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On Nonnegative Solution of Multi-Linear System with Strong $\mathcal{M}_z$-Tensors

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Abstract. A class of structured multi-linear system defined by strong $\mathcal{M}_z$-tensors is considered. We prove that the multi-linear system with strong $\mathcal{M}_z$-tensors always has a nonnegative solution under certain condition by the fixed point theory. We also prove that the zero solution is the only solution of the homogeneous multi-linear system for some structured tensors, such as strong $\mathcal{M}$-tensors, $\mathcal{H}^+$-tensors, strictly diagonally dominant tensors with positive diagonal elements. Numerical examples are presented to illustrate our theoretical results.

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1. Introduction

A basic problem in both pure and applied mathematics is solving various kinds of equations. Those like linear system, Sylvester and Riccati equations are well-known [15]. As a generalization of the matrix to higher-order case, tensor has received considerable attention in recent years. Let $\mathcal{A}$ be an m-th order n-dimensional tensor in $\mathbb{C}^{n\times n\times\cdots\times n} := \mathbb{C}^{[m,n]}$ and vector $\mathbf{b} \in \mathbb{C}^n$, then the linear system could be generalized to a higher-order case represented by tensors, that is,

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1.1}$$

where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and the left-hand side $\mathcal{A}\mathbf{x}^{m-1}$ is a vector with entries

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$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2}\cdots x_{i_m}$$

for all $i = 1,\ldots,n$, [25]. The multi-linear system (1.1) arises in various applications, such as numerical partial differential equations [12], tensor complementarity problems [6,7], data mining [18] and so on.

Up to now, the multi-linear system (1.1) has been studied by many researchers. If $\mathcal{A}$ is an $\mathcal{M}$-tensor (mainly symmetric or strong $\mathcal{M}$-tensors [11,39]), some results corresponding to (1.1) have been presented; see [12,14,17,20,30,36]. Wang *et al.* considered this problem in a more general case, like with $\mathcal{H}^+$-tensors [29] or nonsingular tensors [32]. Besides, some other type equations, like tensor absolute value equation [13], sparse nonnegative tensor equations [18], $\mathcal{H}$-tensor equation [33], are also studied by researchers. More details could be found in the monographs [9,26,27,34] and the references therein.

Many different definitions on eigenvalues of tensors are proposed and studied recently, such as M-eigenvalues [22] and C-eigenvalues [37] and so on. One of the most well-known is the Z-eigenvalue given by Qi [25] and Lim [19] independently. Recently, Mo *et al.* [23] gave a new class of tensors called (strong) $\mathcal{M}_z$-tensors based on Z-eigenvalues and proved that an even-order (strong) $\mathcal{M}$-tensor must be an (strong) $\mathcal{M}_z$-tensor and the converse is not true in general. Some results about positive semi-definiteness and co-positivity are also given.

Motivated by these works, we consider the multi-linear system (1.1) with strong $\mathcal{M}_z$-tensor and we call it strong $\mathcal{M}_z$-tensor equation. Before giving the definition of (strong) $\mathcal{M}_z$-tensor, let us see the definition of Z-eigenvalue first.

Definition 1.1 ([19,25]). We call λ as a Z-eigenvalue of tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if there exists a nonzero real vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}, \quad \mathbf{x}^\top\mathbf{x} = 1.$$

Such an $\mathbf{x}$ is called a Z-eigenvector associated with λ, and $(\lambda,\mathbf{x})$ is called a Z-eigenpair.

The Z-spectral radius of a tensor is defined as

$$\rho_z(\mathcal{B}) := \sup\left\{|\lambda| : \lambda \text{ is } Z\text{-eigenvalue of } \mathcal{B}\right\}.$$

By using the Z-spectral radius, the definition of $\mathcal{M}_z$-tensor [23] is recalled as follows. And it is a generalization of the M-matrix to the high-order case and also contains the even-order $\mathcal{M}$-tensor as a proper subset.

Definition 1.2. ([23, Definition 3.1]). Assume that m is even. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an $\mathcal{M}_z$-tensor, if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $s \geq \rho_z(\mathcal{B})$ such that

$$\mathcal{A} = s\mathcal{I}_z - \mathcal{B},$$

where $\mathcal{I}_z = (e_{i_1\dots i_m}) \in \mathbb{R}^{[m,n]}$ is the Z-identity tensor defined as a nonnegative tensor such that

$$\mathcal{I}_z\mathbf{x}^{m-1} = \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}^\top\mathbf{x} = 1$. In addition if $s > \rho_z(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}_z$-tensor.

The topic on the nonnegative (or positive) solutions of a linear system and also multi-linear is always interesting. For example, in studying the following tensor complementarity problems [6, 21, 28, 31]:

$$\mathrm{TCP}(\mathcal{A},\mathbf{b}) : \mathbf{x} \geq 0, \quad \mathcal{A}\mathbf{x}^{m-1} - b \geq 0, \quad \left\langle \mathbf{x}, \mathcal{A}\mathbf{x}^{m-1} - b \right\rangle = 0. \tag{1.2}$$

We always want to find the nonnegative solution $\mathbf{x}$. Thus the corresponding nonnegative solution of the multi-linear system (1.1) is crucial. In view of this and other applications, some focus on the nonnegative (or positive) solutions of the system (1.1). Ding and Wei [12] proved that if $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then (1.1) possessed a unique positive solution for any positive right-hand side $\mathbf{b}$. Wang *et al.* [29] showed that (1.1) with $\mathcal{H}^+$-tensor $\mathcal{A}$ and positive vector $\mathbf{b}$ also has a unique positive solution.

Inspired by the work about multi-linear system and corresponding TCP, especially about $\mathcal{M}$-tensors, here we mainly consider the nonnegative solutions of the strong $\mathcal{M}_z$-tensor equation. We give results about the nonnegative solution of this system. Some examples are also given to illustrate our theoretical results.

The organization of this paper is as follows. Some definitions and classical results are recalled in the next section. In Section 3, we prove that strong $\mathcal{M}_z$-tensor equation always has a nonnegative (positive) solution if the right-hand is nonnegative (positive). We also give some important results like the nonsingular matrices do, that is, the zero solution is the only solution to the homogeneous system if the coefficient matrices are nonsingular. We show that this result holds for some structured tensors, such as strong $\mathcal{M}$-tensors, a class of $\mathcal{M}_z$-tensors, $\mathcal{H}^+$-tensors, strictly diagonally dominant tensors with positive diagonal elements. Some numerical examples are given in Section 4. We conclude our work in Section 5.

2. Preliminaries

A tensor $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $K \subset N := \{1,\dots,n\}$ such that $a_{i_1 i_2\dots i_m} = 0$ for $i_1 \in K$ and $i_2,\dots,i_m \notin K$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible [3]. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called symmetric [3, 25] if its entries are invariant under any permutation of their indices and it is called weakly symmetric [4] if

$$\nabla f_{\mathcal{A}}(\mathbf{x}) = m\mathcal{A}\mathbf{x}^{m-1}, \quad \forall\, \mathbf{x} \in \mathbb{R}^n$$

and the right-hand side $m\mathcal{A}\mathbf{x}^{m-1}$ is not identical to zero.

Remark 2.1. It is shown by Chang *et al.* in [4] that a symmetric tensor is necessarily weakly symmetric for $m > 2$, but the converse is not true in general. Example that weakly symmetric tensor is not symmetric is given in [4, Remark 3.4].

Eigenvalues and H-eigenvalues are different from E- and Z-eigenvalues in general.

Definition 2.1 ([19, 25])**.** We call λ as an eigenvalue of tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if there exists a nonzero vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$$

for all $i \in N$, and $\mathbf{x}^{[m-1]}$ is a vector in $\mathbb{C}^n$ with entries defined as $(\mathbf{x}^{[m-1]})_i = x_i^{m-1}$ for all $i \in N$. Such $\mathbf{x}$ is called an eigenvector associated with λ, and $(\lambda, \mathbf{x})$ is called an eigenpair. In particular, if $\mathbf{x}$ is real, then λ is also real. In this case, λ is called an H-eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ is its corresponding H-eigenvector.

On the basis of the above definition, Zhang *et al.* [39] and Ding *et al.* [11, 34] gave the definition of $\mathcal{M}$-tensors.

Definition 2.2. ([39, Definition 3.1]). A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an $\mathcal{M}$-tensor, if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $s \geq \rho(\mathcal{B})$ where

$$\rho(\mathcal{B}) = \max \big\{ |\lambda| : \lambda \text{ is eigenvalue of } \mathcal{B} \big\}$$

such that

$$\mathcal{A} = s\mathcal{I} - \mathcal{B}, \tag{2.1}$$

in which $\mathcal{I} = (\delta_{i_1 \ldots i_m})$ is the m-th order n-dimensional identity tensor with

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if} \quad i_1 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, if $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

Remark 2.2. We know that all non-diagonal entries of an $\mathcal{M}$-tensor are non-positive, however, those non-positive entries can be positive in an $\mathcal{M}_z$-tensor. Thus, an $\mathcal{M}_z$-tensor may be not an $\mathcal{M}$-tensor; for concrete examples, see [23]. However, an even-order (strong) $\mathcal{M}$-tensor must be an (strong) $\mathcal{M}_z$-tensor which has been proved in [23, Theorem 3.1].

The tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \ldots i_m})$ is called the comparison tensor of $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ if

$$m_{i_1 i_2 \ldots i_m} = \begin{cases} + |a_{i_1 i_2 \ldots i_m}|, & \text{if} \quad (i_2, i_3, \ldots, i_m) = (i_1, i_1, \ldots, i_1), \\ - |a_{i_1 i_2 \ldots i_m}|, & \text{if} \quad (i_2, i_3, \ldots, i_m) \neq (i_1, i_1, \ldots, i_1). \end{cases}$$

We call a tensor an $\mathcal{H}$-tensor, if its comparison tensor is an $\mathcal{M}$-tensor; we call it as a strong $\mathcal{H}$-tensor, if its comparison tensor is a strong $\mathcal{M}$-tensor. Wang *et al.* defined the $\mathcal{H}^+$-tensor in [30] and it is a subset of $\mathcal{H}$-tensor [11].

Definition 2.3. ([30, Definition 2.5]). Let $\mathcal{A}$ be a strong $\mathcal{H}$-tensor with all the positive diagonal elements, then we call $\mathcal{A}$ as an $\mathcal{H}^+$-tensor.

Remark 2.3. Example 5.2 of [23] illustrates that a strong $\mathcal{M}_z$-tensor may not be an $\mathcal{H}$-tensor, thus an $\mathcal{M}_z$-tensor is not necessarily an $\mathcal{H}^+$-tensor.

Strictly diagonally dominant tensors, which is an extension of strictly diagonally dominant matrices, play an important role in studying tensor problems for its good properties. Its definition is given below.

Definition 2.4. ([39, Definition 3.14]). Let $\mathcal{A}$ be an m-th order n-dimensional tensor. $\mathcal{A}$ is strictly diagonally dominant if

$$|a_{ii\dots i}| > \sum_{(i_2,\dots,i_m)\neq(i,\dots,i)} |a_{ii_2\dots i_m}|$$

hold for all $i = 1,\dots,n$.

The definition of Z-identity tensor is given in [23, Definition 1.4]. There is no odd-order Z-identity tensor and the even-order Z-identity tensor is not unique; see [23, Remark 1.1, Examples 3.1 and 3.2]. Let δ be the standard Kronecker delta and Π_m be the set of all permutations of $(1,\dots,m)$. Consider the tensor $\mathcal{I}_z = (e_{i_1\dots i_m}) \in \mathbb{R}^{[m,n]}$, if

$$e_{i_1\dots i_m} = \delta_{i_1 i_2}\dots\delta_{i_{m-1}i_m} \tag{2.2}$$

or

$$e_{i_1\dots i_m} = \frac{1}{m!}\sum_{p\in\Pi_m} \delta_{i_{p(1)}i_{p(2)}}\dots\delta_{i_{p(m-1)}i_{p(m)}}, \tag{2.3}$$

then $\mathcal{I}_z$ is a nonsymmetric or symmetric Z-identity tensor, respectively.

Due to the differences of the H- and Z-eigenvalues, many results based on these two definitions differ too much. One result is about the famous Perron-Frobenius type theorem for tensors. We give the conclusion about Z-eigenvalues. The corresponding one for H-eigenvalues could be found in [38].

Lemma 2.1. ([5, Theorems 2.5, 2.6, 4.7]). *The following conclusions hold for $\mathcal{A} \in \mathbb{R}^{[m,n]}$:*

a) If $\mathcal{A}$ is a nonnegative tensor, then there exist a nonnegative Z-eigenvalue λ and a nonnegative nonzero Z-eigenvector $\mathbf{x}$ of $\mathcal{A}$ such that $\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}$.

b) If $\mathcal{A}$ is nonnegative and irreducible, then the Z-eigenvalue λ and the corresponding Z-eigenvector $\mathbf{x}$ in a) are positive.

c) If $\mathcal{A}$ is nonnegative and weakly symmetric irreducible, then the Z-spectral radius $\rho_z(\mathcal{A})$ is a positive Z-eigenvalue with a positive corresponding Z-eigenvector.

The following result shows that the Z-eigenvalues of an $\mathcal{M}_z$-tensor are nonnegative.

Lemma 2.2. ([23, Theorem 4.3]) *If $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ is an $\mathcal{M}_z$-tensor and η is a Z-eigenvalue of $\mathcal{A}$, then η is nonnegative. If $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ is a strong $\mathcal{M}_z$-tensor and η is a Z-eigenvalue of $\mathcal{A}$, then η is positive.*

3. Strong $\mathcal{M}_z$-tensor equation

We are going to analyze our main problem deeply and prove our main results in this section. That is, we consider the nonnegative solution of the strong $\mathcal{M}_z$-tensor equation. That is,

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b} \quad \text{where} \quad \mathcal{A} = s\mathcal{I}_z - \mathcal{B}. \tag{3.1}$$

If the coefficient tensor $\mathcal{A}$ is of even-order and the vector $\mathbf{b}$ is nonpositive, then the system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ is equivalent to $\mathcal{A}(-\mathbf{x})^{m-1} = -\mathbf{b}$, which is a system with nonnegative right-hand side. Thus we only consider the strong $\mathcal{M}_z$-equation with nonnegative right-hand side.

The authors drew the following conclusion which gave the crucial relationship between the determinant of the tensor and the solutions of the corresponding multi-linear system like (1.1) in [16].

Lemma 3.1. ([16, Theorem 3.1]). *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. The multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a solution in $\mathbb{C}^n$ for any $\mathbf{b} \in \mathbb{C}^n$ if*

$$\det(\mathcal{A}) \neq 0,$$

in which $\det(\mathcal{A})$ denotes the product of all eigenvalues of $\mathcal{A}$.

Eigenvalue and Z-eigenvalue of tensors are two different concepts. The determinant $\det(\mathcal{A})$ is the product of all eigenvalues of $\mathcal{A}$. However, an $\mathcal{M}_z$-tensor is defined based on Z-eigenvalues from which we can not get the determinant directly. Thus, we prove the following lemma.

Lemma 3.2. *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}_z$-tensor, then its eigenvalues are non-zero. Thus the strong $\mathcal{M}_z$-equation has a solution for any $\mathbf{b} \in \mathbb{C}^n$.*

Proof. Suppose that $\mathcal{A}$ has a eigenvalue $\lambda = 0$, then there exists a nonzero vector $\mathbf{x}$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]} = \mathbf{0}. \tag{3.2}$$

If $\mathbf{x}^\top\mathbf{x} = 1$, then we have $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$ which means that $\mathcal{A}$ has a zero Z-eigenvalue. Notice that the eigenvector $\mathbf{x}$ is real since the tensor $\mathcal{A}$ and right-hand of (3.2) are both real. If the eigenvector $\mathbf{x}$ is not unit, we let $\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$, then the vector $\mathbf{y}$ is unit. Moreover, we have

$$\mathcal{A}\mathbf{y}^{m-1} = \frac{1}{\|\mathbf{x}\|_2^{m-1}}\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0},$$

which means that $\mathcal{A}$ has a Z-eigenpair $(0, \mathbf{y})$. Above all, we know that this is contradicting with the property given in Lemma 2.2 that if η is a Z-eigenvalue of a strong $\mathcal{M}_z$-tensor then η is positive. ☐

Now, we present one main theorem of this section.

Theorem 3.1. *Let $\mathcal{B}$ be a nonnegative weakly symmetric irreducible tensor, and $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ be a strong $\mathcal{M}_z$-tensor, then the multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a positive (nonnegative) solution for every positive (nonnegative) vector* $\mathbf{b}$.

We will prove the above result by fixed point theory. Firstly, the strong $\mathcal{M}_z$-tensor equation (3.1) can be rewritten as follows:

$$\mathbf{0} = (\mathcal{B} - s\mathcal{I}_z)\mathbf{x}^{m-1} + \mathbf{b}.$$

Adding $\alpha\mathcal{I}\mathbf{x}^{m-1}$, where α is a positive scalar, on both sides, we get

$$\alpha\mathcal{I}\mathbf{x}^{m-1} = (\mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I})\mathbf{x}^{m-1} + \mathbf{b}.$$

Thus the following fixed point iteration follows:

$$\begin{aligned} \mathbf{x}_{k+1} &= T_{\alpha,s,\mathcal{B},\mathbf{b}}(\mathbf{x}_k) \\ &:= \left(\alpha^{-1}(\mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I})\mathbf{x}_k^{m-1} + \alpha^{-1}\mathbf{b}\right)^{\left[\frac{1}{(m-1)}\right]}, \quad k = 0, 1, \ldots . \end{aligned}$$

It is easy to see that each fixed point of the above iteration is a solution of the strong $\mathcal{M}_z$-equation (3.1), and vice versa.

The following concepts are quoted from [1] and it is useful for the proof of Theorem 3.1.

Let V be a real vector space. An ordering in V is called linear if $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{x} + \mathbf{z} \leq \mathbf{y} + \mathbf{z}$ for all $\mathbf{z} \in V$ and $\alpha\mathbf{x} \leq \alpha\mathbf{y}$ for all $\alpha \in \mathbb{R}^+ := [0, \infty)$. A real vector space together with a linear ordering is called an ordered vector space (OVS). A nonempty subset P of a real vector space V is called a cone if it satisfies

$$\begin{aligned} &\mathbf{x}, \mathbf{y} \in P \text{ and } \lambda \geq 0 \ \text{ imply } \ \mathbf{x} + \mathbf{y}, \quad \lambda\mathbf{x} \in P, \\ &\mathbf{x} \in P \text{ and } -\mathbf{x} \in P \text{ imply } \ \mathbf{x} = \mathbf{0}. \end{aligned}$$

Every cone P in a real vector space V defines a linear ordering in V by $\mathbf{x} \leq \mathbf{y}$ iff $\mathbf{y} - \mathbf{x} \in P$. Clearly, the cone $P = \{\mathbf{x} \in V | \mathbf{x} \geq \mathbf{0}\}$ is convex. The elements in $\dot{P} = P \backslash \{0\} = \{\mathbf{x} \in V | \mathbf{x} > \mathbf{0}\}$ are called positive and P is said to be the positive cone of the ordering.

Let $E = (E, \|\cdot\|)$ be a Banach space ordered by a cone P. Then E is called an ordered Banach space (OBS), if the positive cone is closed. Next, we denote by the symbol (E, P) an arbitrary OBS with open unit ball B. For every $\gamma > 0$, we denote by P_γ the positive part of γB, that is, $P_\gamma := \gamma B \bigcap P = \gamma B^+$. Due to the convexity of B and P and observe that P_γ is an open neighborhood of 0 in P, then the closure $\bar{P}_\gamma$ of P_γ in P coincides with $\gamma\bar{B} \bigcap P$. Hence the boundary S_γ^+ in P equals $\gamma S \bigcap P$, where S denotes the unit sphere in E.

Let X be a nonempty subset of some Banach space and let f be a map from X into a second Banach space. Then f is called compact if it is continuous and if $f(X)$ is relatively compact. For arbitrary compact maps, Amann [1] gave the fixed point theorem by making use of the fixed point index.

Lemma 3.3. ([1, Theorem 12.2]). *Let $f : \bar{P}_\gamma \to P$ be a compact map such that $f(x) \neq \lambda x$ for $x \in S_\gamma^+$ and $\lambda \geq 1$. Then f has a fixed point in P_γ.*

Remark 3.1 ([1]). It is easy to see that the conditions, $f(x) \neq \lambda x$ for $x \in S_\gamma^+$ and $\lambda \geq 1$, of Lemma 3.3 are satisfied if $f(\mathbf{x}) \ngeq \mathbf{x}$ for all $\mathbf{x} \in S_\gamma^+$.

Before going further, we give an important lemma.

Lemma 3.4. *Suppose that $\mathcal{B} \in \mathbb{R}^{[m,n]}$ is a nonnegative tensor such that*

$$\rho_z(\mathcal{B}) \geq \min_{x_i>0} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i \tag{3.3}$$

for any nonnegative $\mathbf{x}$ satisfying $\|\mathbf{x}\|_2 = 1$. If $\mathcal{A} = s\mathcal{I}_z - \mathcal{B} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}_z$-tensor, then for any given $\mathbf{x} \in \mathbb{R}^n_+$ satisfied $\|\mathbf{x}\|_2 = 1$, there exists $i \in \{1, \ldots, n\}$ such that $(\mathcal{A}\mathbf{x}^{m-1})_i > 0$.

Proof. We want to prove that there is a i such that $(\mathcal{A}\mathbf{x}^{m-1})_i > 0$ which means that we need to find i such that $sx_i > (\mathcal{B}x^{m-1})_i$. Note that we have $\rho_z(\mathcal{B}) \geq \min_{x_i>0} \frac{(\mathcal{B}\mathbf{x}^{m-1})_i}{x_i}$ for any nonnegative $\mathbf{x}$ satisfied $\|\mathbf{x}\|_2 = 1$. Without loss of generality, suppose

$$\rho_z(\mathcal{B}) \geq \min_{x_i>0} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i = \frac{1}{x_1}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_1.$$

Then we have

$$\left(\mathcal{B}\mathbf{x}^{m-1}\right)_1 \leq \rho_z(\mathcal{B})x_1 < sx_1,$$

which means that $(\mathcal{A}\mathbf{x}^{m-1})_1 > 0$. □

Not all nonnegative tensors satisfy the condition (3.3), as shown in the following example. We use the MATLAB Tensor Toolbox (Version 2.6) [2] to compute the Z-eigenvalues here.

Example 3.1. Let $\mathcal{B} = (b_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,3]}$ with all zero entries except that $b_{1221} = b_{3222} = b_{1332} = b_{2323} = 100$ and let $\mathbf{x} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^\top$. We have

$$\min_{x_i>0} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i \approx 33.3333 > 4.8751 \times 10^{-6} \approx \rho_z(\mathcal{B}).$$

Furthermore, we choose $\mathcal{I}_z$ as (2.2) and let $s = \rho_z(\mathcal{B}) + 10^{-6}$, then $\mathcal{A} = s\mathcal{I}_z - \mathcal{B} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}_z$-tensor. However,

$$\mathcal{A}\mathbf{x}^{m-1} = (-38.4900, -19.2450, -19.2450)^\top,$$

whose entries are negative.

In order to ensure (3.3) holds for any nonnegative $\mathbf{x}$ satisfying $\|\mathbf{x}\|_2 = 1$, we have the following lemma.

Lemma 3.5. *If $\mathcal{B} \in \mathbb{R}^{[m,n]}$ is a nonnegative weakly symmetric irreducible tensor, then* (3.3) *holds for any nonnegative* $\mathbf{x}$ *satisfied* $\|\mathbf{x}\|_2 = 1$.

Proof. We only need to show

$$\rho_z(\mathcal{B}) \geq \max_{\mathbf{x}\in S^{n-1}\cap\mathbb{R}^n_+} \min_{1\leq i\leq n} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i,$$

where S^{n-1} is the standard unit sphere in $\mathbb{R}^n$, this can be followed from [5, Theorem 4.7] where it has

$$\rho_z(\mathcal{B}) = \max_{\mathbf{x}\in S^{n-1}\cap\mathbb{R}^n_+} \min_{1\leq i\leq n} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i.$$

We complete the proof. □

A more general case of the Theorem 3.1 is given as follows.

Theorem 3.2. *Let $\mathcal{B}$ be a nonnegative tensor such that*

$$\rho_z(\mathcal{B}) \geq \min_{x_i>0} \frac{1}{x_i}\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i$$

for any nonnegative $\mathbf{x}$ *satisfied* $\|\mathbf{x}\|_2 = 1$*, and* $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ *be a strong* $\mathcal{M}_z$*-tensor, then the multi-linear system* $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ *has a positive (nonnegative) solution for every positive (nonnegative) vector* $\mathbf{b}$.

Proof. Let $\alpha > 0$ and $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ be a strong $\mathcal{M}_z$-tensor. Consider the following map derived in the beginning of this section,

$$T_{\alpha,s,\mathcal{B},b} : \mathbf{x} \rightarrowtail \left(\alpha^{-1}(\mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I})\mathbf{x}^{m-1} + \alpha^{-1}\mathbf{b}\right)^{\left[\frac{1}{(m-1)}\right]}. \tag{3.4}$$

We prove next that this map is continuous and could be compact. Notice that each entry of $T_{\alpha,s,\mathcal{B},b}(\mathbf{x})$ is summation of the product of these entries $x_1, \ldots, x_n$ and plus the scalar b_i and then scale it, that is,

$$y_i := \left(T_{\alpha,s,\mathcal{B},b}(\mathbf{x})\right)_i = \left(\frac{1}{\alpha}\sum_{i_2,\ldots,i_m=1}^{n}(\alpha\mathcal{I}_{ii_2\ldots i_m} - \mathcal{A}_{ii_2\ldots i_m})x_{i_2}\ldots x_{i_m} + \frac{b_i}{\alpha}\right)^{\left[\frac{1}{m-1}\right]}$$

for all $i \in N$. A simple example is $y_1 = T(x_1, x_2, x_3) = x_1x_1 + 2x_1x_2 + x_2x_2 + b_1$. Certainly, it is continuous. By Lemma 3.2, we know that the strong $\mathcal{M}_z$-tensor equation has a solution. Undoubtedly, all entries of the given right-hand side vector $\mathbf{b}$ are bounded. We have

$$b_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}x_{i_2}\ldots x_{i_m}$$

for all $i \in N$. If there is a entry x_i that is unbounded, such as x_1, then the term $a_{1\ldots1}x_1^{m-1}$ is unbounded since $a_{1\ldots1}$ is nonzero (this is because $s > \rho_z(\mathcal{B}) \geq b_{i\ldots i}$ for this

$\mathcal{B}$) by the condition (3.3) which contradicts with the fact that b_i is bounded. Thus all the entries of $\mathbf{x}$ are bounded which means that $T_{\alpha,s,\mathcal{B},b}$ is actually a map from a bounded set of $\mathbb{R}^n$ to another bounded set since $T_{\alpha,s,\mathcal{B},b}$ is continuous. Then we can consider $T_{\alpha,s,\mathcal{B},b} : O_{r_1} \to O_{r_2}$ where r_1 and r_2 are the radii. By the continuous property and also notice that the closure of $T_{\alpha,s,\mathcal{B},b}(X)$, where X is a subset of O_{r_1}, is compact, then the map $T_{\alpha,s,\mathcal{B},b}$ is compact.

Denote $\mathcal{A}_\alpha = \mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I}$. For any $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n_+$, let

$$\Delta = \big\{i : x_i = 0, 1 \le i \le n\big\}, \quad \Delta^c = \big\{j : x_j > 0, 1 \le j \le n\big\},$$

and denote $\mathbf{y} = \frac{\mathbf{x}}{\beta}$ where $\beta = \|\mathbf{x}\|_2$. Note that

$$\mathcal{A}_\alpha \mathbf{x}^{m-1} = \mathcal{B}\mathbf{x}^{m-1} - s\beta^{m-1}\mathbf{y} + \alpha\mathbf{x}^{[m-1]}.$$

Then for any $i \in \Delta$, we have

$$\big(\mathcal{A}_\alpha \mathbf{x}^{m-1}\big)_i = \big(\mathcal{B}\mathbf{x}^{m-1}\big)_i - s\beta^{m-1}y_i + \alpha x_i^{m-1} = \big(\mathcal{B}\mathbf{x}^{m-1}\big)_i \ge 0.$$

For any $j \in \Delta^c$ with sufficiently large α, we have

$$(\mathcal{A}_\alpha \mathbf{x}^{m-1})_j \ge \alpha x_j^{m-1} - s\beta^{m-1}y_j \ge 0.$$

Thus for the map (3.4), we have

$$T_{\alpha,s,\mathcal{B},b} : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \mathbf{x} \rightarrowtail \big(\alpha^{-1}(\mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I})\mathbf{x}^{m-1} + \alpha^{-1}\mathbf{b}\big)^{[\frac{1}{(m-1)}]} \tag{3.5}$$

for any nonnegative vector $\mathbf{b}$.

By Remark 3.1, we want to find γ such that $T(\mathbf{x}) \ngeq \mathbf{x}$ for each $\mathbf{x} \in S^+_\gamma$. Equivalently for each $\mathbf{x} \in S^+_\gamma$, we have

$$\alpha^{-1}(\mathcal{B} - s\mathcal{I}_z + \alpha\mathcal{I})\mathbf{x}^{m-1} + \alpha^{-1}\mathbf{b} \ngeq \mathbf{x}^{[m-1]},$$

i.e.,

$$s\mathcal{I}_z\mathbf{x}^{m-1} \nleq \mathcal{B}\mathbf{x}^{m-1} + \mathbf{b},$$

which is equivalent to

$$\mathcal{A}\mathbf{x}^{m-1} \nleq \mathbf{b}.$$

If we can find i such that $(\mathcal{A}\mathbf{x}^{m-1})_i > b_i$, then the above inequality is satisfied.

For any nonnegative $\mathbf{x} \in S^+_\gamma$, we have

$$\big(\mathcal{A}\mathbf{x}^{m-1}\big)_i = \gamma^{m-1}\big(\mathcal{A}\mathbf{z}^{m-1}\big)_i,$$

where $\mathbf{z} = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$. Therefore, by Lemma 3.4, we see that there exists $(\mathcal{A}\mathbf{z}^{m-1})_i$ being positive which means that $(\mathcal{A}\mathbf{x}^{m-1})_i$ is also positive. For this i, we can find γ such that $(\mathcal{A}\mathbf{x}^{m-1})_i > b_i$. This is because if

$$b_i < \big(\mathcal{A}\mathbf{x}^{m-1}\big)_i = \gamma^{m-1}\big(\mathcal{A}\mathbf{z}^{m-1}\big)_i,$$

we only need to choose

$$\gamma > \left(\frac{b_i}{(\mathcal{A}\mathbf{z}^{m-1})_i} \right)^{\frac{1}{m-1}}.$$

If there exists other i such that $(\mathcal{A}\mathbf{x}^{m-1})_i$ is positive, we may get different γ. Finally, we choose the maximum one as the ultimate γ. Hence, for any nonnegative $\mathbf{b}$, we can find γ such that $T(\mathbf{x}) \ngeq \mathbf{x}$ for each $\mathbf{x} \in S_\gamma^+$. Thus, by Lemma 3.3, there is a fixed point in $[\mathbf{0}, \mathbf{x}_\gamma)$ where $\|\mathbf{x}_\gamma\|_2 = \gamma$. This means that the strong $\mathcal{M}_z$-tensor multilinear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a nonnegative solution for every nonnegative vector $\mathbf{b}$.

Next, we prove that the strong $\mathcal{M}_z$-tensor multilinear system has a positive solution for every positive vector $\mathbf{b}$. Firstly, it has a nonnegative solution $\mathbf{x}$ by the above proof. Denote $\mathbf{y} = \frac{\mathbf{x}}{\beta}$ where $\beta = \|\mathbf{x}\|_2$. If there is i such that $x_i = 0$, then

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = s\left(\mathcal{I}_z\mathbf{x}^{m-1}\right)_i - \left(\mathcal{B}\mathbf{x}^{m-1}\right)_i = s\gamma^{m-1}y_i - \left(\mathcal{B}\mathbf{x}^{m-1}\right)_i = -\left(\mathcal{B}\mathbf{x}^{m-1}\right)_i \leq 0,$$

which is impossible since $(\mathcal{A}\mathbf{x}^{m-1})_i = b_i > 0$. Thus it has a positive solution. We complete the proof. □

Remark 3.2. By Lemma 3.5 and Theorem 3.2, we can get the result Theorem 3.1 easily.

Notice that a symmetric tensor is necessarily weakly symmetric which is given in Remark 2.1 and by the discussions in the beginning of this section. We can immediately derive the following corollaries.

Corollary 3.1. *Let $\mathcal{B}$ be a nonnegative symmetric irreducible tensor and $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ be a strong $\mathcal{M}_z$-tensor, then the multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a positive (nonnegative) solution for every positive (nonnegative) vector* $\mathbf{b}$.

Corollary 3.2. *Let $\mathcal{B}$ be a nonnegative weakly symmetric irreducible tensor, and $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ be a strong $\mathcal{M}_z$-tensor, then the multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a non-positive (negative) solution for every non-positive (negative) vector* $\mathbf{b}$.

As we all know, the linear equations $A\mathbf{x} = \mathbf{0}$ only has one zero solution if the matrix A is nonsingular. By the above discussions and Theorem 3.2 and Corollaries 3.1 and 3.2, we can easily get the following interesting theorem which shows similar result like the nonsingular matrices do.

Theorem 3.3. *Let $\mathcal{B}$ be a nonnegative (weakly) symmetric irreducible tensor, and $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ be a strong $\mathcal{M}_z$-tensor, then $\mathbf{x} = \mathbf{0}$ is the unique solution of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$.*

The system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a unique positive solution for any positive right-hand side $\mathbf{b}$ if $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor [12, Theorem 3.2] or $\mathcal{H}^+$-tensor or strictly diagonally dominant tensors with positive diagonal elements [29, Theorem 3.1 and Corollary 3.2]. By their proof, we can also know that its solution is nonnegative (but may not unique) for any nonnegative right-hand side. If the order of the tensor is even, then the result also holds for strong $\mathcal{M}$-tensors or $\mathcal{H}^+$-tensors. We get the following theorem.

Theorem 3.4. *The multi-linear system* $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$ *has a unique solution* $\mathbf{x} = \mathbf{0}$ *if* $\mathcal{A}$ *is a strong* $\mathcal{M}$*-tensor or* $\mathcal{H}^+$*-tensor or strictly diagonally dominant tensors with positive diagonal elements.*

Proof. We only need to deal with the odd-order case, if the order m is odd, then $\mathcal{A}(-\mathbf{x})^{m-1} = \mathcal{A}\mathbf{x}^{m-1}$ since $m-1$ is even. Let $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} \geq \mathbf{0}$ and $-\mathbf{x} \geq \mathbf{0}$ which means that $\mathbf{x} = \mathbf{0}$. □

Whether or not the zero solution is the only solution of the homogeneous multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$ for any nonsingular tensors $\mathcal{A}$ (that is $\det(\mathcal{A}) \neq 0$), more study is needed.

In recent years, the TCP given in (1.2) has attracted considerable attention. It is obvious that every nonnegative solution of multi-linear equations (1.1) is a solution of the $TCP(\mathcal{A}, \mathbf{b})$. Thus the existence of nonnegative (positive) solution to multi-linear system also implies the existence of solution to the corresponding $TCP(\mathcal{A}, \mathbf{b})$. Therefore, based on the results in this section, mainly Theorem 3.1, the solution of tensor complementarity problem with a class of strong $\mathcal{M}_z$-tensors has been characterized.

4. Numerical examples

We give some numerical examples to illustrate our theoretical analysis for the strong $\mathcal{M}_z$-tensor equation (3.1). All experiments are implemented in a laptop with precision 2.2204×10^{-16} on a personal computer with 3.1 GHz central processing unit (Intel Core i5), 16 GB memory and macOS system with version 10.13.6. The MATLAB Tensor Toolbox (Version 2.6) [2] is used for computing the Z-eigenvalues and also the tensor-vector product.

There are many methods on computing the solution of the multi-linear system (1.1), but most of them are concerning with $\mathcal{M}$ tensors, such as the classical iterative methods for linear systems [12], homotopy method [14] and tensor methods [20,36]. Recently, Wang *et al.* [32] proposed a general approach which can compute the solution of all nonsingular multi-linear system. It is based on neural network and has the advantage of easy implementation. The method of this type is also used for computing the generalized eigenpairs of tensors [24].

We basically introduce neural network approach [32] and apply it to compute the solution of the strong $\mathcal{M}_z$-tensor equation (3.1). We firstly define an error function as follows:

$$E\big(\mathbf{x}(t)\big) = \mathcal{A}\mathbf{x}^{m-1}(t) - \mathbf{b}.$$

The residual error is represented by

$$\varepsilon(t) = \varepsilon\big(x(t)\big) = \frac{1}{2}\,\|E(\mathbf{x}(t))\|_2^2 = \frac{1}{2}\big\|\mathcal{A}\mathbf{x}^{m-1}(t) - \mathbf{b}\big\|_2^2.$$

To let the error converges to zero, the gradient neural network based on gradientdescent direction is a good choice, that is, we set

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\gamma \frac{\partial\big(\varepsilon(t)\big)}{\partial \mathbf{x}},$$

where $\gamma > 0$ is a parameter to control the convergence rate. Then we get the linear gradient neural network model:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\gamma(m-1)\big(\overline{\mathcal{A}}\mathbf{x}^{m-2}(t)\big)^\top\big(\mathcal{A}\mathbf{x}^{m-1}(t)-\mathbf{b}\big). \tag{4.1}$$

Generally, the convergence rate of the model can be improved by using an appropriate activation function. For illustration, we introduce two nonlinear activation functions, they are:

▷ power-sigmoid activation function (PSAF) defined as

$$f_{\mathrm{ps}}(x,p,q) = \begin{cases} x^p, & \text{if } |x| \geq 1, \\ \dfrac{1+\exp(-q)}{1-\exp(-q)}\dfrac{1-\exp(-qx)}{1+\exp(-qx)}, & \text{otherwise} \end{cases} \quad \text{with } q \geq 2 \text{ and } p \geq 3;$$

▷ smooth power-sigmoid activation function (SPSAF) defined as

$$f_{\mathrm{sps}}(x) = \frac{1}{2}x^p + \frac{1+\exp(-q)}{1-\exp(-q)}\frac{1-\exp(-qx)}{1+\exp(-qx)} \quad \text{with } q > 2 \text{ and } p \geq 3.$$

And we get the nonlinear gradient neural network of model (4.1) by using the nonlinear activation functions, that is

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\alpha\left(\overline{\mathcal{A}}\mathbf{x}^{m-2}(t)\right)^\top f_{**}\left(\mathcal{A}\mathbf{x}^{m-1}(t)-\mathbf{b}\right), \tag{4.2}$$

where $f_{**}(\mathbf{x}) := (f_{**}(x_1),\ldots,f_{**}(x_n))^\top$ and one can choose f_{**} as f_{ps} or f_{sps}.

Preconditioning is a good alternative in designing the methods. For a given nonzero $\mathbf{y} \in \mathbb{R}^n$, let $D(\mathbf{y}) = \mathrm{diag}(\mathbf{y}) + \lambda I$, then it can be nonsingular by choosing appropriate λ. We consider $D(\mathbf{y})\mathcal{A}\mathbf{x}^{m-1} = D(\mathbf{y})\mathbf{b}$ let $\mathbf{y} = \mathbf{x}$. By similar processes as above and let $\mathbf{y} = \mathbf{x}$ for $D(\mathbf{y})$, we get another model based on preconditioning technical, that is,

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = -\gamma B(\mathbf{x})^\top\left(D(\mathbf{x})\mathcal{A}\mathbf{x}^{m-1}(t) - D(\mathbf{x})\mathbf{b}\right) \tag{4.3}$$

where $B(\mathbf{x}) = \mathrm{diag}(\mathcal{A}\mathbf{x}^{m-1}(t)-\mathbf{b}) + (m-1)D(\mathbf{x}), \overline{\mathcal{A}}\mathbf{x}^{m-2}(t)$.

We are ready for the numerical examples for illustrating our theoretical results.

Example 4.1. Similar to [12, Example 4.2], let $\mathcal{B} = (b_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$ be a tensor with entries

$$b_{i_1i_2i_3i_4} = |\tan(i_1+i_2+i_3+i_4)|.$$

Then tensor $\mathcal{B}$ is symmetric. It can be computed that $\rho_z(\mathcal{B}) \approx 7.098$. Let $s = 7.2$ and choose the symmetric Z-identity tensor $\mathcal{I}_z$ as (2.3). Then $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$ is a symmetric strong $\mathcal{M}_z$-tensor. Undoubtedly, this tensor is not an $\mathcal{M}$-tensor since the entries like $\mathcal{A}(i,i,j,j)$ may not non-positive by (2.3). We choose different right-hand side vector $\mathbf{b}$, the nonnegative solutions of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ are shown in Table 1, where $rand_{num}(2,1)$ means that we choose "num" as the seed before generated $rand(2,1)$.

Table 1: Nonnegative solution of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ of Example 4.1.

$\mathbf{b}$	Nonnegative solutions of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$
$(0,0)^\top$	$(0,0)^\top$
$(0,1)^\top$	$(0.3926, 2.0876)^\top$
$(1,0)^\top$	$(0.8425, 0.6148)^\top$
$(1,1)^\top$	$(0.4827, 2.1899)^\top$
$rand_{100}(2,1)$	$(0.5775, 0.4696)^\top$
$rand_{1000}(2,1)$	$(0.8750, 0.8710)^\top$, $(0.6278, 1.1979)^\top$, $(0.4367, 1.7554)^\top$

Multi-linear system with strong $\mathcal{M}_z$-tensor $\mathcal{A} = s\mathcal{I}_z - \mathcal{B}$, where $\mathcal{B}$ is weakly symmetric and irreducible is considered below.

Example 4.2. The Z-identity tensor $\mathcal{I}_z \in \mathbb{R}^{[4,2]}$ given by (2.2) is weakly symmetric; see Remark 3.4 of [4]. Suppose that $\mathcal{B}_1 \in \mathbb{R}^{[4,2]}$ is a tensor with all entries are ones and let $\mathcal{B} = \mathcal{I}_z + \mathcal{B}_1$, then $\mathcal{B}$ is weakly symmetric and irreducible. This tensor is not an $\mathcal{M}$-tensor by (2.2). It can be computed that $\rho_z(\mathcal{B}) = 5$. Then $\mathcal{A} = 5.2 * \mathcal{I}_z - \mathcal{B}$ is a strong $\mathcal{M}_z$-tensor. We choose different nonnegative vector $\mathbf{b}$, the nonnegative solutions of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ are shown in Table 2.

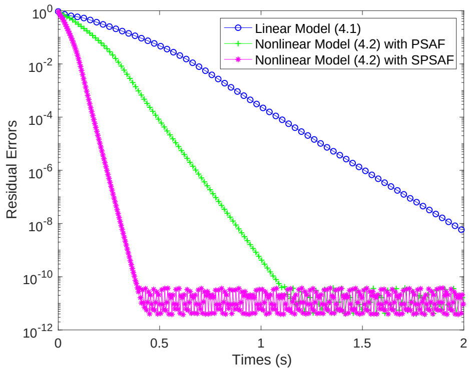
Table 2: Nonnegative solution of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ of Example 4.2.

$\mathbf{b}$	Nonnegative solutions of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$
$(0,0)^\top$	$(0,0)^\top$
$(0,1)^\top$	$(0.5296, 0.5854)^\top$
$(1,0)^\top$	$(0.7156, 0.1195)^\top$
$(1,1)^\top$	$(0.8120, 0.4084)^\top$
$rand_{100}(2,1)$	$(0.4863, 0.1178)^\top$
$rand_{1000}(2,1)$	$(0.7466, 0.2820)^\top$

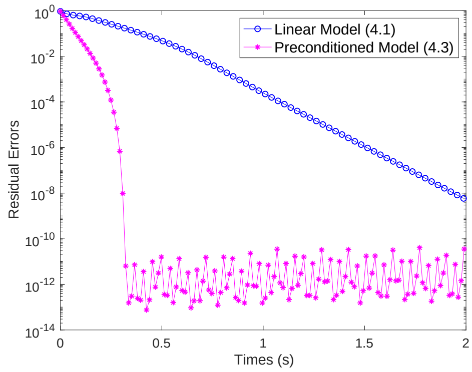
An example arises from discretizing the differential equations of the general form $F(t, u(t), u'(t) = 0)$ is given below.

Example 4.3. Let us consider $u(t)^\lambda (u'(t))^2 = -f$, where λ is a positive integer. By using Euler's method for discretization, we have

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)(t) \approx \frac{1}{h}\big(u(t+h) - u(t)\big) := \frac{1}{h}\big(u_{i+1} - u_i\big).$$



a) Models (4.1) and (4.2)



b) Models (4.1) and (4.3)

Figure 1: Residual error's comparison of different models (4.1), (4.2) and (4.3) for Example 4.3.

Then we get

$$u_i^\lambda u_{i+1}^2 - 2u_i^{\lambda+1}u_{i+1} + u_i^{\lambda+2} = -h^2 f_i.$$

For simplicity, let $\lambda = 1$, then we get the coefficient tensor $\mathcal{A}$ with all zero entries except that

$$\begin{cases} \mathcal{A}(i,i,i,i) = 1, & \text{for} \quad i = 1,\dots,n, \\ \mathcal{A}(i,i,i+1,i+1) = 1, & \text{for} \quad i = 1,\dots,n-1, \\ \mathcal{A}(i,i,i,i+1) = -2, & \text{for} \quad i = 1,\dots,n-1. \end{cases}$$

Let $s > 1$ and $\mathcal{B} = s\mathcal{I}_z - \mathcal{A}$, then $\mathcal{B}$ is nonnegative. This tensor $\mathcal{A}$ with order four and dimension j, where $j = 2,3,\dots,10$ is an strong $\mathcal{M}_z$-tensor if we set $s = 1$; refer to [23] for more details. By giving different right-hand side vector $\mathbf{b}$, we compute the solution by using the methods introduced in the beginning of this section, that is, models (4.1), (4.2) and (4.3).

We define the residual error as $\mathrm{RES} = \|\mathcal{A}\mathbf{x}(t)^{m-1} - \mathbf{b}\|_2$. Let $p = 3$ and $q = 5$ for the power-sigmoid and smooth power-sigmoid activation functions. We choose the strong $\mathcal{M}_z$-tensor with order $m = 4$ and dimension $n = 8$. The parameters for controlling the convergence rate are set as $\alpha = \gamma = 1000$. We firstly give a solution which is $\mathbf{x} = [1,2,\dots,8]^\top$, and then we get $\mathbf{b}$ by computing $\mathcal{A}\mathbf{x}^{m-1}$. We need one initial value by using the three models and it is $\mathbf{x}_0 = [2,3,\dots,9]^\top$. The solution is computed on MATLAB and the residual results are given in Figs. 1(a) and 1 (b).

5. Concluding remarks

We prove that the multi-linear system with strong $\mathcal{M}_z$-tensors always has a nonnegative solution under certain condition and some numerical examples are given for illustration. As a by-product of the conclusion about strong $\mathcal{M}_z$-tensor equation, we find that the homogeneous multi-linear system always and only has the zero solution for some structured tensors, which is an interesting result as it is the same as the homogeneous nonsingular linear equations. However, whether or not this result holds for tensors with nonzero determinant, further studies are needed. The randomized algorithms [8–10,35] for solving the multi-linear system will be reported in the forthcoming papers.

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