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Finite Difference Schemes for the Tempered Fractional Laplacian

Zhijiang Zhang[1], Weihua Deng[1,*] and Hongtao Fan[2]

[1] *School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P. R. China*

[2] *College of Science, Northwest A&F University, Yangling 712100, Shaanxi, P. R. China*

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Abstract. The second and all higher order moments of the β-stable Lévy process diverge, the feature of which is sometimes referred to as shortcoming of the model when applied to physical processes. So, a parameter λ is introduced to exponentially temper the Lévy process. The generator of the new process is tempered fractional Laplacian $(\Delta+\lambda)^{\beta/2}$ [W. H. Deng, B. Y. Li, W. Y. Tian and P. W. Zhang, Multiscale Model. Simul., 16(1), 125-149, 2018]. In this paper, we first design the finite difference schemes for the tempered fractional Laplacian equation with the generalized Dirichlet type boundary condition, their accuracy depending on the regularity of the exact solution on $\bar{\Omega}$. Then the techniques of effectively solving the resulting algebraic equation are presented, and the performances of the schemes are demonstrated by several numerical examples.

AMS subject classifications: 35R11, 65M06, 65F08

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1. Introduction

The fractional Laplacian $\Delta^{\beta/2}$ is the generator of the β-stable Lévy process, in which the random displacements executed by jumpers are able to walk to neighboring or nearby sites, and also perform excursions to remote sites by way of Lévy flights [4, 23, 24]. The distribution of the jump length of β-stable Lévy process obeys the isotropic power-law measure $|x|^{-n-\beta}$, where n is the dimension of the space. The extremely long jumps of the process make its second and higher order moments divergent, sometimes being referred to as a shortcoming when it is applied to physical model in which one expects regular behavior of moments [29]. The natural idea to damp the extremely long jumps is to introduce a small damping parameter λ to the distribution of jump lengths, i.e., $e^{-\lambda|x|}|x|^{-n-\beta}$. With small λ, for short time, it displays the dynamics of Lévy process, while for sufficiently long

*Corresponding author. *Email address:* dengwh@lzu.edu.cn (W. H. Deng)

time the dynamics will transit slowly from superdiffusion to normal diffusion. The generator of the tempered Lévy process is the tempered fractional Laplacian $(\Delta+\lambda)^{\beta/2}$ [9]. The tempered fractional Laplacian equation governs the probability distribution function of the position of the particles.

This paper focuses on developing the finite difference schemes for the tempered fractional Laplacian equation

$$\begin{cases} -(\Delta+\lambda)^{\beta/2}u(x)=f(x), & x\in\Omega, \\ u(x)=g(x), & x\in\mathbb{R}\backslash\Omega, \end{cases} \tag{1.1}$$

where $\beta\in(0,2)$, $\lambda\geq 0$, $\Omega=(a,b)$, and

$$(\Delta+\lambda)^{\beta/2}u(x):=-c_\beta\int_{\mathbb{R}}\frac{u(x)-u(y)}{e^{\lambda|x-y|}|x-y|^{1+\beta}}dy \tag{1.2}$$

with

$$c_\beta=\begin{cases} \frac{\beta\Gamma(\frac{1+\beta}{2})}{2^{1-\beta}\pi^{1/2}\Gamma(1-\beta/2)} & \text{for } \lambda=0 \text{ or } \beta=1, \\ \frac{\Gamma(\frac{1}{2})}{2\pi^{\frac{1}{2}}|\Gamma(-\beta)|} & \text{for } \lambda>0 \text{ and } \beta\neq 1. \end{cases} \tag{1.3}$$

Note that the integral in (1.2) must be regarded as the principal value integral when $\beta\in[1,2)$, but an improper one is enough when $\beta\in(0,1)$. Model (1.1) corresponds to the one-dimensional case of the initial and boundary value problem in Eq. (49) recently proposed in [9], and its well posedness is discussed in [30]. Obviously, when $\lambda=0$, (1.2) reduces to the fractional Laplacian [24]

$$(\Delta)^{\beta/2}u(x):=-c_\beta\ \text{P.V.}\int_{\mathbb{R}}\frac{u(x)-u(y)}{|x-y|^{1+\beta}}dy. \tag{1.4}$$

It is well known that for the proper classes of functions that decay quickly enough at infinity, the fractional Laplacian can be rewritten as the combination of the left and right Riemann-Liouville fractional derivatives ${}_{-\infty}D_x^\beta u(x)$ and ${}_xD_\infty^\beta u(x)$ (the so-called Riesz fractional derivative) [28], i.e.,

$$-(\Delta)^{\beta/2}u(x)=\frac{{}_{-\infty}D_x^\beta u(x)+{}_xD_\infty^\beta u(x)}{2\cos(\beta\pi/2)},\quad \beta\neq 1. \tag{1.5}$$

The similar result also holds for the tempered fractional Laplacian. In fact, letting $u(x)\in H^\beta(\mathbb{R})$ and $\mathscr{F}[u(x)](\omega):=\int_{\mathbb{R}}u(x)e^{-ix\omega}dx$ be its Fourier transform, we have [30, Propositions 2.1 and 2.2]

$$\begin{aligned}&\mathscr{F}\left[(\Delta+\lambda)^{\beta/2}u(x)\right](\omega)\\ =&(-1)^{\lfloor\beta\rfloor}\left(\lambda^\beta-\left(\lambda^2+|\omega|^2\right)^{\frac{\beta}{2}}\cos\left(\beta\arctan\left(\frac{|\omega|}{\lambda}\right)\right)\right)\mathscr{F}[u(x)](\omega),\quad \beta\neq 1,\end{aligned} \tag{1.6}$$