Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2017-0149

Vol. **12**, No. 2, pp. 370-402
May 2019

Proximal ADMM for Euler's Elastica Based Image Decomposition Model

Zhifang Liu[1], Samad Wali[1,*], Yuping Duan[2], Huibin Chang[3], Chunlin Wu[1] and Xue-Cheng Tai[4]

[1] *School of Mathematical Sciences, Nankai University, Tianjin, China*
[2] *Center for Applied Mathematics, Tianjin University, Tianjin, China*
[3] *School of Mathematical Sciences, Tianjin Normal University, Tianjin, China*
[4] *Department of Mathematics, Hong Kong Baptist University, Hong Kong, China*

Received 5 December 2017; Accepted (in revised version) 27 February 2018

Abstract. This paper studies image decomposition models which involve functional related to total variation and Euler's elastica energy. Such kind of variational models with first order and higher order derivatives have been widely used in image processing to accomplish advanced tasks. However, these non-linear partial differential equations usually take high computational cost by the gradient descent method. In this paper, we propose a proximal alternating direction method of multipliers (ADMM) for total variation (TV) based Vese-Osher's decomposition model [L. A. Vese and S. J. Osher, J. Sci. Comput., 19.1 (2003), pp. 553-572] and its extension with Euler's elastica regularization. We demonstrate that efficient and effective solutions to these minimization problems can be obtained by proximal based numerical algorithms. In numerical experiments, we present numerous results on image decomposition and image denoising, which conforms significant improvement of the proposed models over standard models.

AMS subject classifications: 68U10, 90C25, 49M37

Key words: Alternating direction method of multipliers, total variation, Euler's elastica, proximal method, image decomposition, image denoising.

1. Introduction

Assume that $\Omega \subset \mathbb{R}^2$ is a bounded, open, and connected subset (usually a rectangle in image processing). The image decomposition task is to decompose a given image $f : \Omega \to \mathbb{R}$ as the sum of two components

$$f = u + v,$$

where u is geometric part or cartoon component, and v is an oscillating one. In general, u models homogeneous regions with sharp boundaries and v contains oscillating patterns

*Corresponding author. *Email addresses:* liuzhifang0628@163.com (Z. F. Liu), samad.walikhan@gmail.com (S. Wali), yuping.duan@tju.edu.cn (Y. P. Duan), changhuibin@gmail.com (H. B. Chang), wucl@nankai.edu.cn (C. L. Wu), xuechengtai@hkbu.edu.hk (X. C. Tai)

such as texture and noise; See, e.g. [3, 15, 25, 26, 34, 39, 44, 45]. The origins of these ideas are the remarkable book by Meyer [34], in which the author showed that the well-known Rudin-Osher-Fatemi (ROF) model [37] does not always represent texture or oscillatory details well.

An eligible and successful choice to detect textures is the generalized functions space $G = G(\Omega)$ [3, 30, 34, 46], where

$$G(\Omega) = \Big\{ v = \nabla \cdot \mathbf{g} = \partial_x g_1 + \partial_y g_2 : \ \mathbf{g} = (g_1, g_2), \ \ g_1, g_2 \in L^\infty(\Omega, \mathbb{R}^2) \Big\}$$

endowed with the norm

$$|v|_G = \inf \Big\{ \| |\mathbf{g}| \|_{L^\infty(\Omega,\mathbb{R}^2)} : v = \nabla \cdot \mathbf{g}, \ \ \mathbf{g} = (g_1, g_2), \ \ g_1, g_2 \in L^\infty(\Omega, \mathbb{R}^2), \ \ |\mathbf{g}| = \sqrt{g_1^2 + g_2^2} \Big\}.$$

The (BV, G) model proposed by Meyer in [34] is to solve the problem

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} \left\{ \int_\Omega |u|_{BV} + \beta |v|_G, \ \ f = u + v \right\},$$

where $BV(\Omega)$ is the *bounded variation* functions space.

There is no standard calculation of the associated Euler-Lagrange equation due to the term coming from an L^∞-norm (in the G-norm), which maps a series of work out to overcome this difficulty; See, e.g., [2, 4, 45, 46].

In [45, 46], Vese and Osher proposed to model oscillatory components v as first order derivatives of vector fields in $L^p, (1 \le p < \infty)$ (approaching to the L^∞-norm) to approximate Meyer's (BV, G) model. As the first practical image decomposition model, *total variation* (TV) based Vese-Osher's decomposition model solves the following convex minimization problem

$$\min_{u,\mathbf{g}} \left\{ \int_\Omega |\nabla u| + \frac{\alpha}{2} \int_\Omega |f - u - v|^2 + \beta \left(\int_\Omega |\mathbf{g}|^\rho \right)^{\frac{1}{\rho}} \right\}, \tag{1.1}$$

where $\alpha, \ \beta > 0$, are tuning parameters, $v = \nabla \cdot \mathbf{g}$, $\mathbf{g} = (g_1, g_2)^T$ and $1 \le \rho < \infty$. The model (1.1) was solved by sequential descent approach to its Euler-Lagrange equation. It has been shown that the advantage of the model (1.1) is that it is not sensitive to the choice of ρ. The authors recommended to set $\rho = 1$ to yield faster calculations per iteration [45]. In fact if we generate the model (1.1) to the case $\rho = \infty$, then we can easily handle the L^∞-norm in the discrete setting by the ADMM method proposed in this paper.

In [12], the authors showed that TV regularization suffers from the undesirable staircase effect for image denoising application, which also exists in image decomposition problems. To overcome this, high order models have been proposed [13, 32, 49]. Euler's elastica is one of the higher order energy functionals, which has a number of interesting applications in elasticity, computer graphics and in image processing. To improve the quality of an image in the sense of cartoon u and texture v and improve other applications like