aligned}
&\begin{vmatrix}
1 & 1 & \cdots & 1 & \cdots & 1 \\
\cos\frac{1\cdot 0\pi}{n+1} & \cos\frac{1\cdot 1\pi}{n+1} & \cdots & \cos\frac{1k\pi}{n+1} & \cdots & \cos\frac{1(n+1)\pi}{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cos\frac{s0\pi}{n+1} & \cos\frac{s1\pi}{n+1} & \cdots & \cos\frac{sk\pi}{n+1} & \cdots & \cos\frac{s(n+1)\pi}{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cos\frac{(n+1)0\pi}{n+1} & \cos\frac{(n+1)1\pi}{n+1} & \cdots & \cos\frac{(n+1)k\pi}{n+1} & \cdots & \cos\frac{(n+1)(n+1)\pi}{n+1}
\end{vmatrix} \\
=&\begin{vmatrix}
1 & 1 & \cdots & 1 & \cdots & 1 \\
2^{1-1}\cos^1\frac{0\pi}{n+1} & 2^{1-1}\cos^1\frac{1\pi}{n+1} & \cdots & 2^{1-1}\cos^1\frac{k\pi}{n+1} & \cdots & 2^{1-1}\cos^1\frac{(n+1)\pi}{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2^{s-1}\cos^s\frac{0\pi}{n+1} & 2^{s-1}\cos^s\frac{1\pi}{n+1} & \cdots & 2^{s-1}\cos^s\frac{k\pi}{n+1} & \cdots & 2^{s-1}\cos^s\frac{(n+1)\pi}{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2^n\cos^{n+1}\frac{0\pi}{n+1} & 2^n\cos^{n+1}\frac{1\pi}{n+1} & \cdots & 2^n\cos^{n+1}\frac{k\pi}{n+1} & \cdots & 2^n\cos^{n+1}\frac{(n+1)\pi}{n+1}
\end{vmatrix} \\
=&2^{n(n+1)/2}\prod_{0\le i<j\le n+1}\left(\cos\frac{j\pi}{n+1} - \cos\frac{i\pi}{n+1}\right) \neq 0.
\end{aligned} \tag{2.21}$$

From (2.21) we know that the solution of the system of the linear equations given by (2.14a)-(2.14c) and (2.18) is unique. Hence we obtain (2.11). This completes the proof of Lemma 2.1. □

To give the worst-case error for T_{n+2} approximating INT_1, we give two lemmas. Let

$$h_k(x) = \left(1-x^2\right)(1-xx_k)\left(\frac{V_n(x)}{(n+1)(x-x_k)}\right)^2, \qquad k=1,\cdots,n, \tag{2.22a}$$

$$\sigma_k(x) = (x-x_k)\left(1-x_k^2\right)(1-x^2)\left(\frac{V_n(x)}{(n+1)(x-x_k)}\right)^2, \qquad k=1,\cdots,n. \tag{2.22b}$$

For $f \in C^{(1)}[-1,1]$, the quasi-Hermite interpolation polynomial $G_n(f,x)$ based on the points system (2.7) is of degree at most $2n+1$ and satisfies conditions

$$G_n(f,x_j) = f(x_j), \qquad j=0,\cdots,n+1, \tag{2.23a}$$

$$G_n'(f,x_j) = f'(x_j), \qquad j=1,\cdots,n. \tag{2.23b}$$

From [10] we know

$$G_n(f,x) = \left(\frac{1+x}{2}f(1) + \frac{1-x}{2}f(-1)\right)\frac{V_n^2(x)}{(n+1)^2} + \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} f'(x_k)\sigma_k(x). \tag{2.24}$$

Lemma 2.2. *If $f \in C^{(1)}[-1,1]$, then*

$$T_{n+2}(f) = \frac{1}{\pi}\int_{-1}^{1} G_n(f,t)\frac{dt}{\sqrt{1-t^2}}. \tag{2.25}$$

Proof. By (2.9a) and a direct computation we obtain

$$\phi_k'(x) = \frac{(-1)^k(xV_n(x)+(n+1)T_{n+1}(x))}{(n+1)(x-x_k)} + \frac{(-1)^k(1-x^2)V_n(x)}{(n+1)(x-x_k)^2}, \qquad 1 \le k \le n, \tag{2.26}$$

where $T_n(x)$ denotes the nth Chebyshev polynomial of the first kind, i.e., $T_n(x) = \cos n\theta, x = \cos\theta$. From (2.26) and a simple computation it follows that

$$\phi_k'(x_k) = \frac{-x_k}{2\left(1-x_k^2\right)}, \qquad 1 \le k \le n, \tag{2.27a}$$

$$\phi_k'(x_j) = \frac{(-1)^{k+j}}{x_j - x_k}, \qquad 1 \le j \ne k \le n. \tag{2.27b}$$

Similarly, by (2.9b) and a simple computation, one obtains

$$\phi_0'(x_j) = \frac{(-1)^j}{2(x_j-1)}, \qquad \phi_{n+1}'(x_j) = \frac{(-1)^{j+n+1}}{2(x_j+1)}, \qquad 1 \le j \le n. \tag{2.28}$$

By the properties of the Lagrange interpolation we obtain

$$Q_{n+2}(f,x_j) = f(x_j), \qquad 0 \le j \le n+1. \tag{2.29}$$

From (2.8) and (2.27a)-(2.28) we obtain that for $1 \le j \le n$,

$$
\begin{aligned}
Q'_{n+2}(f, x_j) =& \sum_{k=0}^{n+1} f(x_k)\phi'_k(x_j) \\
=& \frac{(-1)^j f(1)}{2(x_j - 1)} - \frac{x_j f(x_j)}{2\left(1 - x_j^2\right)} + \sum_{1 \le k \ne j \le n} f(x_k)\frac{(-1)^{k+j}}{x_j - x_k} + f(-1)\frac{(-1)^{j+n+1}}{2(x_j + 1)}.
\end{aligned}
\tag{2.30}
$$

Let $C(x) = Q_{n+2}(f, x) - G_n(f, x)$. Then from (2.23a), (2.23b), (2.29) and (2.30) we obtain

$$
C(x_j) = 0, \qquad 0 \le j \le n+1, \tag{2.31}
$$

and for $1 \le j \le n$,

$$
\begin{aligned}
C'(x_j) = \frac{(-1)^j f(1)}{2(x_j - 1)} - \frac{x_j f(x_j)}{2\left(1 - x_j^2\right)} + \sum_{1 \le k \ne j \le n} f(x_k)\frac{(-1)^{k+j}}{x_j - x_k} \\
+ f(-1)\frac{(-1)^{j+n+1}}{2(x_j + 1)} - f'(x_j).
\end{aligned}
\tag{2.32}
$$

From the interpolation properties of $G_n(f, x)$ and the fact that $C(x)$ is a polynomial of degree at most $2n+1$ we obtain

$$
\begin{aligned}
C(x) =& G_n(C, x) \\
=& \left(\frac{1+x}{2}C(1) + \frac{1-x}{2}C(-1)\right)\frac{V_n^2(x)}{(n+1)^2} + \sum_{k=1}^{n} C(x_k)h_k(x) + \sum_{k=1}^{n} C'(x_k)\sigma_k(x) \\
=& \sum_{k=1}^{n} C'(x_k)\sigma_k(x).
\end{aligned}
\tag{2.33}
$$

For an arbitrary $1 \le k \le n$, from (2.22b) and the orthonormality of $\{V_n(x)\}_{n=1}^{\infty}$ with respect to the weight function $\sqrt{1-x^2}$ we obtain

$$
\int_{-1}^{1} \sigma_k(t)\frac{dt}{\sqrt{1-t^2}} = \frac{1-x_k^2}{(n+1)^2}\int_{-1}^{1} V_n(t)\cdot\frac{V_n(t)}{t - x_k}\cdot\sqrt{1-t^2}\,dt = 0. \tag{2.34}
$$

From (2.33) and (2.34) we obtain

$$
\int_{-1}^{1} C(t)\frac{dt}{\sqrt{1-t^2}} = 0,
$$

and which means

$$
\int_{-1}^{1} Q_{n+2}(f, t)\frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} G_n(f, t)\frac{dt}{\sqrt{1-t^2}}. \tag{2.35}
$$

From (2.10) and (2.35) we obtain (2.25). This completes the proof of Lemma 2.2. □

For $f \in C^{(1)}[-1,1]$, denote

$$R_n(f,x) = f(x) - G_n(f,x).$$

Lemma 2.3. *Let $f \in C^{(2n+2)}[-1,1]$. Then*

$$R_n(f,x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x^2-1)\prod_{j=1}^{n}(x-x_j)^2, \qquad x \in [-1,1] \tag{2.36}$$

for some $\xi \in (-1,1)$ depending on x.

Proof. Since (2.36) is trivially satisfied if x coincides with one of the interpolation points $x_0, \cdots, x_{n+1}$, we need be concerned only with the case where x does not coincide with one of the interpolation points. We define

$$q_n(x) := (x^2-1)\prod_{j=1}^{n}(x-x_j)^2, \tag{2.37}$$

and keeping x fixed, consider $g : [-1,1] \to \mathbb{R}$ given by

$$g(y) := R_n(f,y) - q_n(y)\frac{R_n(f,x)}{q_n(x)}, \qquad y \in [-1,1]. \tag{2.38}$$

By the assumption on f we know $g \in C^{(2n+2)}[-1,1]$. From (2.23a) and (2.23b) we know that g has at least $2n+3$ zeros (count multiplicity), namely single zero $x, 1, -1$ and double zero $\{x_k\}_{k=1}^n$. Then, by Rolle's theorem the derivative g' has at least $2n+2$ zeros. Repeating the argument, by induction we deduce that the derivative $g^{(2n+2)}$ has at least one zero in $[-1,1]$, which we denote by ξ. For this zero we have that

$$0 = f^{(2n+2)}(\xi) - (2n+2)!\frac{R_n(f,x)}{q_n(x)},$$

and from this we obtain (2.36). This completes the proof of Lemma 2.3. □

Noticed that $\prod_{j=1}^n (x-x_j)$ has the same zeros $\{x_j\}_{j=1}^n$ with $V_n(x)$ and this implies that there exists C such that

$$V_n(x) = C\prod_{j=1}^{n}(x-x_j).$$

From [11, p. 87] we know that the leading coefficient of V_n is 2^n. Hence by comparing the leading coefficients of V_n with $\prod_{j=1}^n (x-x_j)$ we obtain

$$V_n(x) = 2^n\prod_{j=1}^{n}(x-x_j). \tag{2.39}$$

From (2.36) and (2.39) we know that for $f \in C^{(2n+2)}[-1,1]$, one has

$$R_n(f,x) = \frac{f^{(2n+2)}(\xi)}{2^{2n}(2n+2)!}(x^2-1)V_n^2(x), \qquad x \in [-1,1] \tag{2.40}$$

for some $\xi \in [-1,1]$ depending on x.

3. Main results and their proofs

For convenience, for any operator $S : F \to G$, we denote

$$\|S\|_{F\to G} = \sup_{f\in F, \|f\|_F \le 1} \|Sf\|_G.$$

The first result is for univariate functions. We obtained the following result.

Theorem 3.1. *Let* T_{n+2} *be given by* (2.10). *Then the following equality holds.*

$$e(INT_1, T_{n+2}, F_1, \mathbb{R}) = \|INT_1 - T_{n+2}\|_{F_1\to\mathbb{R}} = \frac{1}{2^{2n+1}(2n+2)!}. \tag{3.1}$$

Proof. Note that if $f \in F_1$, then $INT_1(f)$ coincides with $I(f)$ given by (2.6). Hence for $f \in F_1$ with $\|f\|_{F_1} \le 1$, from (2.6), (2.25) and (2.40) it follows that

$$\begin{aligned} INT_1(f) - T_{n+2}(f) &= \frac{1}{\pi}\int_{-1}^{1} (f(t) - G_n(f,t))\frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{2^{2n}(2n+2)!\pi}\int_{-1}^{1} f^{(2n+2)}(\xi)(t^2-1)V_n^2(t)\frac{dt}{\sqrt{1-t^2}} \\ &= -\frac{1}{2^{2n}(2n+2)!\pi}\int_{-1}^{1} f^{(2n+2)}(\xi)\sqrt{1-t^2}V_n^2(t)dt. \end{aligned} \tag{3.2}$$

From $|f^{(2n+2)}(\xi)| \le \|f^{(2n+2)}\|_\infty \le 1$ and (3.2) we obtain

$$\begin{aligned} |INT_1(f) - T_{n+2}(f)| &= \frac{1}{2^{2n}(2n+2)!\pi}\left|\int_{-1}^{1} f^{(2n+2)}(\xi)\sqrt{1-t^2}V_n^2(t)dt\right| \\ &\le \frac{1}{2^{2n}(2n+2)!\pi}\int_{-1}^{1} |f^{(2n+2)}(\xi)|\sqrt{1-t^2}V_n^2(t)dt \\ &\le \frac{1}{2^{2n}(2n+2)!\pi}\int_{-1}^{1} \sqrt{1-t^2}V_n^2(t)dt \\ &= \frac{1}{2^{2n}(2n+2)!\pi}\int_{0}^{\pi} \sin^2(n+1)\theta d\theta = \frac{1}{2^{2n+1}(2n+2)!}. \end{aligned} \tag{3.3}$$

On the other hand, let $g(x)=-\frac{x^{2n+2}}{(2n+2)!}$. Then $g\in F_1, \|g\|_{F_1}=1$ and by the computation of (3.3) we obtain

$$INT_1(g)-T_{n+2}(g)=\frac{1}{2^{2n+1}(2n+2)!}. \tag{3.4}$$

From (3.3) and (3.4) we obtain (3.1). This completes the proof of Theorem 3.1. □

Now we begin to consider multivariate functions. The Smolyak algorithm based on the Lagrange interpolation was introduced in [12]. Afterwards, [7] and [13] used this algorithm to consider the tractability of the non-weighted integral problem and the non-weighted L_p-approximation problems for $1\le p<\infty$ of an infinitely differentiable multivariate function class, respectively. In this paper we used the corresponding Smolyak algorithm to approximate the integral INT_d. Now we introduce specific algorithms.

For $d=1$, we define the sequence of quadrature rules $\{U^l\}_{l=1}^{\infty}$, where for $l=1$ there is only one node $x_1^1=0$ with weight $a_1^1=1$, and for $l>1$ we define

$$U^l(f)=T_{2^{l-1}+1}(f)=\sum_{j=0}^{2^{l-1}}a_j^l f(x_j^l) \tag{3.5}$$

with the nodes and weights given by

$$x_j^l=\cos\frac{j\pi}{2^{l-1}},\qquad j=0,1,\cdots,2^{l-1}\quad\text{and}\quad a_j^l=\begin{cases}\frac{1}{2^l}, & j=0,2^{l-1};\\ \frac{1}{2^{l-1}}, & j=1,2,\cdots,2^{l-1}-1.\end{cases} \tag{3.6}$$

Observe that the nodes of the U^l are nested, since

$$x_{2j}^{l+1}=x_j^l\ \text{ for }\ j=0,1,\cdots,2^{l-1}.$$

For $d>1$ we first define tensor product formulas as follow: for $\mathbf{i}=(i_1,\cdots,i_d)\in\mathbb{N}^d$,

$$\big(U^{i_1}\otimes\cdots\otimes U^{i_d}\big)(f)=\sum_{j_1=0}^{m_{i_1}}\cdots\sum_{j_d=0}^{m_{i_d}}\prod_{s=1}^{d}a_{j_s}^{i_s}\cdot f\big(x_{j_1}^{i_1},\cdots,x_{j_d}^{i_d}\big),$$

where $m_1=0$ and $m_i=2^{i-1}$ for $i>1$.

With $U^0=0$, we define

$$\Delta^i=U^i-U^{i-1} \tag{3.7}$$

for $i\in\mathbb{N}$. Moreover, we put $|\mathbf{i}|=i_1+\cdots+i_d$ for $\mathbf{i}=(i_1,\cdots,i_d)\in\mathbb{N}^d$. Then the Smolyak algorithm is given by

$$A(q,d)=\sum_{|\mathbf{i}|\le q}\left(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right) \tag{3.8}$$

for integers $q\ge d$. To compute $A(q,d)(f)$, from [7] we know that one only needs to know function values at the points set

$$H(q,d)=\bigcup_{|\mathbf{i}|=q}\left(X^{i_1}\times\cdots\times X^{i_d}\right),$$

where $X^i = \{x_0^i, \cdots, x_{m_i}^i\} \subset [-1,1]$ denotes the set of points used by U^i. The points $x \in H(q,d)$ are called hyperbolic cross and $H(q,d)$ is also called a sparse grid. Let

$$N_d(k) := |H(d+k,d)|$$

be the number of points used by $A(d+k,d)$. We use $\approx$ to denote the strong equivalence of sequences, i.e., $u_n \approx v_n$ iff $\lim_{n\to\infty} u_n/v_n = 1$. Then, for $k \to \infty$ and fixed d, from Th. Műller-Gronbach [14, Lemma 1] we know

$$N_d(k) \approx \frac{2^k k^{d-1}}{(d-1)! \cdot 2^{d-1}}. \tag{3.9}$$

Lemma 3.1. *Let Δ^i for $i \in \mathbb{N}$ be given by* (3.7). *Then the following relations hold.*

$$\|\Delta^i\|_{F_1\to\mathbb{R}} = \frac{2}{2^{2^{i-1}} 2^{i-1}!} \quad \textit{for } i \in \mathbb{N}. \tag{3.10}$$

Proof. For $i=1$, by the definition of Δ^1 we know that if $f \in F_1$ with $\|f\|_{F_1} \le 1$, then one has

$$|\Delta^1 f| = |f(0)| \le \|f\|_\infty \le 1. \tag{3.11}$$

On the other hand, let $g(x)=1$. Then $g \in F_1, \|g\|_{F_1} = 1$ and

$$\Delta^1 g = g(0) = 1. \tag{3.12}$$

From (3.11) and (3.12) we obtain (3.10) for $i=1$.

For $i=2$ and $f \in F_1$ with $\|f\|_{F_1} \le 1$, from (3.5)-(3.7) we know that

$$\Delta^2(f) = \frac{f(1)+2f(0)+f(-1)}{4} - f(0) = \frac{f(1)-2f(0)+f(-1)}{4}. \tag{3.13}$$

From the properties of difference we know that there exists a $\xi \in [-1,1]$ such that

$$|f(1)-2f(0)+f(-1)| = |f''(\xi)|. \tag{3.14}$$

From (3.13), (3.14) and $\|f''\|_\infty \le 1$ one obtains

$$|\Delta^2 f| \le \frac{1}{4}. \tag{3.15}$$

On the other hand, let $g = \frac{x^2}{2}$. Then $g \in F_1$ with $\|g\|_{F_1} = 1$ and from (3.13) we obtain

$$\Delta^2 g = 1/4. \tag{3.16}$$

From (3.15) and (3.16) we obtain (3.10) for $i=2$.

Now we consider $i \ge 3$. Assume that $f \in F_1$ with $\|f\|_{F_1} \le 1$. Let

$$\left\{ t_k = \cos\left(\frac{2k-1}{2(n+1)}\pi\right)\right\}_{k=1}^{n+1}$$

be the zeros of the nth Chebyshev polynomials $T_{n+1}(x)$. Then from (3.5)-(3.7) it follows that

$$\begin{aligned}\Delta^i(f) =& \frac{1}{2^{i-1}} \sum_{k=1}^{2^{i-2}} f\left(\cos \frac{2k-1}{2^{i-1}}\pi\right) - \frac{U^{i-1}(f)}{2} \\ =& \frac{INT_1(f) - U^{i-1}(f)}{2} - \frac{1}{2}\left(INT_1(f) - \frac{1}{2^{i-2}} \sum_{k=1}^{2^{i-2}} f\left(t_k\right)\right). \end{aligned} \tag{3.17}$$

Let $n = 2^{i-2} - 1$ in (3.1). Then one has

$$\left|INT_1(f) - U^{i-1}(f)\right| \le \frac{2}{2^{2^{i-1}} 2^{i-1}!}. \tag{3.18}$$

From [9, p. 106] we obtain that there exists a $\xi \in [-1,1]$ such that

$$INT_1(f) - \frac{1}{2^{i-2}} \sum_{k=1}^{2^{i-2}} f\left(t_k\right) = \frac{f^{(2^{i-1})}(\xi)}{2^{2^{i-1}-1} 2^{i-1}!}. \tag{3.19}$$

From (3.17)-(3.19) and $\left|f^{(2^{i-1})}(\xi)\right| \le 1$ it follows that

$$|\Delta^i(f)| \le \frac{2}{2^{2^{i-1}} 2^{i-1}!}. \tag{3.20}$$

On the other hand, let $g(x) = -\frac{x^{2^{i-1}}}{2^{i-1}!}$. Then $g \in F_1$ with $\|g\|_{F_1} = 1$ and from (3.4), (3.17) and (3.19) it follows that

$$\Delta^i(g) = \frac{2}{2^{2^{i-1}} 2^{i-1}!}. \tag{3.21}$$

From (3.20) and (3.21) we obtain (3.10) for $i \ge 3$. This completes the proof of Lemma 3.1. □

From the well known inequality

$$n! > \left(\frac{n+1}{e}\right)^n,$$

we obtain that for $i \ge 2$, one has (where we used $\ln 10 > 2$)

$$\begin{aligned} 2^{2^i} 2^i! \ge& 2^{2^i} \left(\frac{2^i+1}{e}\right)^{2^i} = \left(\frac{2^{i+1}+2}{e}\right)^{2^i} \\ =& e^{2^i\left(\ln(2^{i+1}+2)-1\right)} \ge e^{2^{i-1}\ln(2^{i+1}+2)} \ge e^{2^{i-1}\ln 2^{i+1}} = 2^{(i+1)2^{i-1}}. \end{aligned}$$

From above relation and a direct inspection for $i = 0, 1$ we obtain that for $i \in \mathbb{N}_0$, one has

$$2^{2^i} 2^i! \ge 2^{(i+1)2^{i-1}}. \tag{3.22}$$

From (3.10) and (3.22) it follows that

$$\left\|\Delta^i\right\|_{F_1\to\mathbb{R}} = \frac{2}{2^{2^{i-1}}2^{i-1}!} \le \frac{2}{2^{i2^{i-2}}} = 2\cdot 2^{-\frac{i2^i}{4}} \quad \text{for } i\in\mathbb{N}. \tag{3.23}$$

From (3.23) we obtain that for an arbitrary $\mathbf{i}=(i_1,i_2,\cdots,i_d)\in\mathbb{N}^d$, one has

$$\left\|\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right\|_{F_d\to\mathbb{R}} \le \prod_{k=1}^{d}\left\|\Delta^{i_k}\right\|_{F_1\to\mathbb{R}} \le 2^d 2^{-\frac{1}{4}\sum_{k=1}^{d} i_k 2^{i_k}}. \tag{3.24}$$

Using Lagrangian multiplier method to compute the minimum value of the function

$$f(x_1,\cdots,x_d) = \sum_{i=1}^{d} x_i 2^{x_i}$$

under the constraint conditions $x_i\ge 1, 1\le i\le d, |x| = \sum_{i=1}^{d} x_i = A\ge d$, we find that

$$\sum_{i=1}^{d} x_i 2^{x_i} \ge A2^{\frac{A}{d}} \quad \text{for all } x_i\ge 1,\ 1\le i\le d,\ |x| = \sum_{i=1}^{d} x_i = A. \tag{3.25}$$

From (3.25) we obtain that for all $\mathbf{i}=(i_1,\cdots,i_d)\in\mathbb{N}^d$, one has

$$\sum_{k=1}^{d} i_k 2^{i_k} \ge |\mathbf{i}|2^{|\mathbf{i}|/d} \ge \frac{1}{4}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}. \tag{3.26}$$

From (3.24) and (3.26) we obtain

$$\left\|\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right\|_{F_d\to\mathbb{R}} \le 2^d 2^{-\frac{1}{16}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}}. \tag{3.27}$$

The second result is for multivariate functions. We obtained the following result.

Theorem 3.2. *Let F_d be defined by* (1.1). *Then the integral problem $INT = \{INT_d\}_{d=1}^{\infty}$ is of exponential convergence, i.e., there exist positive $C(d)$ and $C_2(d)$ depending on d such that*

$$e(n,INT_d,F_d,\mathbb{R}) \le C(d)2^{-(n/C_2(d))^{1/d}} \quad \textit{for all } n,d\in\mathbb{N}. \tag{3.28}$$

Proof. From

$$INT_d = \sum_{\mathbf{i}\in\mathbb{N}^d}\left(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right)$$

and (3.8) it follows that

$$INT_d - A(q,d) = \sum_{|\mathbf{i}|>q}\left(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right). \tag{3.29}$$

From (3.29) and (3.27) it follows that

$$\begin{aligned}&\|INT_d - A(q,d)\|_{F_d\to\mathbb{R}}\\ \le&\sum_{|\mathbf{i}|>q}\left\|\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}\right\|_{F_d\to\mathbb{R}}\le 2^d\sum_{|\mathbf{i}|>q}2^{-\frac{1}{16}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}}.\end{aligned}\tag{3.30}$$

For an arbitrary $\mathbf{i}=(i_1,i_2,\cdots,i_d)\in\mathbb{N}^d$, let

$$D(\mathbf{i})=\otimes_{j=1}^d[i_j,i_j+1).$$

Then for $\mathbf{x}=(x_1,\cdots,x_d)\in D(\mathbf{i})$, one has $|\mathbf{i}|+d\ge|\mathbf{x}|$ and hence one has

$$2^{-\frac{1}{16}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}}\le 2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}\qquad\text{for }\ \mathbf{x}\in D(\mathbf{i}).\tag{3.31}$$

(3.31) means that

$$2^{-\frac{1}{16}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}}=\int_{D(\mathbf{i})}2^{-\frac{1}{16}(|\mathbf{i}|+d)2^{|\mathbf{i}|/d+1}}d\mathbf{x}\le\int_{D(\mathbf{i})}2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}d\mathbf{x}.\tag{3.32}$$

From (3.30) and (3.32) one obtains

$$\begin{aligned}&\|INT_d - A(q,d)\|_{F_d\to\mathbb{R}}\\ \le&2^d\sum_{|\mathbf{i}|>q}\int_{D(\mathbf{i})}2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}d\mathbf{x}=2^d\int_{\bigcup_{|\mathbf{i}|>q}D(\mathbf{i})}2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}d\mathbf{x}.\end{aligned}\tag{3.33}$$

It is easily checked that $\mathbf{x}\in D(\mathbf{i})$ for $|\mathbf{i}|>q$ implies $|\mathbf{x}|>q$. This implies $\bigcup_{|\mathbf{i}|>q}D(\mathbf{i})\subset\{\mathbf{x}||\mathbf{x}|\ge q\}$. Hence (3.33) means

$$\|INT_d - A(q,d)\|_{F_d\to\mathbb{R}}\le 2^d\int_{|\mathbf{x}|\ge q}2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}d\mathbf{x}.\tag{3.34}$$

Let $t=|\mathbf{x}|$. Then by a variable transformation we obtain

$$\int_{|\mathbf{x}|\ge q}2^{-\frac{1}{16}|\mathbf{x}|2^{|\mathbf{x}|/d}}d\mathbf{x}=\frac{1}{(d-1)!}\int_q^{+\infty}t^{d-1}2^{-\frac{t}{16}2^{t/d}}dt,\tag{3.35}$$

where we used the fact that the volume of d-dimensional simplex is $(d!)^{-1}$. It is easy to check that

$$\begin{aligned}&\int_q^{+\infty}t^{d-1}2^{-\frac{t}{16}2^{t/d}}dt\\ =&16d\int_q^{+\infty}\frac{t^{d-1}}{2^{t/d}(d+t\ln 2)}\cdot\frac{2^{t/d}(d+t\ln 2)}{16d}2^{-\frac{t}{16}2^{t/d}}dt\\ \le&\frac{16d}{\ln 2}\int_q^{+\infty}\frac{t^{d-2}}{2^{t/d}}\cdot\frac{2^{t/d}(d+t\ln 2)}{16d}2^{-\frac{t}{16}2^{t/d}}dt.\end{aligned}\tag{3.36}$$

For $d=1,2$, it is easy to see that $\frac{t^{d-2}}{2^{t/d}} \le 1$ on $[d,+\infty)$. Hence by (3.36) we obtain

$$\int_q^{+\infty} t^{d-1} 2^{-\frac{t}{16}2^{t/d}} dt \le \frac{16d}{\ln 2} \int_q^{+\infty} \frac{2^{t/d}(d+t\ln 2)}{16d} 2^{-\frac{t}{16}2^{t/d}} dt = \frac{16d}{(\ln 2)^2} 2^{-\frac{q}{16}2^{q/d}}. \tag{3.37}$$

For $d \ge 3$, by a direct computation we obtain that the maximum value point of $\frac{t^{d-2}}{2^{t/d}}$ on $[d,+\infty)$ is $t=\frac{d^2-2d}{\ln 2}$, and its value is $\left(\frac{d^2-2d}{e\ln 2}\right)^{d-2}$. Hence (3.36) means

$$\begin{aligned}
&\int_q^{+\infty} t^{d-1} 2^{-\frac{t}{16}2^{t/d}} dt \\
\le & \frac{16d}{\ln 2} \cdot \left(\frac{d^2-2d}{e\ln 2}\right)^{d-2} \int_q^{+\infty} \frac{2^{t/d}(d+t\ln 2)}{16d} 2^{-\frac{t}{16}2^{t/d}} dt \\
= & \frac{16d}{(\ln 2)^2} \cdot \left(\frac{d^2-2d}{e\ln 2}\right)^{d-2} 2^{-\frac{q}{16}2^{q/d}}.
\end{aligned} \tag{3.38}$$

Denote $C(d) = \max\left\{\frac{2^{d+4}d}{(d-1)!(\ln 2)^2}, \frac{2^{d+4}d}{(d-1)!(\ln 2)^2} \cdot \left(\frac{d^2-2d}{e\ln 2}\right)^{d-2}\right\}$. Then from (3.34)-(3.38) it follows that

$$\|INT_d - A(q,d)\|_{F_d\to\mathbb{R}} \le C(d) 2^{-\frac{q}{16}2^{q/d}}. \tag{3.39}$$

Let $q=k+d$. Then from (3.9) it follows that

$$N_d(k) \approx \frac{(k+d)^{d-1}2^{k+d}}{(d-1)!\cdot 2^{2d-1}} = \frac{q^{d-1}2^q}{(d-1)!\cdot 2^{2d-1}} \le \frac{q^d 2^q}{d!\cdot 2^{2d-1}}. \tag{3.40}$$

From (3.40) we know that there exists $C_1(d)$ such that

$$q^d 2^q \ge C_1(d)\cdot N_d(k). \tag{3.41}$$

Denote $C_2(d) = \frac{16^d}{C_1(d)}$. Then (3.41) implies that

$$q 2^{q/d} \ge 16\left(N_d(k)/C_2(d)\right)^{1/d}. \tag{3.42}$$

From (3.39) and (3.42) it follows that

$$\|INT_d - A(q,d)\|_{F_d\to\mathbb{R}} \le C(d) 2^{-\left(N_d(k)/C_2(d)\right)^{1/d}}. \tag{3.43}$$

It is obviously that (3.43) leads to (3.28). This completes the proof of Theorem 3.2. □

Remark 3.1. The result (3.28) is obviously better than (2.4). The quadrature nodes and weights used by Smolyak algorithm have explicit expressions (see (3.6)), while the so-called digital nets used by [5] involved in a very complicated computation. Furthermore, the Smolyak algorithm is constructive, however the algorithms used by [5] are non-constructive.

Remark 3.2. The error estimates in [12] are obtained by a proof of induction. The error estimates in [7, 13] are based on the operator norms and the best polynomial approximation errors. In this paper we used a completely different method and obtained a completely different result with that of [7, 12-13].

Acknowledgments This work was supported by the National Natural Science Foundation of China (Grant No. 11471043, 11671271) and by the Beijing Natural Science Foundation (Grant No. 1172004)

References

[1] J. Dick, G. Larcher, F. Pillichshammer and H. Woźniakowski, *Exponential convergence and tractability of multivariate integration for Korobov spaces*, Math. Comp., 80 (2011), pp. 905–930.

[2] P. Kritzer, F. Pillichshammer and H. Woźniakowski, *Multivariate integration of infinitely many times differentiable functions in weighted Korobov spaces*, Math. Comp., 83 (2014), pp. 1189–1206.

[3] J. Dick, P. Kritzer, F. Pillichshammer and H. Woźniakowski, *Approximation of analytic functions in Korobov spaces*, J. Complexity, 30 (2014), pp. 2–28.

[4] C. Irrgeher, P. Kritzer, G. Leobacher and F. Pillichshammer, *Integration in Hermite spaces of analytic functions*, J. Complexity, 41 (2017), pp. 102–125.

[5] K. Suzuki, *Super-polynomial convergence and tractability of multivariate integration for infinitely times differentiable functions*, J. Complexity, 39 (2017), pp. 51–68.

[6] F. L. Huang and S. Zhang, *Approximation of infinitely differentiable multivariate functions is not strongly tractable*, J. Complexity, 23 (2007), pp. 73–81.

[7] A. Hinrichs, E. Novak and M. Ullrich, *On weak tractability of the Clenshaw-Curtis Smolyak algorithm*, J. Approx. Theory, 183 (2014), pp. 31–44.

[8] A. K. Varma and P. Vértesi, *Some Erdös and E. Feldheim type theorems on mean convergence of Lagrange interpolation*, J. Math. Anal. Appl., 91 (1983), pp. 68–79.

[9] Rainer Kress, Numerical Analysis, Springer-Verlag, New York, 2003.

[10] A. K. Varma and J. Prasad, *An analogue of a problem of P. Erdös and E. Feldheim on L_p convergence of interpolatory processes*, J. Approx. Theory, 56 (1989), pp. 225–240.

[11] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, New York, 1993.

[12] V. Barthelmann, E. Novak and K. Ritter, *High dimensional polynomial interpolation on sparse grids*, Adv. Comput. Math., 12 (2000), pp. 273–288.

[13] G. Q. Xu, *On weak tractability of the Smolyak algorithm for approximation problems*, J. Approx. Theory, 192 (2015), pp. 347–361.

[14] T. Műller-Gronbach, *Hyberbolic cross designs for approximation of random fields*, J. Statist. Plann. Inference, 66 (1998), pp. 321–344.