Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2018-0060

Vol. **12**, No. 3, pp. 906-922
August 2019

The Cubic Spline Rule for the Hadamard Finite-Part Integral on an Interval

Gendai Gu[1], Sheng An[1] and Meiling Zhao[1,*]

[1] *School of Mathematics and Physics, North China Electric Power University, Baoding, 071003, China*

Received 27 April 2018; Accepted (in revised version) 21 July 2018

Abstract. We propose a cubic spline rule for the calculation of the Hadamard finite-part integral on an interval. The error estimate is presented in theory, and the superconvergence result of the cubic spline rule for Hadamard finite-part integral is derived. When the singular point coincides with a prior known point, the convergence rate is one order higher than what is globally possible. Numerical experiments are given to demonstrate the efficiency of the theoretical analysis.

AMS subject classifications: 65M10, 78A48

Key words: hypersingular integral, cubic spline rule, superconvergence.

1. Introduction

The hypersingular integral exists in variety of boundary integral equations, which are widely applied to some physical problems, such as acoustics, electromagnetic scattering and fracture mechanics [1, 2, 7, 10, 25, 32, 33]. Hence it is an important issue to develop some numerical methods for calculating such integrals.

In this work, we consider the following hypersingular integral

$$I(f,s) = \fint_a^b \frac{f(x)}{(x-s)^2}dx, \quad s \in (a,b), \tag{1.1}$$

where its kernel has a higher order singularity than the dimension of the integral. Here $\fint$ denotes a Hadamard finite-part integral and s is the singular point. There are several forms of Hadamard finite-part integral and these definitions can be proved mathematically equivalent [5, 8, 20, 22, 26, 31]. Here we adopt the following form:

$$\fint_a^b \frac{f(x)}{(x-s)^2}dx = \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon} \frac{f(x)}{(x-s)^2}dx + \int_{s+\varepsilon}^b \frac{f(x)}{(x-s)^2}dx - \frac{2f(s)}{\varepsilon}\right\}, \quad s \in (a,b),$$

*Corresponding author. *Email addresses:* `ggd999@126.com` (G. Gu), `anshengmath@163.com` (S. An), `meilingzhaocn@yahoo.com` (M. Zhao)

where $f(x)$ is called Hadamard finite-part integrable with respect to the weight $(x-s)^{-2}$ if the limit on the right hand side of the above equation exists. A sufficient condition for $f(x)$ to be finite-part integrable is that the derivative of $f(x)$ is a Hölder continuous function. In recent years, the Gaussian method [8, 9, 21, 24], the Newton-Cotes method [4, 15, 16, 18, 20, 23, 28], the transformation method [3, 5] and other numerical quadrature methods [11–13, 35] have been developed for the calculation of hypersingular integrals.

Hadamard first proposed the concept of finite-part integral in [6]. Kutt made further improvement of this concept and emphasized the Gaussian method in [14]. If the integrand function $f(x)$ is smooth enough, the Gaussian method and the transformation method are efficient. However, in many physical applications, hypersingular integral equations coupled with some domain equations are required to be solved, where $f(x)$ is less smooth or even unknown. Since the integrals are often approximated at Gaussian points, the use of the Gaussian method is restricted by the mesh selection. Linz first used the general trapezoidal formula to solve the Hadamard integral in [20], and the error estimates of the trapezoidal rule and Simpson's rule were obtained. The Newton-Cotes method is invalid when the singular point is close to the grid point. It is even totally invalid if the singular point coincides with the grid points. The analysis and numerical results of Kutt and Linz both show that the mesh selection significantly influences the accuracy of the integral. In the early stage, many researchers select meshes very carefully to make sure that the singular point is located at the center of some subinterval. Later, some researchers are committed to seeking better algorithms [30, 31] which the location of the singular point is no longer restricted. The superconvergence phenomenons of the trapezoidal rule, Simpson's rule and Newton-Cotes rules for hypersingular integrals were studied in [17, 27–29].

Recently, the Hermite rule and the corresponding superconvergence phenomenon for evaluating hypersingular integrals were discussed in [34]. The convergence order of the Hermite rule for the Hadamard finite-part integrals is higher than the composite trapezoidal rule and the composite Simpson's rule. However, it is not always available because of its requirement for the derivative of the integrand function. In this paper, we present a cubic spline rule for the evaluation of the Hadamard finite-part integral. Theoretical analysis and numerical experiments both confirm that the error estimate has the same order with the Hermite rule, while the cubic spline rule requires no information about the derivatives of the integrand functions. Furthermore, we obtain the superconvergence rate of the cubic spline rule. At the non-superconvergence point, the convergence order is $\mathcal{O}(h^3)$. When the local coordinate of the singular point s is the zero of a special function, the convergence rate at the superconvergence points is one order higher than the global convergence rate.

The rest of this paper is organized as follows. In Section 2, the basic cubic spline formulas and some notions are introduced. Furthermore, we develop the cubic spline rule for the hypersingular integrals. In Section 3, we give the results of error estimates and the proofs. In Section 4, the superconvergence phenomenon of the cubic spline rule is investigated. In Section 5, numerical examples are presented to verify our theoretical analysis. Finally, we give the concluding remarks in the last section.