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The Generalized Order Tensor Complementarity Problems

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Abstract. The main propose of this paper is devoted to study the solvability of the generalized order tensor complementarity problem. We define two problems: the generalized order tensor complementarity problem and the vertical tensor complementarity problem and show that the former is equivalent to the latter. Using the degree theory, we present a comprehensive analysis of existence, uniqueness and stability of the solution set of a given generalized order tensor complementarity problem.

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1. Introduction

Over the past decade, the research of finite-dimensional variational inequality and complementarity problems [9, 19, 24, 36] has been rapidly developed in the theory of existence, uniqueness and sensitivity of solutions, theory of algorithms and the application of these techniques to transportation planning, regional science, socio-economic analysis, energy modeling and game theory. The tensor complementarity problem (TCP), which is a natural generalization of the linear complementarity problem (LCP) and a special case of the nonlinear complementarity problem (NCP), is a new topic emerged from the tensor community, inspired by the growing research on structured tensors. Huang and Qi [26] reformulated the multilinear game (a class of N-person noncooperative games) as the TCP and showed that finding a Nash equilibrium point of

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the multilinear game is equivalent to finding a solution of the resulted TCP. The readers can be recommended [2,6,21,26,35,41,42] for a thorough survey of the existence of the solution set of the TCP.

Recently, some researchers focus on numerical algorithms for solving the TCPs and the interested readers can be recommended, e.g., [15,16,23,26,33,47]. Che et al. [7] considered the stochastic tensor complementarity problem via the theory of stochastic R_0 tensors. Barbagallo et al. [3] studied some variational inequalities on a class of structured tensors. Wang et al. [46] introduced the tensor variational inequality, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor. The interested readers can be referred to [27–29] for the basic theory, solution methods and applications of tensor complementarity problems.

Gowda and Sznajder [22] studied the solution set of the generalized order linear complementarity problem (GOLCP) and showed that this problem is equivalent to the generalized linear complementarity problem (henceforth called the vertical linear complementarity problem, abbreviated by VLCP, as in [11]) considered by Cottle and Dantzig [10], which established basic existence results via Lemke's algorithms. The readers can be recommended [1,13,17,25,30,37,45] and their references for a survey of the GOLCP. Gowda [20] introduced the concept of degree of an $\mathbb{R}_0$ tensor and showed that the degree of an $\mathbb{R}_0$ tensor is one.

In this paper, we study a special case of the generalized order complementarity problem, named as the generalized order tensor complementarity problem (GOTCP). GOTCPs can be viewed as the generalizations of GOLCPs and TCPs. In order to study the solution set of the GOTCP, we define the degree of any arbitrary tensor and the classes of tensors, based on the degree of an $\mathbb{R}_0$ tensor. We give a comprehensive analysis of existence, uniqueness and stability issue connected with the GOTCP. As we see, the degree theory approach allows us to prove existence results directly and under relaxed assumptions. Degree theory plays an important role in stability results as well.

It is a brief description of various sections. In Section 2, we define the GOTCP the vertical tensor complementarity problem (VTCP) and introduce the classes of $\mathbb{R}_0$ tensors, $\mathbb{P}$ tensors, $\mathbb{Q}$ tensors and so on, which can be used to study the solution set of the TCP. In Section 3, we introduce block tensors of various types and prove existence results in a general setting. In Section 4 we obtain the necessary and sufficient conditions for the zero vector being the unique solution of any given GOTCP. In Section 5, we give the sufficient conditions for the solution set and an isolated solution of the GOTCPs being stable. In Section 6, we conclude this paper and present some issues which will be considered in the future.

2. Preliminaries

Higher-order equivalents of vectors (first order) and matrices (second order) are called *higher-order tensors*, *multi-dimensional matrices*, or *multi-way arrays*. A tensor is an N-order array with dimension I of numbers denoted by script notation $\mathcal{A} \in$

$\mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ with entries given by

$$a_{i_1i_2\cdots i_N}\in\mathbb{R}\quad\text{for}\ \ i_n=1,\cdots,I_n\quad\text{ and }\quad n=1,\cdots,N.$$

Throughout this paper, the norm $\|\cdot\|_2$ denotes the Euclidean norm. We use $\mathbb{R}^I_+=\{\mathbf{x}\in\mathbb{R}^I: x_i\geq 0,\ i=1,\cdots,I\}$, $\mathbb{R}^I_{++}=\{\mathbf{x}\in\mathbb{R}^I:\ x_i>0,\ i=1,\cdots,I\}$ and $\mathbb{R}^I_-=\{\mathbf{x}\in\mathbb{R}^I: x_i\leq 0,\ i=1,\cdots,I\}$ to denote the non-negative orthant, the positive orthant and the non-positive orthant, respectively. For given two vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^I$, the ith components of $\mathbf{x}\wedge\mathbf{y}$ and $\mathbf{x}\vee\mathbf{y}$ are $\min\{x_i,y_i\}$ and $\max\{x_i,y_i\}$, respectively, for $i=1,\cdots,I$. Given two vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^I$, $\mathbf{x}>\mathbf{y}$ and $\mathbf{x}\geq\mathbf{y}$ mean $x_i>y_i$ and $x_i\geq y_i$, respectively, with $i=1,\cdots,I$. Similarly, we can define $\mathbf{x}<\mathbf{y}$ and $\mathbf{x}\leq\mathbf{y}$ for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^I$. We use $\mathbf{0}_I$ and $\mathbf{I}_I$ to denote the zero vector in $\mathbb{R}^I$ and the identity matrix in $\mathbb{R}^{I\times I}$, respectively.

The following notations will be adopted. We assume that I, J and N will be reserved to denote the index upper bounds, unless stated otherwise. We use small letters $x,u,v,\cdots$ for scalars, small bold letters $\mathbf{x},\mathbf{u},\mathbf{v},\cdots$ for vectors, bold capital letters $\mathbf{A},\mathbf{B},\mathbf{C},\cdots$ for matrices and calligraphic letters $\mathcal{A},\mathcal{B},\mathcal{C},\cdots$ for higher-order tensors. The entry with row index i and column index j in a matrix $\mathbf{A}$, i.e., $(\mathbf{A})_{ij}$, is symbolized by a_{ij} (also $(\mathbf{x})_i=x_i$ and $(\mathcal{A})_{i_1i_2\cdots i_N}=a_{i_1i_2\cdots i_N}$).

In this paper, we assume that all I_n are the same, that is, $I_n=I$ for $n=1,\cdots,N$. The set of all Nth order I-dimensional real tensors is denoted by $T_{N,I}$. We first introduce two denotations [39] as follows. For any $\mathcal{A}\in T_{N,I}$ and $\mathbf{x}\in\mathbb{R}^I$, $\mathcal{A}\mathbf{x}^{N-1}$ is an I-dimensional real vector whose ith component is

$$(\mathcal{A}\mathbf{x}^{N-1})_i=\sum_{i_2,\cdots,i_N=1}^{I}a_{ii_2\cdots i_N}x_{i_2}\cdots x_{i_N},$$

and $\mathcal{A}\mathbf{x}^N$ is a scalar given by

$$\mathcal{A}\mathbf{x}^N=\sum_{i_1,i_2,\cdots,i_N=1}^{I}a_{i_1i_2\cdots i_N}x_{i_1}x_{i_2}\cdots x_{i_N}.$$

The Frobenius norm of $\mathcal{A}$ is given by

$$\|\mathcal{A}\|_F=\sqrt{\langle\mathcal{A},\mathcal{A}\rangle}$$

and the scalar product $\langle\mathcal{A},\mathcal{B}\rangle$ is defined by [8,32]

$$\langle\mathcal{A},\mathcal{B}\rangle=\sum_{i_1,i_2,\cdots,i_N=1}^{I}b_{i_1i_2\cdots i_N}a_{i_1i_2\cdots i_N}.$$

For the given $i_n\in\{1,\cdots,I\}$ and $n\in\{1,\cdots,N\}$, the *i_nth mode-n slice* [4,5,12,14, 31] of a tensor $\mathcal{A}\in T_{N,I}$ is defined as a tensor in $T_{N-1,I}$, by fixing the mode-n index of $\mathcal{A}$ to i_n: $\mathcal{A}(:,\cdots,:,i_n,:,\cdots,:)$.

2.1. Formulation of the GOTCP

For a given tensor $\mathcal{A} \in T_{N,I}$ and a vector $\mathbf{q} \in \mathbb{R}^I$, the TCP, denoted by $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$, is to find a vector $\mathbf{x} \in \mathbb{R}^I_+$ such that

$$\mathcal{A}\mathbf{x}^{N-1} + \mathbf{q} \in \mathbb{R}^I_+, \qquad \mathcal{A}\mathbf{x}^N + \mathbf{x}^\top \mathbf{q} = 0.$$

$\mathrm{TCP}(\mathcal{A}, \mathbf{q})$ can be formulated equivalently as the system of nonlinear equations: to find a vector $\mathbf{x} \in \mathbb{R}^I$ such that

$$\mathbf{x} \wedge (\mathcal{A}\mathbf{x}^{N-1} + \mathbf{q}) = \mathbf{0}_I.$$

As seen in [44], a vector $\mathbf{x} \in \mathbb{R}^I_+$ with $\mathcal{A}\mathbf{x}^{N-1} + \mathbf{q} \geq \mathbf{0}_I$ $(> \mathbf{0}_I)$ is called a feasible (respectively, strictly feasible) vector for $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$. If there is such a vector, we say that $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$ is feasible (respectively, strictly feasible). Let

$$\begin{aligned} &\mathbb{F}(\mathcal{A}) = \{\mathbf{q} : \mathcal{A}\mathbf{x}^{N-1} + \mathbf{q} \geq \mathbf{0}_I \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^I_+\}, \\ &\mathbb{K}(\mathcal{A}) = \{\mathbf{q} : \mathrm{SOL}(\mathcal{A}, \mathbf{q}) \neq \emptyset\}, \end{aligned}$$

where $\mathrm{SOL}(\mathcal{A}, \mathbf{q})$ denotes the solution set of $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$.

Let $K \geq 1$ be any positive number. Given K matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_K \in \mathbb{R}^{I \times I}$ and K vectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_K \in \mathbb{R}^I$, the GOLCP [22] is to find a vector $\mathbf{x} \in \mathbb{R}^I$ such that

$$\mathbf{x} \wedge (\mathbf{A}_1\mathbf{x} + \mathbf{q}_1) \wedge (\mathbf{A}_2\mathbf{x} + \mathbf{q}_2) \wedge \cdots \wedge (\mathbf{A}_K\mathbf{x} + \mathbf{q}_K) = \mathbf{0}_I. \tag{2.1}$$

For given K tensors $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_K \in T_{N,I}$ and K vectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_K \in \mathbb{R}^I$, we denote

$$\widehat{\mathcal{A}} := \{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_K\}, \qquad \widehat{\mathbf{q}} := \{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_K\}. \tag{2.2}$$

For $\widehat{\mathcal{A}}$ and $\widehat{\mathbf{q}}$ given in (2.2), the GOTCP, denoted by $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$, is to find a vector $\mathbf{x} \in \mathbb{R}^I_+$ such that

$$\mathbf{x} \wedge (\mathcal{A}_1\mathbf{x}^{N-1} + \mathbf{q}_1) \wedge (\mathcal{A}_2\mathbf{x}^{N-1} + \mathbf{q}_2) \wedge \cdots \wedge (\mathcal{A}_K\mathbf{x}^{N-1} + \mathbf{q}_K) = \mathbf{0}_I.$$

For brevity, we introduce three denotations

$$\begin{aligned} &\mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}}) := \mathbf{x} \wedge (\mathcal{A}_1\mathbf{x}^{N-1} + \mathbf{q}_1) \wedge (\mathcal{A}_2\mathbf{x}^{N-1} + \mathbf{q}_2) \wedge \cdots \wedge (\mathcal{A}_K\mathbf{x}^{N-1} + \mathbf{q}_K), \\ &\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} := \mathbf{x} \wedge (\mathcal{A}_1\mathbf{x}^{N-1}) \wedge (\mathcal{A}_2\mathbf{x}^{N-1}) \wedge \cdots \wedge (\mathcal{A}_K\mathbf{x}^{N-1}), \\ &\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1} := \mathbf{x} \vee (\mathcal{A}_1\mathbf{x}^{N-1}) \vee (\mathcal{A}_2\mathbf{x}^{N-1}) \vee \cdots \vee (\mathcal{A}_K\mathbf{x}^{N-1}). \end{aligned}$$

We use the notation $\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}} \geq \mathbf{0}_I$ to mean $\mathcal{A}_k\mathbf{x}^{N-1} + \mathbf{q}_k \geq \mathbf{0}_I$ for $k = 1, \cdots, K$. A vector $\mathbf{x} \in \mathbb{R}^I_+$ with $\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}} \geq \mathbf{0}_I$ $(> \mathbf{0}_I)$ is called a feasible (respectively, strictly feasible) vector for $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$. If there is such a vector, we say that $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ is feasible (respectively, strictly feasible). Let

$$\begin{aligned} &\mathbb{F}(\widehat{\mathcal{A}}) = \{\widehat{\mathbf{q}} : \widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}} \geq \mathbf{0}_I \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^I_+\}, \\ &\mathbb{K}(\widehat{\mathcal{A}}) = \{\widehat{\mathbf{q}} : \mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) \neq \emptyset\}, \end{aligned}$$

where $\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ denotes the solution set of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. It is obvious that $\mathbb{K}(\widehat{\mathcal{A}}) \subseteq \mathbb{F}(\widehat{\mathcal{A}})$.

Borrowing the terminology from the LCP theory, we say that these sets are, respectively (when $\widehat{\mathcal{A}}$ is fixed), the set of all "feasible" $\widehat{\mathbf{q}}$'s and "solvable" $\widehat{\mathbf{q}}$'s. Note that

$$\mathrm{int}\mathbb{F}(\widehat{\mathcal{A}}) = \{\widehat{\mathbf{q}} : \widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}} > \mathbf{0}_I \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^I_+\}.$$

2.2. The VTCP

In this subsection, we introduce the vertical TCP (VTCP), which is the generalization of the VLCP [10,11,45] and show that it is equivalent to the GOTCP.

Consider an N-order tensor $\mathcal{B} \in \mathbb{R}^{J\times I\times\cdots\times I}$ with $J \geq I$ and $\mathbf{p} \in \mathbb{R}^J$. Let $J = J_1 + J_2 + \cdots + J_I$ and $J_0 = 0$. Suppose that $\mathcal{B}_i \in \mathbb{R}^{J_i\times I\times\cdots\times I}$ and $\mathbf{p}_i \in \mathbb{R}^{J_i}$ satisfy

$$\begin{aligned}
&\mathcal{B}_i = \mathcal{B}(J_1 + \cdots + J_{i-1} + 1 : J_1 + \cdots + J_i, :, \cdots, :) \in \mathbb{R}^{J_i\times I\times\cdots\times I},\\
&\mathbf{p}_i = \mathbf{p}(J_1 + \cdots + J_{i-1} + 1 : J_1 + \cdots + J_i) \in \mathbb{R}^{I_i},
\end{aligned}$$

for $i = 1, \cdots, I$. Then $\mathrm{VTCP}(\mathcal{B},\mathbf{p})$ is to find a vector $\mathbf{x} \in \mathbb{R}^I_+$ such that

$$\mathcal{B}\mathbf{x}^{N-1} + \mathbf{p} \in \mathbb{R}^J_+, \qquad x_i \prod_{j=1}^{J_i} \left(\mathcal{B}_i\mathbf{x}^{N-1} + \mathbf{p}_i\right)_j = 0,$$

for $i = 1, \cdots, I$, where the jth component of $\mathcal{B}\mathbf{x}^{N-1}$ is defined by

$$(\mathcal{B}\mathbf{x}^{N-1})_j = \sum_{i_2,\cdots,i_N=1}^{I} b_{ji_2\cdots i_N} x_{i_2}\cdots x_{i_N}, \qquad j = 1, \cdots, J.$$

We show that this problem can be formulated as a GOTCP. Let $K = \max\{J_i : i = 1, \cdots, I\}$. Let $\mathcal{B}_i^j := \mathcal{B}_i(j, :, \cdots, :)$ is the jth mode-1 slice of $\mathcal{B}_i$ where $j = 1, \cdots, J_i$ and $i = 1, \cdots, I$. We define tensors $\widehat{\mathcal{B}}_1, \widehat{\mathcal{B}}_2, \cdots, \widehat{\mathcal{B}}_I \in \mathbb{R}^{K\times I\times\cdots\times I}$ of order N in the following way. For each i, the jth mode-1 slice of $\widehat{\mathcal{B}}_i$ is $\mathcal{B}_i^j$ if $j \leq J_i$ and $\mathcal{B}_i^1$ if $j > J_i$. Correspondingly, we define vectors $\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2, \cdots, \widehat{\mathbf{p}}_I \in \mathbb{R}^K$ in the following way. For each i, the jth element of $\widehat{\mathbf{p}}_i$ is the jth element of $\mathbf{p}_i$ if $j \leq J_i$ and the first element of $\mathbf{p}_i$ if $j > J_i$.

It is clear that $\mathrm{VTCP}(\widehat{\mathcal{B}},\widehat{\mathbf{p}})$ is equivalent to the $\mathrm{VTCP}(\mathcal{B},\mathbf{p})$. Let the jth mode-1 slice of $\mathcal{A}_i \in T_{N,I}$ be the ith mode-1 slice of $\widehat{\mathcal{B}}_j$ with $i = 1, \cdots, K$ and $j = 1, \cdots, I$. For example, $\mathcal{A}_1$ is formed by considering $\widehat{\mathcal{B}}_j^1$ with $j = 1, \cdots, I$. Similarly, let $\mathbf{q}_i$ be the vector of size $I \times 1$ whose jth component is the ith component in the vector $\widehat{\mathbf{p}}_j$. Hence we can verify that the $\mathrm{VTCP}(\widehat{\mathcal{B}},\widehat{\mathbf{p}})$ is equivalent to the $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$.

We shall show that every GOTCP can be formulated as a VTCP. For a given $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$, let the ith mode-1 slice of $\mathcal{B}_j \in \mathbb{R}^{K\times I\times\cdots\times I}$ be the jth mode-1 slice of $\mathcal{A}_i$ with $i = 1, \cdots, K$ and $j = 1, \cdots, I$. For example, $\mathcal{B}_1$ is formed by considering $\widehat{\mathcal{A}}_i^1$ with $i = 1, \cdots, K$. Correspondingly, we define $\mathbf{p}_i \in \mathbb{R}^K$ with $i = 1, \cdots, I$. This construction leads to the pair $(\mathcal{B},\mathbf{p})$ and the corresponding VTCP is easily seen to be equivalent to $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$.

2.3. Basic definitions

Before considering the solution set of any given $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$, we introduce the definitions of some special tensors, which are used to study the solution set of a given $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$.

Definition 2.1. *Suppose that $\mathcal{A} \in T_{N,I}$ and $\mathbf{q} \in \mathbb{R}^I$. We say that $\mathcal{A}$ is*

(a) an $\mathbb{R}$ tensor if the following system is inconsistent

$$\begin{cases} \mathbf{0}_I \neq \mathbf{x} \in \mathbb{R}^I_+, & t \geq 0, \\ (\mathcal{A}\mathbf{x}^{N-1})_i + t = 0, & \text{if } x_i > 0, \\ (\mathcal{A}\mathbf{x}^{N-1})_j + t \geq 0, & \text{if } x_j = 0, \end{cases} \tag{2.3}$$

(b) an $\mathbb{R}_0$ tensor if the system (2.3) *is inconsistent for $t = 0$,*

(c) a semi-positive tensor if any $\mathbf{q} > \mathbf{0}_I$, the solution set of $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$ has a unique element $\mathbf{0}_I$,

(d) a strictly semi-positive tensor if for each nonzero $\mathbf{x} \in \mathbb{R}^I_+$, there exists an index $k \in \{1, \cdots, I\}$ such that

$$x_k > 0 \quad \textit{and} \quad (\mathcal{A}\mathbf{x}^{N-1})_k > 0,$$

(e) a $\mathbb{P}$ tensor if for each nonzero $\mathbf{x} \in \mathbb{R}^I$, there exists an index $k \in \{1, \cdots, I\}$ such that

$$x_k \neq 0 \quad \textit{and} \quad x_k(\mathcal{A}\mathbf{x}^{N-1})_k > 0,$$

(f) a $\mathbb{P}_0$ tensor if for each nonzero $\mathbf{x} \in \mathbb{R}^I$, there exists an index $k \in \{1, \cdots, I\}$ such that

$$x_k \neq 0 \quad \textit{and} \quad x_k(\mathcal{A}\mathbf{x}^{N-1})_k \geq 0,$$

(g) a $\mathbb{Q}$ tensor if for any $\mathbf{q} \in \mathbb{R}^I$, the $\mathrm{TCP}(\mathcal{A}, \mathbf{q})$ is solvable,

(h) a $\mathbb{Q}_0$ tensor if $\mathbb{F}(\mathcal{A}) \subseteq \mathbb{K}(\mathcal{A})$.

Remark 2.1. If for each nonzero $\mathbf{x} \in \mathbb{R}^I_+$, there exists an index $k \in \{1, \cdots, I\}$ such that

$$x_k > 0 \quad \text{and} \quad (\mathcal{A}\mathbf{x}^{N-1})_k \geq 0,$$

then $\mathcal{A} \in T_{N,I}$ is also a semi-posivite tensor.

In this paper, we sometimes call a semi-positive tensor as an $\mathbb{E}_0$ tensor and a strictly semi-positive tensor as an $\mathbb{E}$ tensor.

3. Existence results

Motivated by the LCP theory and the TCP theory, we introduce the following definitions.

Definition 3.1. *We say that $\widehat{\mathcal{A}}$ is of*

1. type $\mathbb{R}_0$ *if* $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} = \mathbf{0}_I \Rightarrow \mathbf{x} = \mathbf{0}_I$,
2. type $\mathbb{G}$ *if for any* $\mathbf{d}_k > \mathbf{0}_I$ *with* $k = 1, \cdots, K$, $\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{d}}) = \{\mathbf{0}_I\}$,
3. type $\mathbb{R}$ *if it is of type* $\mathbb{R}_0$ *and type* $\mathbb{G}$,
4. type $\mathbb{E}$ *if* $\mathbf{x} \in \mathbb{R}^I_+$, $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} \leq \mathbf{0}_I \Rightarrow \mathbf{x} = \mathbf{0}_I$,
5. type $\mathbb{P}$ *if* $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} \leq \mathbf{0}_I \leq \mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1} \Rightarrow \mathbf{x} = \mathbf{0}_I$,
6. type $\mathbb{Q}$ *if for every* $\mathbf{q}_k \in \mathbb{R}^I$ *with* $k = 1, \cdots, K$, $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ *has a solution,*
7. type $\mathbb{Q}_0$ *if* $\mathbb{F}(\widehat{\mathcal{A}}) \subseteq \mathbb{K}(\widehat{\mathcal{A}})$.

Remark 3.1. We use the same symbol to denote the class of $\widehat{\mathcal{A}}$'s of a given type. For example, type $\mathbb{Q}$ denotes the set of all $\widehat{\mathcal{A}}$'s are of type $\mathbb{Q}$. When $K = 1$, items 1, 2, 3, 6 and 7 in Definition 3.1 reduce, respectively, to those of $\mathbb{R}_0$ tensors [43], semi-positive tensors, $\mathbb{R}$ tensors [43], $\mathbb{Q}$ tensors [43] and $\mathbb{Q}_0$ tensors; items 4 and 5 in Definition 3.1 reduce to these of strictly semi-positive tensors [43] and $\mathbb{P}$ tensors [41], respectively.

To illustrate the above definitions, we give two examples dealing with $\mathbb{R}^2$.

Example 3.1. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\}$ where

$$\mathcal{A}_1(:,:,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_1(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\mathcal{A}_2(:,:,1) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2(:,:,2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

By some tedious computation, for any $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, we have

$$\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^2 = (\min(x_1, x_2^2, -x_1^2), \min(x_2, x_1^2 + x_2^2, x_2^2))^\top.$$

It is easy to verify that $\widehat{\mathcal{A}}$ is of type $\mathbb{R}_0$.

Suppose that $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$ satisfies $x_1 > x_2 = 0$, it is easy to verify that all entries of $\mathbf{x} \wedge \mathcal{A}_1\mathbf{x}^2$ are zero. Hence $\mathcal{A}_1$ is not an $\mathbb{R}_0$ tensor.

Example 3.2. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\}$ where

$$\mathcal{A}_1(:,:,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_1(:,:,2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\mathcal{A}_2(:,:,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_2(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By some tedious computation, for any $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2_+$, we have

$$\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^2 = (\min(x_1, x_1^2 + x_2^2, x_1^2 + x_2^2), \min(x_2, x_2^2, x_1^2 + x_2^2))^\top.$$

It is obvious to check that $\widehat{\mathcal{A}}$ is of type $\mathbb{E}$ and hence of type $\mathbb{R}$ and $\mathbb{R}_0$ by Proposition 3.1.

Proposition 3.1. *For Definition 3.1, we have* $\text{type } \mathbb{P} \subseteq \text{type } \mathbb{E} \subseteq \text{type } \mathbb{R} \subseteq \text{type } \mathbb{R}_0$.

Proof. If $\mathbf{x} \in \mathbb{R}^I_+$, then for any vector $\mathbf{y} \in \mathbb{R}^I$, we have $\mathbf{x} \vee \mathbf{y} \in \mathbb{R}^I_+$. This observation shows that $\text{type } \mathbb{P} \subseteq \text{type } \mathbb{E}$. Since $\mathbf{x} \in \mathbb{R}^I_+$ and $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} \in \mathbb{R}^I_-$, then for any $\mathbf{d} > \mathbf{0}_I$, there exists $t \geq 0$ such that $\mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + t\widehat{\mathbf{d}}) \in \mathbb{R}^I_+$. Hence we have the inclusion $\text{type } \mathbb{E} \subseteq \text{type } \mathbb{R}$. The last inclusion $\text{type } \mathbb{R} \subseteq \text{type } \mathbb{R}_0$ follows from Definition 3.1. $\square$

In this paper, we use degree-theoretical ideas. All necessary results concerning degree theory are given in [19, 20, 34, 38]. Here is a short review. Suppose that Ω is a bounded open set in $\mathbb{R}^I$ and $g : \overline{\Omega} \to \mathbb{R}^I$ is continuous and $\mathbf{p} \notin g(\partial\Omega)$, where $\overline{\Omega}$ and $\partial\Omega$ denote, respectively, the closure and boundary of Ω. Then the degree of g over Ω with respect to $\mathbf{p}$ is defined; it is an integer and will be denoted by $\deg(g, \Omega, \mathbf{p})$. When this degree is nonzero, the equation $g(\mathbf{x}) = \mathbf{p}$ has a solution in Ω. Suppose that $g(\mathbf{x}) = \mathbf{p}$ has a unique solution, say, $\mathbf{x}_* \in \Omega$. Then $\deg(g, \Omega', \mathbf{p})$ is constant over all bounded open sets Ω' containing $\mathbf{x}_*$ and contained in Ω.

Let $\mathcal{A} \in T_{N,I}$ be an $\mathbb{R}_0$ tensor. Then with $F(\mathbf{x}) = \mathcal{A}\mathbf{x}^{N-1}$ and $\widehat{F}(\mathbf{x}) = \mathbf{x} \wedge \mathcal{A}\mathbf{x}^{N-1}$, we have $\widehat{F}(\mathbf{x}) = \mathbf{0}_I \Rightarrow \mathbf{x} = \mathbf{0}_I$; hence the TCP-degree of $\mathcal{A}$ is defined. Corresponding to $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$, we define two functions $F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})}$ and $F_{\widehat{\mathcal{A}}}$ by

$$F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})} = \mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}}), \qquad F_{\widehat{\mathcal{A}}} = \mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1}.$$

Note that $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ can be formulated as an equation $F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})} = \mathbf{0}_I$. Suppose that $\widehat{\mathcal{A}}$ is of $\text{type } \mathbb{R}_0$ so that the zero vector is the only solution of $F_{\widehat{\mathcal{A}}} = \mathbf{0}_I$. Let $\Omega \in \mathbb{R}^I$ be any bounded open set containing the zero vector. Then the integer $\deg(F_{\widehat{\mathcal{A}}}, \Omega, \mathbf{0}_I)$ (the degree of $F_{\widehat{\mathcal{A}}}$ over Ω relative to zero) is defined. Furthermore, this is independent of the bounded open set Ω. We call this integer the GOTCP-degree of $\widehat{\mathcal{A}}$ and denote it by GOTCP-deg $\widehat{\mathcal{A}}$.

Before stating our first existence result, we present a necessary and sufficient condition for boundedness of the solution set of $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ for all $\widehat{\mathbf{q}}$. We omit the proof as it is identical to the one in the classical case [44, Theorem 3.3]. Note that the solution set $\text{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ may be empty for some particular $\widehat{\mathbf{q}}$.

Proposition 3.2. $\widehat{\mathcal{A}}$ *is of* $\text{type } \mathbb{R}_0$ *if and only if for any* $\mathbf{q}_k \in \mathbb{R}^I$ *with* $k = 1, \cdots, K$, $\text{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ *is bounded.*

Remark 3.2. When $\widehat{\mathcal{A}}$ is of $\text{type } \mathbb{R}_0$, it is easy to see that the solution sets of the problem $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ are uniformly bounded as $\mathbf{q}_k$ varies over a bounded set in $\mathbb{R}^I$ with $k = 1, \cdots, K$.

Here is our first existence result.

Theorem 3.1. *Suppose that $\widehat{\mathcal{A}}$ is of* $\text{type } \mathbb{R}_0$ *and* GOTCP-degree *of $\widehat{\mathcal{A}}$ is nonzero. Then $\widehat{\mathcal{A}}$ is of* $\text{type } \mathbb{Q}$.

Proof. Let $\widehat{\mathbf{q}}$ be arbitrary. For any t in the interval $[0,1]$, we consider

$$F_{(\widehat{\mathcal{A}},t\widehat{\mathbf{q}})} = \mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + t\widehat{\mathbf{q}}).$$

In view of the remark above, the set $\{\mathbf{x} : F_{(\widehat{\mathcal{A}},t\widehat{\mathbf{q}})} = \mathbf{0}_I \text{ for some } t \in [0,1]\}$ is bounded in $\mathbb{R}^I$ and hence contained in some bounded open set, say, Ω. Note that $F_{(\widehat{\mathcal{A}},t\widehat{\mathbf{q}})}$ is a homotopy connecting the mappings $F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})}$ and $F_{\widehat{\mathcal{A}}}$. Using the homotopy invariance property of the degree [34, Theorem 2.1.2], we see that $\deg(F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})}, \Omega, \mathbf{0}_I) = \deg(F_{\widehat{\mathcal{A}}}, \Omega, \mathbf{0}_I) =$ GOTCP-deg $\widehat{\mathcal{A}}$. Since the last integer is assumed to be nonzero, by the well-known property of the degree [34, Theorem 2.1.1], the equation $F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})} = \mathbf{0}_I$ has a solution in Ω, i.e., $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ has a solution. Since $\widehat{\mathbf{q}}$ is arbitrary, the result follows. $\square$

Theorem 3.2. *Suppose that $\widehat{\mathcal{A}}$ is of* $\text{type } \mathbb{R}$. *Then its* GOTCP-degree *is one and hence it is of* $\text{type } \mathbb{Q}$. *In particular, If $\widehat{\mathcal{A}}$ is of* $\text{type } \mathbb{P}$ *or of* $\text{type } \mathbb{E}$*, then it is of* $\text{type } \mathbb{Q}$.

Proof. For any t in the interval $[0,1]$, we consider

$$F_{(\widehat{\mathcal{A}},t\widehat{\mathbf{d}})} = \mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + t\widehat{\mathbf{d}}),$$

where $\widehat{\mathbf{d}}$ is as in the definition of the class $\text{type } \mathbb{R}$. As in the proof of the previous theorem, we have $\deg(F_{(\widehat{\mathcal{A}},\widehat{\mathbf{d}})}, \Omega, \mathbf{0}_I) = \deg(F_{\widehat{\mathcal{A}}}, \Omega, \mathbf{0}_I) = \text{GOTCP} - \deg\widehat{\mathcal{A}}$, where Ω is any bounded open set containing $\mathbf{0}_I$. Since $\mathbf{d}_k > \mathbf{0}_I$ for $k = 1, \cdots, K$, we have for all $\mathbf{x}$ near $\mathbf{0}_I$, the entries of $\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{d}}$ are positive and $F_{(\widehat{\mathcal{A}},\widehat{\mathbf{d}})}(\mathbf{x}) = \mathbf{x}$. This implies that $\deg(F_{\widehat{\mathcal{A}}}, \Omega, \mathbf{0}_I) = 1$. Now Theorem 3.1 completes the proof. $\square$

4. Uniqueness results

The following theorems characterize uniqueness of the solution of the GOTCP.

Theorem 4.1. *The following statements are equivalent:*

(a) $\widehat{\mathcal{A}}$ *is of* $\text{type } \mathbb{E}$*,*

(b) For every $\mathbf{q}_k \in \mathbb{R}^I_+$ *with* $k = 1, \cdots, K$*, the zero vector is the only solution of* $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$.

Proof. Suppose (a) holds and $\mathbf{x}$ is a solution of $\text{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$, where $\mathbf{q}_k \in \mathbb{R}^I_+$ with $k = 1, \cdots, K$. It follows that $\mathbf{x} \in \mathbb{R}^I_+$ and $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} \leq \mathbf{x} \wedge (\widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}}) = \mathbf{0}_I$. We see that $\mathbf{x} = \mathbf{0}_I$. On the other hand, suppose (b) holds and let $\mathbf{z}$ be a solution of the system

$$\mathbf{z} \in \mathbb{R}^I_+, \qquad \mathbf{z} \wedge \widehat{\mathcal{A}}\mathbf{z}^{N-1} \in \mathbb{R}^I_-.$$

Then $\mathbf{z}$ is a solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$, where $\mathbf{q}$ is defined by $\mathbf{q}_k=(\mathcal{A}_k\mathbf{z}^{N-1})^+-\mathcal{A}_k\mathbf{z}^{N-1}$ with $k=1,\cdots,K$, where the ith element of $(\mathcal{A}_k\mathbf{z}^{N-1})^+$ is

$$\max\left\{\sum_{i_2,\cdots,i_N=1}^{I} a_{ii_2\cdots i_N}x_{i_2}\cdots x_{i_N},0\right\}.$$

Since $\mathbf{q}_k\in\mathbb{R}^I_+$, we get $\mathbf{z}=\mathbf{0}_I$. $\square$

Remark 4.1. When $K=1$, the above theorem reduces to Theorems 3.2 and 3.4 in [42]: $\mathcal{A}_1$ is strictly semi-positive if and only if for every $\mathbf{q}_1\in\mathbb{R}^I_+$, the zero vector is the only solution of $\mathrm{TCP}(\mathcal{A}_1,\mathbf{q}_1)$.

When $K=1$, statement (b) in Theorem 4.1 describes strictly semi-positive tensors. Analogous to the classes of semi-positive tensors and $\mathbb{P}_0$ tensors, we introduce two classes of $\widehat{\mathcal{A}}$. Let

$$\widehat{\mathcal{A}}+\varepsilon\widehat{\mathcal{I}}:=\{\mathcal{A}_1+\varepsilon\mathcal{I},\mathcal{A}_2+\varepsilon\mathcal{I},\cdots,\mathcal{A}_K+\varepsilon\mathcal{I}\},$$

where $\mathcal{I}\in T_{N,I}$ is the identity tensor[†] and $\varepsilon\in\mathbb{R}$. We say that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{E}_0$ ($\mathrm{type}\ \mathbb{P}_0$) if $\widehat{\mathcal{A}}+\varepsilon\widehat{\mathcal{I}}$ is of $\mathrm{type}\ \mathbb{E}$ (respectively, $\mathrm{type}\ \mathbb{P}$) for any $\varepsilon>0$.

Theorem 4.2. *Consider the statements:*

(a) $\widehat{\mathcal{A}}$ *is of* $\mathrm{type}\ \mathbb{E}_0$*,*

(b) For every $\mathbf{q}_k>\mathbf{0}_I$ *with* $k=1,\cdots,K$*, the zero vector is the only solution of* $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$.

Then $(a)\Rightarrow(b)$.

Proof. Suppose that $\mathbf{q}_k>\mathbf{0}_I$ with $k=1,\cdots,K$ and $\mathbf{x}\wedge(\widehat{\mathcal{A}}\mathbf{x}^{N-1}+\widehat{\mathbf{q}})=\mathbf{0}_I$. Let $\widehat{\mathbf{x}}=\{\mathbf{x}^{[N-1]},\mathbf{x}^{[N-1]},\cdots,\mathbf{x}^{[N-1]}\}$ and $\varepsilon>0$ be small so that $\mathbf{q}_k-\varepsilon\mathbf{x}^{[N-1]}>\mathbf{0}_I$. Then $\mathbf{x}\in\mathbb{R}^I_+$ and

$$\mathbf{x}\wedge(\widehat{\mathcal{A}}\mathbf{x}^{N-1}+\varepsilon\widehat{\mathbf{x}})\le\mathbf{x}\wedge(\widehat{\mathcal{A}}\mathbf{x}^{N-1}+\widehat{\mathbf{q}})=\mathbf{0}_I,$$

which implies that $\mathbf{x}=\mathbf{0}_I$ in view of (a). $\square$

Remark 4.2. In the following, we show that the reverse implication $(\mathrm{b})\Rightarrow(\mathrm{a})$ holds. The above theorem shows that $\mathrm{type}\ \mathbb{E}_0\subseteq\mathrm{type}\ \mathbb{G}$ and therefore $\mathrm{type}\ \mathbb{E}_0\cap\mathrm{type}\ \mathbb{R}_0\subseteq\mathrm{type}\ \mathbb{R}$.

[†]A tensor $\mathcal{A}\in T_{N,I}$ is called the identity tensor if its entries satisfy

$$a_{i_1i_2\cdots i_N}=\begin{cases}1, & \text{if } i_1=i_2=\cdots=i_N,\\ 0, & \text{otherwise.}\end{cases}$$

Let $(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ be given by (2.1), where $\mathcal{A}_k \in T_{N,I}$ and $\mathbf{q}_k \in \mathbb{R}^I$ with $k = 1, \cdots, K$. We say that the pair $(\mathcal{A}, \mathbf{p}) \in T_{N,I} \times \mathbb{R}^I$ is a representative of $(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ if for each $j = 1, \cdots, I$, the jth mode-1 slice of $\mathcal{A}$ is the jth mode-1 slice of some $\mathcal{A}_i$ and the jth component of $\mathbf{p}$ is the jth component of the corresponding $\mathbf{q}_i$. In other words, $\mathcal{A}^j \in \{\mathcal{A}_1^j, \cdots, \mathcal{A}_K^j\}$, where the subscript refers to the corresponding mode-1 slice and so on. We say that $\mathcal{A}$ is a representative tensor of $\widehat{\mathcal{A}}$ and $\mathbf{p}$ is a representative vector of $\widehat{\mathbf{q}}$.

It is immediate that $\mathbf{x} \in \mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ if and only if $\mathbf{x}$ is a feasible vector for $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ and $\mathbf{x} \in \mathrm{SOL}(\mathcal{A}, \mathbf{p})$ for some representative of $(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$. We record this formally in the following proposition.

Proposition 4.1. *It holds that*

$$\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) = \mathbb{N}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) \cap (\cup \mathrm{SOL}(\mathcal{A}, \mathbf{p})),$$

where

$$\mathbb{N}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) = \{\mathbf{x} \in \mathbb{R}_+^I : \widehat{\mathcal{A}}\mathbf{x}^{N-1} + \widehat{\mathbf{q}} \in \mathbb{R}_+^I\}$$

and the union is over all representatives $(\mathcal{A}, \mathbf{p}) \in T_{N,I} \times \mathbb{R}^I$ of $(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$.

We say that $\widehat{\mathcal{A}}$ has the $\mathbb{T}$ property if every representative tensor of $\widehat{\mathcal{A}}$ is a $\mathbb{T}$ tensor, where $\mathbb{T}$ denotes a class of tensors. For example, $\widehat{\mathcal{A}}$ has the $\mathbb{P}$ property if every representative tensor of $\widehat{\mathcal{A}}$ is a $\mathbb{P}$ tensor. The standard classes in the TCP theory are the classes of copositive tensors, $\mathbb{R}_0$ tensors, $\mathbb{R}$ tensors and so on.

Proposition 4.2. *If $\widehat{\mathcal{A}}$ has the $\mathbb{R}_0$ property, then it is of* $\mathrm{type}\ \mathbb{R}_0$.

Proof. Suppose that $\widehat{\mathcal{A}}$ has the $\mathbb{R}_0$ property and let $\mathbf{x}$ be a vector such that $\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1} = \mathbf{0}_I$. This leads to

$$x_i \wedge (\mathcal{A}_1\mathbf{x}^{N-1})_i \wedge \cdots \wedge (\mathcal{A}_K\mathbf{x}^{N-1})_i = 0 \qquad \text{for} \quad i = 1, \cdots, I.$$

Let $l(i)$ be an index such that $(\mathcal{A}_{l(i)}\mathbf{x}^{N-1})_i = (\mathcal{A}_1\mathbf{x}^{N-1})_i \wedge \cdots \wedge (\mathcal{A}_K\mathbf{x}^{N-1})_i$. Let $\mathcal{A}$ be a representative tensor of $\widehat{\mathcal{A}}$ whose ith mode-1 slice is $\mathcal{A}_{l(i)}^i$. Clearly, $\mathbf{x} \wedge \mathcal{A}\mathbf{x}^{N-1} = \mathbf{0}_I$. By assumption, $\mathcal{A}$ is an $\mathbb{R}_0$ tensor and so $x_i = 0$ for all $i = 1, \cdots, I$. Thus, $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{R}_0$. $\square$

Remark 4.3. Example 3.1 shows that the converse statement in the above proposition may not hold. There does not seem to be any connection between $\mathrm{type}\ \mathbb{R}$ and the $\mathbb{R}_0$ property. However, suppose that there is a positive vector $\mathbf{e} \in \mathbb{R}^I$ such that for every representative tensor $\mathcal{A}$ of $\widehat{\mathcal{A}}$ and for every $t \geq 0$, $\mathrm{SOL}(\mathcal{A}, t\mathbf{e}) = \{\mathbf{0}_I\}$. Then it is easily verified that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{R}$. In particular, if $\widehat{\mathcal{A}}$ has the $\mathbb{E}_0$ property and the $\mathbb{R}_0$ property, then it is of $\mathrm{type}\ \mathbb{R}$.

Before moving on to the uniqueness aspect, we present two important examples.

Example 4.1. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\}$ where

$$\mathcal{A}_1(:,:,1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{A}_1(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathcal{A}_2(:,:,1) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \mathcal{A}_2(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let t be any nonnegative real number and $\widehat{\mathbf{e}} = \{\mathbf{e}, \mathbf{e}\}$ with $\mathbf{e} = (1,1)^\top$. By some tedious computation, for any $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, we have

$$\min\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1^2 - x_2^2 + t \\ x_1^2 - x_2^2 + t \end{pmatrix}, \begin{pmatrix} x_1^2 - 2x_2^2 + t \\ x_1^2 + x_2^2 + t \end{pmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is obvious that if $x_1 = 0$, then $x_2 = 0$; if $x_2 = 0$, then $x_1 = 0$. We now consider the case of $x_1^2 - x_2^2 + t = 0$. Then we have

$$x_1^2 - 2x_2^2 + t = x_2^2 - t - 2x_2^2 + t = -x_2^2 \le 0.$$

Hence $x_1 = 0$ and $x_2 = 0$. We finally consider the case of $x_1^2 - 2x_2^2 + t = 0$. Then

$$x_1^2 - x_2^2 + t = 2x_2^2 - t - x_2^2 + t = x_2^2 \ge 0.$$

Hence $x_1 = 0$ and $x_2 = 0$. That is, $\mathrm{SOL}(\widehat{\mathcal{A}}, t\widehat{\mathbf{e}}) = \{\mathbf{0}_I\}$. While verifying this, we have used the fact that the tensor

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \mathcal{A}(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is an $\mathbb{R}_0$ tensor. This means that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{R}$ and hence is of $\mathrm{type}\ \mathbb{Q}$ by Theorem 3.2. However, $\mathcal{A}_1$ is neither an $\mathbb{R}_0$ tensor nor a $\mathbb{Q}$ tensor, i.e., $\widehat{\mathcal{A}}$ has neither the $\mathbb{R}$ property nor the $\mathbb{Q}$ property.

Example 4.2. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\}$ and $\widehat{\mathbf{q}} = \{\mathbf{q}_1, \mathbf{q}_2\}$, where

$$\mathcal{A}_1(:,:,1) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \mathcal{A}_1(:,:,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathcal{A}_2(:,:,1) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_2(:,:,2) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\mathbf{q}_1 = (6,3)^\top, \qquad \mathbf{q}_2 = (-1,3)^\top.$$

It can be verified that $\widehat{\mathcal{A}}$ has the $\mathbb{Q}$ property but that $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ has no solution, i.e., $\widehat{\mathcal{A}}$ is not of $\mathrm{type}\ \mathbb{Q}$. Note also that $\widehat{\mathcal{A}}$ has the $\mathbb{R}_0$ property.

Proposition 4.3. *$\widehat{\mathcal{A}}$ has the $\mathbb{P}$ property if and only if it is of* $\mathrm{type}\ \mathbb{P}$.

Proof. Suppose that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{P}$ and let $\mathcal{A}$ be a representative tensor of $\widehat{\mathcal{A}}$. To show that $\mathcal{A}$ is a $\mathbb{P}$ tensor, we start with a vector $\mathbf{x} \in \mathbb{R}^I$ such that $x_i(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ for $i = 1, \cdots, I$ and show that $\mathbf{x} = \mathbf{0}_I$. If $x_i > 0$, then $(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ and hence

$$(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0 \leq (\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i.$$

If $x_i < 0$, then $(\mathcal{A}\mathbf{x}^{N-1})_i \geq 0$ and hence $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0 \leq (\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i$. If $x_i = 0$, then

$$(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0 \leq (\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i.$$

Therefore, $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0 \leq (\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i$. Since $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{P}$, then $\mathbf{x} = \mathbf{0}_I$. We show that each representative of $\widehat{\mathcal{A}}$ is a $\mathbb{P}$ tensor and so $\widehat{\mathcal{A}}$ has the $\mathbb{P}$ property.

To see the converse, assume that $\widehat{\mathcal{A}}$ has the $\mathbb{P}$ property. Let $\mathbf{x}$ be a vector such that $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1}) \leq \mathbf{0}_I \leq (\mathbf{x} \vee \widehat{\mathcal{A}}\mathbf{x}^{N-1})$. We construct a representative tensor $\mathcal{A}$ as follows. For any index j, let $\mathcal{A}^j$ be the jth mode-1 slice in the set $\{\mathcal{A}_1^j, \mathcal{A}_2^j, \cdots, \mathcal{A}_K^j\}$ such that $(\mathcal{A}\mathbf{x}^{N-1})_j \leq 0$ if $x_j > 0$ and $(\mathcal{A}\mathbf{x}^{N-1})_j \geq 0$ if $x_j < 0$. Since $\mathcal{A}$ is a $\mathbb{P}$ tensor and $x_i(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ for $i = 1, \cdots, I$, we see that $x_i = 0$ for $i = 1, \cdots, I$. Therefore $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{P}$. $\square$

Similarly, we have the following proposition.

Proposition 4.4. *$\widehat{\mathcal{A}}$ has the $\mathbb{E}$ property if and only if it is of* $\mathrm{type}\ \mathbb{E}$.

Proof. Suppose that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{E}$ and let $\mathcal{A}$ be a representative tensor of $\widehat{\mathcal{A}}$. To show that $\mathcal{A}$ is a strictly semi-positive tensor, we start with a vector $\mathbf{x} \in \mathbb{R}_+^I$ such that $x_i(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ for $i = 1, \cdots, I$ and show that $\mathbf{x} = \mathbf{0}_I$. If $x_i > 0$, then $(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ and hence $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0$. If $x_i = 0$, then $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0$. Therefore, $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1})_i \leq 0$. Since $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{E}$, then $\mathbf{x} = \mathbf{0}_I$. We show that each representative of $\widehat{\mathcal{A}}$ is a strictly semi-positive tensor and so $\widehat{\mathcal{A}}$ has the $\mathbb{E}$ property.

To see the converse, assume that $\widehat{\mathcal{A}}$ has the $\mathbb{E}$ property. Let $\mathbf{x}$ be a vector such that $(\mathbf{x} \wedge \widehat{\mathcal{A}}\mathbf{x}^{N-1}) \leq \mathbf{0}_I$. We construct a representative tensor $\mathcal{A}$ as follows. For any index j, let $\mathcal{A}^j$ be the jth mode-1 slice in the set $\{\mathcal{A}_1^j, \mathcal{A}_2^j, \cdots, \mathcal{A}_K^j\}$ such that $(\mathcal{A}\mathbf{x}^{N-1})_j \leq 0$ if $x_j > 0$. Since $\mathcal{A}$ is a strictly semi-positive tensor and $x_i(\mathcal{A}\mathbf{x}^{N-1})_i \leq 0$ for $i = 1, \cdots, I$, we see that $x_i = 0$ for $i = 1, \cdots, I$. Therefore $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{E}$. $\square$

By slightly modifying the proof of the above propositions, we can show that $\widehat{\mathcal{A}}$ is of $\mathrm{type}\ \mathbb{E}_0$ (respectively, $\mathrm{type}\ \mathbb{P}_0$) if and only if $\widehat{\mathcal{A}}$ has the $\mathbb{E}_0$ property (property, $\mathbb{P}_0$ property). We omit the details.

Remark 4.4. Combining with Theorems 4.1 and 4.2, we see that the zero vector is a unique solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ for every $\mathbf{q}_k \in \mathbb{R}_+^I$ with $k = 1, \cdots, K$ if and only if every representative tensor of $\widehat{\mathcal{A}}$ is an $\mathbb{E}$ tensor; the zero vector is a unique solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ for every $\mathbf{q}_k > \mathbf{0}_I$ with $k = 1, \cdots, K$ if and only if every representative tensor of $\widehat{\mathcal{A}}$ is a semi-positive tensor.

Now we prove the reverse implication (b) $\Rightarrow$ (a) in Theorem 4.2. Because of the previous remark, we need to show that if (b) holds, then every representative tensor of $\widehat{\mathcal{A}}$ is a semi-positive tensor. Without loss of generality, we consider the tensor $\mathcal{A}_1$ in $\widehat{\mathcal{A}}$. Let $\mathbf{q}_1 > \mathbf{0}_I$ in $\mathbb{R}^I$. If the problem $\mathrm{TCP}(\mathcal{A}_1, \mathbf{q}_1)$ has two solutions, say $\mathbf{x}$ and $\mathbf{y}$, then we can take $\mathbf{r} > \mathbf{0}_I$ and define $\mathbf{q}_i = (-\mathcal{A}_i\mathbf{x}^{N-1}) \vee (-\mathcal{A}_i\mathbf{y}^{N-1}) \vee \mathbf{r}$ for $i = 2, \cdots, K$. Clearly, $\mathbf{q}_k > \mathbf{0}_I$ and $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ has two solutions. So, if (b) holds, then $\mathrm{TCP}(\mathcal{A}_1, \mathbf{q}_1)$ has a unique solution for all $\mathbf{q}_1 > \mathbf{0}_I$, that is, $\mathcal{A}_1$ is a semi-positive tensor.

5. Stability

In the stability aspect of the GOTCP, we are interested in the solution behavior as the data changes. First, we deal with the behavior of the entire solution set.

Definition 5.1. *The problem* $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ *is said to be stable, if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\mathrm{SOL}(\widehat{\mathcal{A}}', \widehat{\mathbf{q}}') \cap (\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) + \varepsilon\mathbb{B}) \neq \emptyset$$

for any $(\widehat{\mathcal{A}}', \widehat{\mathbf{q}}')$ *with* $\|\widehat{\mathcal{A}} - \widehat{\mathcal{A}}'\|_F + \|\widehat{\mathbf{q}} - \widehat{\mathbf{q}}'\|_2 < \delta$*, where* $\mathbb{B}$ *is the open unit ball in* $\mathbb{R}^I$*. Here we define*

$$\|\widehat{\mathcal{A}} - \widehat{\mathcal{A}}'\|_F^2 = \sum_{k=1}^{K} \|\mathcal{A}_k - \mathcal{A}_k'\|_F^2, \qquad \|\widehat{\mathbf{q}} - \widehat{\mathbf{q}}'\|_2^2 = \sum_{k=1}^{K} \|\mathbf{q}_k - \mathbf{q}_k'\|_2^2.$$

The following theorem is the basis of our stability analysis.

Theorem 5.1. *Suppose that* $\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ *is nonempty and bounded. Suppose that for any same open set* Ω *containing* $\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$*,* $\deg(F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}, \Omega, \mathbf{0}_I)$ *is nonzero. Then* $\mathrm{GOTCP}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})$ *is stable.*

Proof. For any given $\varepsilon > 0$, we consider the open set $\mathbb{D} = \mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{q}}) + \varepsilon\mathbb{B}$. Without loss of generality, we can assume that $\mathbb{D} \subset \Omega$. Since there are no solutions of $F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}(\mathbf{x}) = \mathbf{0}_I$ in $\Omega \backslash \mathbb{D}$, by the excision property of the degree [34, Theorem 2.1.2], we have $\deg(F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}, \Omega, \mathbf{0}_I) = \deg(F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}, \mathbb{D}, \mathbf{0}_I) \neq 0$. Now for a suitable $\delta > 0$, we have

$$\sup_{\mathbb{D}} \left\| F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}(\mathbf{x}) - F_{(\widehat{\mathcal{A}}', \widehat{\mathbf{q}}')}(\mathbf{x}) \right\|_2 < \mathrm{dist}\left(\mathbf{0}_I, F_{(\widehat{\mathcal{A}}, \widehat{\mathbf{q}})}(\partial\mathbb{D})\right)$$

for any $(\widehat{\mathcal{A}}', \widehat{\mathbf{q}}')$ with $\|\widehat{\mathcal{A}} - \widehat{\mathcal{A}}'\|_F + \|\widehat{\mathbf{q}} - \widehat{\mathbf{q}}'\|_2 < \delta$. Hence the equation $F_{(\widehat{\mathcal{A}}', \widehat{\mathbf{q}}')}(\mathbf{x}) = \mathbf{0}_I$ has a solution in $\mathbb{D}$. The stated conclusion follows. $\square$

We say that $\widehat{\mathcal{A}}$ is of type $\mathbb{D}$ if there is a $\widehat{\mathbf{d}}$ such that (a) $\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{d}})$ is nonempty and bounded and (b) $\deg(F_{\widehat{\mathcal{A}}, \widehat{\mathbf{d}}}, \Omega, \mathbf{0}_I) \neq 0$, where $\Omega \in \mathbb{R}^I$ is some bounded open set containing $\mathrm{SOL}(\widehat{\mathcal{A}}, \widehat{\mathbf{d}})$.

Conjecture 5.1. *If $\widehat{\mathcal{A}}$ is of* $\mathrm{type}\ \mathbb{G}$*, then it is of* $\mathrm{type}\ \mathbb{D}$*.*

Theorem 5.2. *Let $\widehat{\mathcal{A}}$ be of type $\mathbb{P}_0$ and $\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ be nonempty and bounded. Then $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ is stable.*

Proof. Let $\Omega\subset\mathbb{R}^I$ be a bounded open set in $\mathbb{R}^I$ containing $\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. Let $\mathbf{x}_*\in\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. For any $t\in[0,1]$, we define $(\widehat{\mathcal{A}}_t,\widehat{\mathbf{q}}_t)$ by

$$\widehat{\mathcal{A}}_t=\widehat{\mathcal{A}}+t\widehat{\mathcal{I}},\qquad \widehat{\mathbf{q}}_t=\widehat{\mathbf{q}}-t\widehat{\mathbf{x}_*},$$

where $\widehat{\mathbf{x}_*}=\{\mathbf{x}_*^{[N-1]},\mathbf{x}_*^{[N-1]},\cdots,\mathbf{x}_*^{[N-1]}\}$. Then for $t>0$, $\mathbf{x}^*\in\mathrm{SOL}(\widehat{\mathcal{A}}_t,\widehat{\mathbf{q}}_t)$ and $\widehat{\mathcal{A}}_t$ is of $\mathrm{type}\ \mathbb{P}$. Therefore, $\mathbf{x}^*\in\mathrm{SOL}(\widehat{\mathcal{A}}_t,\widehat{\mathbf{q}}_t)$ for all $t>0$. Clearly, we have a homotopy between $(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ and $(\widehat{\mathcal{A}}_1,\widehat{\mathbf{q}}_1)$. Hence $\deg(F_{(\widehat{\mathcal{A}},\widehat{\mathbf{d}})},\Omega,\mathbf{0}_I)=\deg(F_{(\widehat{\mathcal{A}}_1,\widehat{\mathbf{d}}_1)},\Omega,\mathbf{0}_I)$. Since $\widehat{\mathcal{A}}_1$ is of $\mathrm{type}\ \mathbb{P}$, it is of $\mathrm{type}\ \mathbb{R}$ by Proposition 3.1. By Theorem 3.2, its GOTCP-degree is one. By following the proof of Theorem 3.1, we see that

$$\deg(F_{(\widehat{\mathcal{A}}_1,\widehat{\mathbf{q}}_1)},\Omega,\mathbf{0}_I)=1.$$

Therefore,

$$\deg(F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})},\Omega,\mathbf{0}_I)\neq 0.$$

Now the stability follows from the previous theorem. □

We now formulate the definition of stability at a solution point.

Definition 5.2. *Let $\mathbf{x}_*$ be a solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. The problem $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ is said to be stable at $\mathbf{x}_*$, if $\mathbf{x}_*$ is an isolated solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ and for every $\varepsilon>0$ there exists a $\delta>0$ such that*

$$\mathrm{SOL}(\widehat{\mathcal{A}}',\widehat{\mathbf{q}}')\cap(\mathbf{x}_*+\varepsilon\mathbb{B})\neq\emptyset$$

for any $(\widehat{\mathcal{A}}',\widehat{\mathbf{q}}')$ with $\|\widehat{\mathcal{A}}-\widehat{\mathcal{A}}'\|_F+\|\widehat{\mathbf{q}}-\widehat{\mathbf{q}}'\|_2<\delta$.

Here we present a result for $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. By modifying the proofs of Theorems 5.1 and 5.2, we are led to the following. We omit the details.

Theorem 5.3. *Let $\mathbf{x}_*$ be an isolated solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. Suppose that there is a bounded open set Ω containing only one solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$, namely, $\mathbf{x}_*$ and $\deg(F_{(\widehat{\mathcal{A}},\widehat{\mathbf{q}})},\Omega,\mathbf{0}_I)$ is nonzero. Then $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ is stable at $\mathbf{x}_*$.*

Theorem 5.4. *Let $\widehat{\mathcal{A}}$ be of type $\mathbb{P}_0$ and $\mathbf{x}_*$ be an isolated solution of $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. Then $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ is stable at $\mathbf{x}_*$.*

From the above theorem, we have an interesting consequence. Suppose that $\widehat{\mathcal{A}}$ is of type $\mathbb{P}_0$ and there are two isolated solutions $\mathbf{x}_*$ and $\mathbf{y}_*$ for $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. By Theorem 5.3, $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ is stable at both the solutions. But if $\widehat{\mathcal{A}}$ is perturbed to $\widehat{\mathcal{A}}'$, then the problem $\mathrm{GOTCP}(\widehat{\mathcal{A}}',\widehat{\mathbf{q}}')$ must have solutions near $\mathbf{x}_*$ as well as $\mathbf{y}_*$. This clearly contradicts the uniqueness property of $\widehat{\mathcal{A}}'$. We conclude that for $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$, there can be at most one isolated solution. This means that when $\widehat{\mathcal{A}}$ is of type $\mathbb{P}_0$ and $\widehat{\mathbf{q}}$ is arbitrary, $\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$ either is empty, or a singleten set, or an infinite set.

6. Concluding remarks

In this paper, we studied the solvability of the GOTCP, which can be viewed as the generalization of the TCP and the GOLCP. In order to study its solution set, we generalized the definition of some structured tensors and introduced the definition of the GOTCP-degree. We also obtained a necessary and sufficient condition for that a given GOTCP has a unique zero solution. Finally, we considered two cases of stability of the GOTCP.

For a given $\mathrm{GOTCP}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$, let $\Phi(\widehat{\mathbf{q}})=\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. When $N=2$, we have that $\Phi(\widehat{\mathbf{q}})$ is a polyhedral multifunction [40]. Gowda and Sznajder [22] indicated the following proposition.

Proposition 6.1. *Suppose that $N=2$. For a given $\widehat{\mathcal{A}}$, consider the mapping $\Phi(\widehat{\mathbf{q}})=\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. Let $\mathbb{E}$ be a compact subset of $\mathbb{R}^I$. If $\Phi(\widehat{\mathbf{q}})$ is bounded for each $\mathbf{q}_k\in\mathbb{E}$ $(k=1,\cdots,K)$, then $\cup_{\mathbf{q}_k\in\mathbb{E}\ (k=1,\cdots,K)}\Phi(\widehat{\mathbf{q}})$ is bounded.*

In general, when $N\geq 3$, we have the following conjecture. Note that $\Phi(\widehat{\mathbf{q}})$ is not a polyhedral multifunction.

Conjecture 6.1. *Suppose that $N\geq 3$. For a given $\widehat{\mathcal{A}}$, consider the mapping $\Phi(\widehat{\mathbf{q}})=\mathrm{SOL}(\widehat{\mathcal{A}},\widehat{\mathbf{q}})$. Let $\mathbb{E}$ be a compact subset of $\mathbb{R}^I$. If $\Phi(\widehat{\mathbf{q}})$ is bounded for each $\mathbf{q}_k\in\mathbb{E}$ $(k=1,\cdots,K)$, then $\cup_{\mathbf{q}_{k\in\mathbb{E}}}$ $(k=1,\cdots,K)$ $\Phi(\widehat{\mathbf{q}})$ is bounded.*

Gowda and Sznajder [22] obtained a general problem when $\mathbf{x}$ appearing at the beginning of the expression (2.1) is replaced by an affine function. The resulting piecewise-linear system is an extended GOLCP: finding $\mathbf{x}\in\mathbb{R}^I$ such that

$$(\mathbf{A}_0\mathbf{x}+\mathbf{q}_0)\wedge(\mathbf{A}_1\mathbf{x}+\mathbf{q}_1)\wedge(\mathbf{A}_2\mathbf{x}+\mathbf{q}_2)\wedge\cdots\wedge(\mathbf{A}_K\mathbf{x}+\mathbf{q}_K)=\mathbf{0}_I. \tag{6.1}$$

It is clear that if $\mathbf{A}_0$ is invertible, then (6.1) can be formulated as a GOLCP. Similarly, a given GOTCP can be generalized as:

(a) an extended GOTCP I: finding $\mathbf{x}\in\mathbb{R}^I$ such that

$$(\mathbf{A}_0\mathbf{x}+\mathbf{q}_0)\wedge(\mathcal{A}_1\mathbf{x}^{N-1}+\mathbf{q}_1)\wedge(\mathcal{A}_2\mathbf{x}^{N-1}+\mathbf{q}_2)\wedge\cdots\wedge(\mathcal{A}_K\mathbf{x}^{N-1}+\mathbf{q}_K)=\mathbf{0}_I, \tag{6.2}$$

(b) an extended GOTCP II: finding $\mathbf{x}\in\mathbb{R}^I$ such that

$$\begin{aligned}&(\mathcal{A}_0\mathbf{x}^{N-1}+\mathbf{q}_0)\wedge(\mathcal{A}_1\mathbf{x}^{N-1}+\mathbf{q}_1)\wedge(\mathcal{A}_2\mathbf{x}^{N-1}+\mathbf{q}_2)\\&\quad\wedge\cdots\wedge(\mathcal{A}_K\mathbf{x}^{N-1}+\mathbf{q}_K)=\mathbf{0}_I.\end{aligned} \tag{6.3}$$

It is clear that if $\mathbf{A}_0$ is invertible, then (6.2) can be formulated as a GOTCP. However, in general, (6.3) cannot be formulated as a GOTCP. One of our goals in the future is to study the solution sets of (6.2) and (6.3).

Consider a polynomial map $f:\mathbb{R}^I\to\mathbb{R}^I$, which is expressed in the following form:

$$f(\mathbf{x})=\mathcal{A}_N\mathbf{x}^{N-1}+\mathcal{A}_{N-1}\mathbf{x}^{N-2}+\cdots+\mathcal{A}_2\mathbf{x},$$

where $\mathcal{A}_n \in T_{n,I}$ with $n = 2, 3, \cdots, N$. For any given $\mathbf{q} \in \mathbb{R}^I$, the polynomial complementarity problem [18, 20], denoted by $\mathrm{PCP}(f, \mathbf{q})$, is to find $\mathbf{x} \in \mathbb{R}^I$ such that

$$\mathbf{x} \wedge f(\mathbf{x}) + \mathbf{q} = \mathbf{0}_I.$$

Meanwhile, consider K polynomial maps $f_k : \mathbb{R}^I \to \mathbb{R}^I$ $(k = 1, \cdots, K)$, which are expressed in the following form:

$$f_k(\mathbf{x}) = \mathcal{A}_{k,N_k}\mathbf{x}^{N_k-1} + \mathcal{A}_{k,N_k-1}\mathbf{x}^{N_k-2} + \cdots + \mathcal{A}_{k,2}\mathbf{x},$$

where $\mathcal{A}_{k,n_k} \in T_{n_k,I}$ with $n_k = 2, 3, \cdots, N_k$. For K given $\mathbf{q}_k \in \mathbb{R}^I$, the generalized order polynomial complementarity problem is to find $\mathbf{x} \in \mathbb{R}^I$ such that

$$\mathbf{x} \wedge (f_1(\mathbf{x}) + \mathbf{q}_1) \wedge (f_2(\mathbf{x}) + \mathbf{q}_2) \wedge \cdots \wedge (f_K(\mathbf{x}) + \mathbf{q}_K) = \mathbf{0}_I.$$

Under certain conditions, Gowda [20] derived the properties of the solution set of any given $\mathrm{PCP}(f, \mathbf{q})$ based on the solution set of $\mathrm{TCP}(\mathcal{A}_N, \mathbf{q})$. Hence, another of our goals in the future is to study the generalized order polynomial complementarity problem via the results about the associated GOTCP.

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