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An Optimized Two-Step Hybrid Block Method Formulated in Variable Step-Size Mode for Integrating $y'' = f(x, y, y')$ Numerically

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Abstract. An optimized two-step hybrid block method is presented for integrating general second order initial value problems numerically. The method considers two intra-step points which are selected adequately in order to optimize the local truncation errors of the main formulas for the solution and the first derivative at the final point of the block. The new proposed method is consistent, zero-stable and has seventh algebraic order of convergence. To illustrate the performance of the method, some numerical experiments are presented for solving this kind of problems, in comparison with methods of similar characteristics in the literature.

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1. Introduction

It is well-known that the formulation of many physical phenomena in mathematical language results in second order differential equations. For instance, the mass movement under the action of a force, problems of orbital dynamics, circuit theory, control theory, chemical kinetics, or in general, any problem involving second Newton's law.

The present article is concerned with approximating on a given interval the solution of a general second order initial value problem (I.V.P) of the form

$$y''(x) = f\left(x, y(x), y'(x)\right), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{1.1}$$

An equation of the form (1.1) can be integrated by reformulating it as a system of two first order ODEs and then applying one the methods available for solving such systems.

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It seems less costly to develop numerical methods in order to integrate (1.1) directly. In this regard, many authors have proposed different methods for integrating the problem (1.1) directly (see for references, Hairer and Wanner [6], Chawla and Sharma [3], and Vigo-Aguiar and Ramos [1] among others). Among those procedures, block methods have been developed in order to obtain the numerical solution at more than one point at a time. One can see one of the pioneering works on block methods in [25]. Some advantages of block methods include (i) overcoming the overlapping of pieces of solutions and (ii) that they are self starting, thus avoiding the use of other methods to get starting values. Some useful references are [1-29].

In this article, we develop a two-step hybrid block method with two intra-step points using interpolation and collocation procedures with a constant step-size. Further, we will formulate the new proposed method in a variable step-size mode in order to make it more efficient from a practical point of view.

The article is organized as follows: Section 2 is concerned with development of the block method. Main characteristics of the block method are presented in Section 3. A formulation in variable step-size mode of the block method is considered in Section 4 using an embedded-type approach. To illustrate the performance of the proposed method, some numerical experiments are presented in Section 5 which show the efficiency of the new method when it is compared with other methods proposed in the scientific literature. Finally, some conclusions are presented in Section 6.

2. Development of the method

We present here the derivation of the block method with a constant step-size, and then a variable step-size formulation will be considered. To derive the block method, consider a polynomial approximation of the true solution $y(x)$ of (1.1) at the grid points $a = x_0 < x_1 < \cdots < x_N = b$ of the integration interval, with constant step-size $h = x_{j+1} - x_j$, $j = 0, 1, \cdots, N-1$. Let

$$y(x) \simeq p(x) = \sum_{n=0}^{8} a_n x^n \tag{2.1}$$

from which we get

$$y'(x) \simeq p'(x) = \sum_{n=1}^{8} a_n n x^{n-1}, \tag{2.2a}$$

$$y''(x) \simeq p''(x) = \sum_{n=2}^{8} a_n n(n-1) x^{n-2}, \tag{2.2b}$$

$$y'''(x) \simeq p'''(x) = \sum_{n=3}^{8} a_n n(n-1)(n-2) x^{n-3}, \tag{2.2c}$$