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Finite Difference Schemes for the Variable Coefficients Single and Multi-Term Time-Fractional Diffusion Equations with Non-Smooth Solutions on Graded and Uniform Meshes

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Abstract. Finite difference scheme for the variable coefficients subdiffusion equations with non-smooth solutions is constructed and analyzed. The spatial derivative is discretized on a uniform mesh, and $L1$ approximation is used for the discretization of the fractional time derivative on a possibly graded mesh. Stability of the proposed scheme is given using the discrete energy method. The numerical scheme is $\mathcal{O}(N^{-\min\{2-\alpha,r\alpha\}})$ accurate in time, where α $(0<\alpha<1)$ is the order of the fractional time derivative, r is an index of the mesh partition, and it is second order accurate in space. Extension to multi-term time-fractional problems with nonhomogeneous boundary conditions is also discussed, with the stability and error estimate proved both in the discrete l^2-norm and the l^∞-norm on the nonuniform temporal mesh. Numerical results are given for both the two-dimensional single and multi-term time-fractional equations.

AMS subject classifications: 35R11, 65M06, 65M12, 65M15

Key words: Fractional diffusion equation, graded mesh, multi-term, variable coefficients, low regularity, stability and convergence analysis.

1. Introduction

Fractional calculus has been established for a long time, its fast development arose in the last decades after many applications have been found in science and engineering, see the monograph [34] and the review paper [32]. In this paper, we consider the following two dimensional time-fractional reaction-diffusion problem with the initial and boundary conditions

$$\begin{cases} {}_0^C D_t^\alpha u(x,y,t) - \dfrac{\partial}{\partial x}\left(a(x,y,t)\dfrac{\partial u}{\partial x}(x,y,t)\right) - \dfrac{\partial}{\partial y}\left(b(x,y,t)\dfrac{\partial u}{\partial y}(x,y,t)\right) \\ \qquad\qquad +c(x,y,t)u(x,y,t) = f(x,y,t), \quad (x,y)\in\Omega, \quad t\in(0,T], \\ u(x,y,0)=w(x,y), \qquad\qquad\qquad\qquad (x,y)\in\Omega, \\ u(x,y,t)=0, \qquad\qquad\qquad\qquad\qquad (x,y)\in\partial\Omega, \quad t\in(0,T] \end{cases} \tag{1.1}$$

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where $\Omega=(0,L)^2$. Here $a(x,y,t)\geq a_0>0$, $b(x,y,t)\geq b_0>0$ and $c(x,y,t)\geq 0$ are given continuous functions for $(x,y,t)\in[0,L]^2\times(0,T]$.

The Caputo fractional derivative ${}_0^C D_t^\alpha u$ of the function $u(x,y,t)$ is defined by [34], i.e.,

$$ {}_0^C D_t^\alpha u(x,y,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u}{\partial \tau}(x,y,\tau)d\tau \qquad \text{for}\ \ \alpha\in(0,1). \tag{1.2}$$

Various numerical methods have been proposed to solve fractional partial differential equations(FPDEs), e.g., finite difference method in papers [2,5–7,10,15,25,28,39,51], finite element and spectral methods [11,23,26,31,45,46], fast numerical methods [41–43,48], second-order numerical methods [13,40,47,49], element free Galerkin methods [1,8,9].

However, solutions to FPDEs are generally non-smooth, they usually have a weak singularity at the boundaries. For one dimensional problems, the authors in paper [37] have obtained the following bounds on the derivatives of u under certain conditions on the right-hand side function $f(x,t)$:

$$\left|\frac{\partial^k u}{\partial x^k}(x,t)\right|\leq C \qquad \text{for}\quad k=0,1,2,3,4, \tag{1.3a}$$

$$\left|\frac{\partial^l u}{\partial t^l}(x,t)\right|\leq C(1+t^{\alpha-l}) \qquad \text{for}\quad l=0,1,2 \tag{1.3b}$$

for all $(x,t)\in[0,L]\times(0,T]$. And the authors pointed out, the essential feature of all typical solutions of time-fractional reaction-diffusion problem is that u has an initial layer at $t=0$ and $\frac{\partial u}{\partial t}(x,t)$ blows up as $t\to 0+$. This conclusion was drawn by solving the exact solution by using separation of variables, see, e.g., [30]. The L1 scheme, and schemes using the backward Euler method and second-order backward difference in time are given in [19,20]. The use of nonuniform grids is an efficient way to solve for FPDEs with non-smooth solutions, and they have been discussed in [4,22,27,37,44,50]. The temporal accuracy is increased from $\mathcal{O}(N^{-\min\{2-\alpha,r\alpha\}})$ to $\mathcal{O}(N^{-\min\{2-\alpha,2r\alpha\}})$ using the decomposition of the exact solution $u(x,t)$ into a new smoother function $v(x,t)$ in paper [36]. It is difficult to solve for problem (1.1) using the method of separation of variable as the coefficients are time-dependent, so we make the hypothesis that its solution has a layer at $t=0$.

In this paper, we give the finite difference scheme for solving the variable coefficients time-fractional diffusion problems with the Dirichlet boundary conditions using possibly nonuniform mesh. The graded temporal meshes offer an efficient way of computing reliable numerical approximations of solutions with initial singularity [21,37].

The motivation of this paper lies in two respects. First, we want to generalize the results of papers [36,37] from constant diffusion coefficient to variable ones. Kopteva discusses this kind of problems when the coefficients depend on the spatial variable x [21], giving a simple framework for the analysis of the error in the $L^\infty(\Omega)$ and $L^2(\Omega)$ norms for L1-type discretizations, and we hope to generalize to the case when the coefficients a, b and c also depend on the time variable t. Therefore, to give the theoretical analysis of the scheme, as the diffusion coefficient depends on both x and t, instead of the Fourier method for constant coefficients problems [44], we use the discrete energy method to give the stability