Numer. Math. Theor. Meth. Appl. Vol. **12**, No. 3, pp. 923-941
doi: 10.4208/nmtma.OA-2018-0124 August 2019

A Weak Galerkin Finite Element Method for the Elliptic Variational Inequality

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Received 10 December 2018; Accepted (in revised version) 11 February 2019

Abstract. In this paper, we discuss the weak Galerkin (WG) finite element method for the obstacle problem and the second kind of the elliptic variational inequality. We use piecewise linear functions to approximate the exact solutions. The WG schemes for the first and the second kind of elliptic variational inequality are established and the well-posedness of the two schemes are proved. Furthermore, we can obtain the optimal order estimates in H^1 norm. Finally, some numerical examples are presented to confirm the theoretical analysis.

AMS subject classifications: 65N30, 65N12, 35J85, 51M16, 26A27
Key words: Obstacle problem, the second kind of elliptic variational inequality, weak Galerkin finite element method, discrete weak gradient.

1. Introduction

Variational inequalities play an important role in nonlinear problems. They have applications not only in mathematics, such as nonlinear optimizations and cybernetics [29, 46], but also in other fields such as mechanics [7, 35], engineering [15, 22], and economics [16, 47]. For instance, in [13], when studying frictionless between linear elastomers and rigid bodies, Signorini proposed the so called Signorini problem [4, 9, 13], which was one of the earliest work involving variational inequalities. However, the rigorous mathematical theory of the variational inequality was not established until about three decades later [28].

Variational inequalities can roughly be divided into three categories: the elliptic variational inequality (EVI) [3, 27], the parabolic variational inequality (PVI) [14, 24] and the hyperbolic variational inequality (HVI) [30]. This paper focuses on the elliptic variational inequality. A variety of numerical methods have been developed for this problem, such as relaxation method [18], multigrid method [23–26, 40], multilevel projection method [49] and so on. In addition, the discontinuous Galerkin method is also used to study variational inequalities because of its flexibility and high efficiency. The relevant work and results have been detailed in [2, 10, 11].

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In this paper, we use weak Galerkin (WG) finite element method to solve elliptic variational inequalities. The WG finite element method was first proposed for the second order elliptic problem by Wang and Ye in [42]. The method has the following characteristics. First, the exact solution is approximated by the weak function. The weak function has the form $v = \{v_0, v_b\}$, where v_0 denotes the interior of each element, and v_b denotes the boundary of the corresponding element. v_b is not necessarily related to the trace of v_0. Second, the key idea of WG method is the definition of weak differential operators which replaces the classical differential operators. Finally, the finite element partition can be arbitrary shape of polygons in two dimensional space or polyhedra in three dimensional space. These features make the WG method more flexible. Therefore, WG method has been used to solve different types of partial differential problems, such as Stokes equation [32, 39, 44, 51, 53], Navier-Stokes equations [56], elasticity problem [38], Brinkman equation [45, 57], biharmonic equation [31, 54], Diffusion Equations [52], Maxwell equation [33], parabolic equation [55], elliptic interface [12] and so on.

As far as the variational inequality, the WG method for the first kind of the elliptic variational inequality was considered in [20]. The authors gave the rigorous and complete theoretical analysis of the obstacle problem. Due to the use of the local BDM_k elements [6], the WG finite element was limit to be triangular or tetrahedral. In addition, a modified weak Galerkin method (MWG) was introduced to deal with the obstacle problem and the Signorini problem in [50]. In MWG method, the boundary part of the weak function is replaced by the average of the internal part. For more details, readers may refer to [21, 32, 37, 41]. As for the second kind of the elliptic variational inequality, to the best of our knowledge, there exists no corresponding work for solving this problem with the WG method.

In this paper, we introduce the WG method based on [43] for the obstacle problem. By applying a stabilization idea, the mesh is extended to allow arbitrary polygons, which is different from the recently work in [20]. In addition, we shall make an initial effort for the second kind of the variational inequality with the WG method.

This paper is organized as follows: In Section 2, we give some preliminary knowledge for subsequent analysis and proof. In Section 3, we introduce the obstacle problem and relevant features, then we establish the WG scheme of the obstacle problem and obtain the optimal order error estimate in H^1 norm. As for the second kind of the elliptic variational inequality, in Section 4, we present the general form of the problem and relevant features, WG scheme and error estimate in H^1 norm are also derived. In Section 5, we provide some numerical examples of the two problems, which confirm the theoretical analysis and further explain the effectiveness of the WG method.

2. Preliminary

For convenience, we first give some notations and inequalities which will be used throughout the rest of the paper. We use standard notations of Sobolev space [1]. Let D be an open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$ ($d = 2,\ 3$). The internal product of $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$ and the boundary product is denoted by $\langle\cdot,\cdot\rangle_{s,\partial D}$. Notations $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ represent the norm and semi-norm in $H^s(D)$, respectively, where