

Convergence and Stability of the Truncated Euler-Maruyama Method for Stochastic Differential Equations with Piecewise Continuous Arguments

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Received 21 July 2019; Accepted (in revised version) 4 July 2020

Abstract. In this paper, we develop the truncated Euler-Maruyama (EM) method for stochastic differential equations with piecewise continuous arguments (SDEPCAs), and consider the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. The order of convergence is obtained. Moreover, we show that the truncated EM method can preserve the exponential mean square stability of SDEPCAs. Numerical examples are provided to support our conclusions.

AMS subject classifications: 65C30

Key words: Stochastic differential equations with piecewise continuous argument, local Lipschitz condition, Khasminskii-type condition, truncated Euler-Maruyama method, convergence and stability.

1. Introduction

Differential equations with piecewise continuous arguments (EPCAs) have the following form

$$dx(t) = a(x(t), x(u(t)))dt, \quad (1.1)$$

where the argument $u(t)$ has the intervals of constancy, such as $u(t) = [t], [t-n], t-n[t]$. EPCAs have plenty of useful applications in the stabilization of hybrid control systems with feedback discrete controller [9, 17, 25, 28]. Note that $[t]$ is a discontinuous function and the solutions are determined by a finite set of initial data, rather than an initial function. Moreover, EPCAs represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations [27].

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However, [21] pointed out that substantial theories of functional differential equations did not exist for differential equations with piecewise constant or piecewise continuous arguments. Therefore, it is necessary to extend the theory of delay differential or functional differential equations with continuous arguments to equations with discontinuous arguments.

Actually, the effects of noise are not negligible for some systems [22]. Therefore, the stochastic differential equations with piecewise continuous arguments (SDEPCAs) are used to describe many practical problems. It has been widely used in the neural networks, control theory and so on [11, 23, 28]. Since there is, in general, no explicit solutions to SDEPCAs, numerical solutions are required in practice. The convergence and stability of numerical methods are the central properties in numerical analysis, and it is well studied for SDEPCAs under global Lipschitz and linear growth conditions, such as [20]. However, many systems do not satisfy the global Lipschitz or the linear growth conditions. Song *et al.* [24] investigated the convergence of the tamed Euler method for SDEPCAs under non-global Lipschitz continuous coefficients.

According to [7], the classical explicit Euler method diverges when the coefficients of stochastic differential equations (SDEs) are super-linearly growing. To overcome this difficulty, some scholars turn to use implicit Euler methods, such as the semi-implicit Euler-Maruyama method, the backward Euler-Maruyama method and the split-step theta (SST) method [2, 4, 6, 19]. However, compared with explicit methods, implicit methods require more additional computational efforts and cost much more time. Therefore, many scholars pay attention to find an explicit method which will converge in the strong sense when the coefficients of SDEs satisfy the super-linear growth condition. At present, there are some effective methods, such as the stopped Euler method [12], the tamed Euler method [1, 8, 26], the truncated Euler-Maruyama method [5, 16, 18], the partially truncated Euler-Maruyama method [3] and the modified truncated Euler-Maruyama method [10]. In this paper, we develop the truncated Euler-Maruyama (EM) method for SDEPCAs, and consider the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. The order of convergence is obtained.

Guo *et al.* [3] investigated the mean square exponential stability of the partially truncated Euler-Maruyama method for SDEs. Later, Lan and Xia [10] gave the exponential stability of the modified truncated Euler-Maruyama method for SDEs. Hu *et al.* [5] obtained the asymptotic stability of the truncated Euler-Maruyama method for SDEs. In this paper, we show that the truncated Euler-Maruyama method preserves the exponential mean square stability of the SDEPCAs.

The rest of this article is organized as follows. Section 2 introduces some basic assumptions, definitions and properties of the exact solution. We construct the truncated Euler-Maruyama method in Section 3. The p -th moment boundedness of the numerical solutions and the convergence theorem are presented in Section 4. Section 5 obtains the convergence order and Section 6 shows the exponential stabilities of both the SDEPCAs and the numerical solutions. Numerical simulations are provided to verify the analytical theory in Section 7.

2. Analysis of exact solutions

Throughout this paper, unless otherwise specified, we will use the following notations. If \mathbf{A} is a vector or matrix, its transpose is denoted by \mathbf{A}^T . If $x \in \mathbf{R}^n$, then $|x|$ is the Euclidean norm. If \mathbf{A} is a matrix, we let $|\mathbf{A}| = \sqrt{\text{trace}(\mathbf{A}^T \mathbf{A})}$ be its trace norm. For two real numbers a and b , we use $a \vee b$ and $a \wedge b$ to denote $\max(a, b)$ and $\min(a, b)$, respectively. If D is a set, its indicator function is denoted by I_D . Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while \mathcal{F}_0 contains all \mathbb{P} -null sets), and let \mathbb{E} denote the expectation corresponding to \mathbb{P} . Denote $\mathcal{L}^p([0, T]; \mathbf{R}^{n \times d})$ by the family of $\mathbf{R}^{n \times d}$ -valued \mathcal{F}_t -adapted processes $\{f(t)\}_{0 \leq t \leq T}$ such that $\int_0^T |f(t)|^p dt < \infty$, *a.s.*. Denote $\mathcal{L}^p([0, \infty), \mathbf{R}^{n \times d})$ by the family of the processes $\{f(t)\}_{t \geq 0}$ such that for every $T > 0$, $\{f(t)\}_{0 \leq t \leq T} \in \mathcal{L}^p([0, T]; \mathbf{R}^{n \times d})$. Let $B(t)$ be a d -dimensional Brownian motion defined on the space. Consider the n -dimensional SDEPCAs

$$dx(t) = f(x(t), x([t]))dt + g(x(t), x([t]))dB(t) \quad (2.1)$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbf{R}^n$, where $f : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$, $g : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^{n \times d}$ are measurable functions and $[t]$ denotes the greatest-integer part of t . Eq. (2.1) is equivalent to the following stochastic integral equation:

$$x(t) = x_0 + \int_0^t f(x(s), x([s]))ds + \int_0^t g(x(s), x([s]))dB(s), \quad \forall t \geq 0. \quad (2.2)$$

Moreover, we also require that the coefficients f and g are sufficiently smooth. To guarantee the existence of the global solution, we present the following assumptions.

Assumption 2.1. (The local Lipschitz condition.) For every integer $R \geq 0$, there exists a positive constant L_R , such that

$$|f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \leq L_R(|x - \bar{x}| + |y - \bar{y}|) \quad (2.3)$$

for all $x, \bar{x}, y, \bar{y} \in \mathbf{R}^n$, $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

Assumption 2.2. (Khasminskii-type condition.) There are two constants $K > 0$ and $p \geq 2$, such that

$$x^T f(x, y) + \frac{p-1}{2} |g(x, y)|^2 \leq K(1 + |x|^2 + |y|^2) \quad (2.4)$$

for all $x, y \in \mathbf{R}^n$.

The definition of the exact solution of (2.1) is given in the following.

Definition 2.1. ([24]) An \mathbf{R}^n -valued stochastic process $\{x(t)\}$ is called a solution of Eq. (2.1) on $[0, \infty)$ if it has the following properties:

- (1) $\{x(t)\}$ is continuous on $[0, \infty)$ and \mathcal{F}_t -adapted;
- (2) $\{f(x(t), x([t]))\} \in \mathcal{L}^1([0, \infty), \mathbf{R}^n)$ and $\{g(x(t), x([t]))\} \in \mathcal{L}^2([0, \infty), \mathbf{R}^{n \times d})$;
- (3) Eq. (2.2) is satisfied on each interval $[n, n + 1) \subset [0, \infty)$ with integer end-points almost surely.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$\mathbb{P}\{x(t) = \bar{x}(t), t \in [0, \infty)\} = 1.$$

We state a known result in [24] as a lemma for the use of this paper.

Lemma 2.1. *Under Assumptions 2.1 and 2.2, the SDEPCAs (2.1) have a unique global solution $x(t)$. Moreover,*

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{2.5}$$

Lemma 2.2. *Let Assumptions 2.1 and 2.2 hold. Define the stopping time*

$$\tau_R = \inf \{t \geq 0 : |x(t)| \geq R\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Then for any $R > |x_0|$, we have

$$\mathbb{P}\{\tau_R \leq T\} \leq \frac{C}{R^2}, \tag{2.6}$$

where C stands for a generic positive real constant dependent on T, K, x_0 , but independent of R and its value may change between occurrences.

Proof. By the Itô formula and Assumption 2.2, we obtain

$$\begin{aligned} \mathbb{E}|x(t \wedge \tau_R)|^2 &\leq |x_0|^2 + \mathbb{E} \int_0^{t \wedge \tau_R} 2K \left(1 + |x(s)|^2 + |x([s])|^2\right) ds \\ &\leq |x_0|^2 + 2KT + 4K \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x(u \wedge \tau_R)|^2 ds, \quad \forall 0 \leq t \leq T. \end{aligned}$$

For any $0 \leq t \leq T$, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t \wedge \tau_R)|^2 \leq |x_0|^2 + 2KT + 4K \int_0^T \sup_{0 \leq u \leq s} \mathbb{E}|x(u \wedge \tau_R)|^2 ds.$$

Applying the Gronwall inequality, we get

$$\mathbb{E}|x(T \wedge \tau_R)|^2 \leq C.$$

This implies

$$R^2 \mathbb{P}\{\tau_R \leq T\} \leq C$$

and the assertion follows. □

3. The truncated Euler-Maruyama method

To define the truncated Euler-Maruyama numerical solutions, we choose a strictly increasing continuous function $\mu : [1, \infty) \rightarrow \mathbf{R}_+$ such that $\mu(r) \rightarrow \infty$ as $r \rightarrow \infty$ and

$$\sup_{|x| \vee |y| \leq r} (|f(x, y)| \vee |g(x, y)|) \leq \mu(r), \quad \forall r \geq 1. \quad (3.1)$$

The inverse function of μ is denoted by μ^{-1} and we see that μ^{-1} is a strictly increasing continuous function. We also choose a constant $\hat{h} \geq 1$ and a strictly decreasing function $h : (0, 1] \rightarrow [\mu(1), \infty)$ such that

$$\lim_{\Delta \rightarrow 0} h(\Delta) = \infty, \quad \Delta^{\frac{1}{4}} h(\Delta) \leq \hat{h}, \quad \Delta \in (0, 1]. \quad (3.2)$$

It follows from $h(\Delta) \geq \mu(1)$ that $\mu^{-1}(h(\Delta)) \geq 1$.

For a given step size $\Delta \in (0, 1]$, define the mapping $\pi_\Delta : \mathbf{R}^n \rightarrow \mathbf{R}^n$ by

$$\pi_\Delta(x) = \left(|x| \wedge \mu^{-1}(h(\Delta)) \right) \frac{x}{|x|},$$

where we set $\frac{x}{|x|} = 0$ when $x = 0$. We then define the truncated functions

$$f_\Delta(x, y) = f(\pi_\Delta(x), \pi_\Delta(y)), \quad g_\Delta(x, y) = g(\pi_\Delta(x), \pi_\Delta(y)), \quad \forall x, y \in \mathbf{R}^n. \quad (3.3)$$

It is easy to see that

$$|f_\Delta(x, y)| \vee |g_\Delta(x, y)| \leq \mu\left(\mu^{-1}(h(\Delta))\right) = h(\Delta), \quad \forall x, y \in \mathbf{R}^n. \quad (3.4)$$

The following lemma shows that these truncated functions preserve the Khasminskii-type condition (2.4) for all $\Delta \in (0, 1]$.

Lemma 3.1. *Let Assumption 2.2 hold. Then for all $\Delta \in (0, 1]$, we have*

$$x^\top f_\Delta(x, y) + \frac{p-1}{2} |g_\Delta(x, y)|^2 \leq 3K \left(1 + |x|^2 + |y|^2 \right), \quad x, y \in \mathbf{R}^n. \quad (3.5)$$

Proof. Case 1. If $x \in \mathbf{R}^n$ with $|x| \leq \mu^{-1}(h(\Delta))$, then $\pi_\Delta(x) = x$. Moreover, for any $y \in \mathbf{R}^n$, we have

$$|\pi_\Delta(y)| = \left| \left(|y| \wedge \mu^{-1}(h(\Delta)) \right) \frac{y}{|y|} \right| \leq |y|.$$

Therefore

$$\begin{aligned} & x^\top f_\Delta(x, y) + \frac{p-1}{2} |g_\Delta(x, y)|^2 \\ &= \pi_\Delta^\top(x) f(\pi_\Delta(x), \pi_\Delta(y)) + \frac{p-1}{2} |g(\pi_\Delta(x), \pi_\Delta(y))|^2 \\ &\leq K \left(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 \right) \\ &\leq K \left(1 + |x|^2 + |y|^2 \right). \end{aligned}$$

Case 2. If $x \in \mathbf{R}^n$ with $|x| > \mu^{-1}(h(\Delta)) \geq 1$, then

$$\pi_{\Delta}(x) = \left(|x| \wedge \mu^{-1}(h(\Delta))\right) \frac{x}{|x|} = \mu^{-1}(h(\Delta)) \frac{x}{|x|}.$$

For any $y \in \mathbf{R}^n$, we have

$$\begin{aligned} & x^T f_{\Delta}(x, y) + \frac{p-1}{2} |g_{\Delta}(x, y)|^2 \\ &= \pi_{\Delta}^T(x) f(\pi_{\Delta}(x), \pi_{\Delta}(y)) + \frac{p-1}{2} |g(\pi_{\Delta}(x), \pi_{\Delta}(y))|^2 \\ &\quad + (x - \pi_{\Delta}(x))^T f(\pi_{\Delta}(x), \pi_{\Delta}(y)) \\ &\leq K \left(1 + |\pi_{\Delta}(x)|^2 + |\pi_{\Delta}(y)|^2\right) \\ &\quad + \left(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\right) \pi_{\Delta}^T(x) f(\pi_{\Delta}(x), \pi_{\Delta}(y)). \end{aligned}$$

By Assumption 2.2, we obtain

$$\pi_{\Delta}^T(x) f(\pi_{\Delta}(x), \pi_{\Delta}(y)) \leq K \left(1 + |\pi_{\Delta}(x)|^2 + |\pi_{\Delta}(y)|^2\right).$$

Thus

$$\begin{aligned} & x^T f_{\Delta}(x, y) + \frac{p-1}{2} |g_{\Delta}(x, y)|^2 \\ &\leq \frac{|x|}{\mu^{-1}(h(\Delta))} K \left(1 + |\pi_{\Delta}(x)|^2 + |\pi_{\Delta}(y)|^2\right) \\ &\leq K \left(|x| + |x|^2 + |x||y|\right) \\ &\leq 3K \left(1 + |x|^2 + |y|^2\right). \end{aligned}$$

The proof is completed. □

Let $\Delta = \frac{1}{m}$ be a given step size with integer $m > 0$. Let $T = N\Delta, N \in \mathbb{N}^+$ and the grid points t_k can be defined by $t_k = k\Delta, k = 0, \dots, N - 1$. Since for each k , there exist $s \in \mathbb{N}$ and $l \in \{0, \dots, m - 1\}$ such that $k = sm + l$. The truncated Euler-Maruyama method is given by

$$\begin{aligned} X_{\Delta}(t_{sm+l+1}) &= X_{\Delta}(t_{sm+l}) + f_{\Delta}(X_{\Delta}(t_{sm+l}), X_{\Delta}(t_{sm}))\Delta \\ &\quad + g_{\Delta}(X_{\Delta}(t_{sm+l}), X_{\Delta}(t_{sm}))\Delta B_{sm+l}, \end{aligned} \tag{3.6}$$

where $X_{\Delta}(0) = x_0, \Delta B_{sm+l} = B(t_{sm+l+1}) - B(t_{sm+l})$ and $X_{\Delta}(t_{sm+l})$ is the approximation to $x(t)$ at $t = t_{sm+l}$.

The two types of continuous-time truncated Euler-Maruyama solutions are defined as follows. For any $t \geq 0$,

$$\bar{x}_\Delta(t) = \sum_{sm+l=0}^{\infty} X_\Delta(t_{sm+l})I_{[t_{sm+l}, t_{sm+l+1})}(t), \tag{3.7}$$

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) ds + \int_0^t g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) dB(s). \tag{3.8}$$

4. Convergence of the truncated Euler-Maruyama method

In this section, we will show that the truncated Euler-Maruyama method converges to the exact solution of Eq. (2.1). In order to get our conclusions, we need some useful lemmas. The first lemma shows that $x_\Delta(t)$ and $\bar{x}_\Delta(t)$ are arbitrarily closed to each other.

Lemma 4.1. *Assume that Assumptions 2.1, 2.2 and (3.2) hold. For any $\Delta \in (0, 1]$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C\Delta^{\frac{p}{2}}(h(\Delta))^p, \quad \forall t \geq 0, \tag{4.1}$$

where C is a positive constant that depends only on p , but is independent of Δ and its value may change between occurrences. Consequently,

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p = 0, \quad \forall t \geq 0. \tag{4.2}$$

Proof. For any fixed $\Delta \in (0, 1]$ and $t \in [t_{sm+l}, t_{sm+l+1})$, we have $\bar{x}_\Delta(t) = X_\Delta(t_{sm+l})$ and

$$\begin{aligned} & x_\Delta(t) - \bar{x}_\Delta(t) \\ &= \int_{t_{sm+l}}^t f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) ds + \int_{t_{sm+l}}^t g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) dB(s). \end{aligned} \tag{4.3}$$

By the properties of the Itô integral, Hölder inequality, Theorem 1.7.1 in [15] and (3.4), we then obtain

$$\begin{aligned} & \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \\ & \leq 2^{p-1} \Delta^{p-1} \mathbb{E} \int_{t_{sm+l}}^t |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^p ds \\ & \quad + 2^{p-1} \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \Delta^{\frac{p-2}{2}} \mathbb{E} \int_{t_{sm+l}}^t |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^p ds \\ & \leq C\Delta^{\frac{p}{2}}(h(\Delta))^p, \end{aligned} \tag{4.4}$$

where

$$C = 2^{p-1} \left(1 + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \right).$$

Moreover, by (3.2), we have

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C\Delta^{\frac{p}{2}}(h(\Delta))^p \leq C\hat{h}^p\Delta^{\frac{p}{4}}. \quad (4.5)$$

We obtain (4.2) from (4.5) as $\Delta \rightarrow 0$. \square

Before giving the p -th moment boundedness of $x_\Delta(t)$, we define the stopping time

$$\rho_R = \inf \{t \geq 0 : |x_\Delta(t)| \geq R\} \quad (4.6)$$

and present the following lemma.

Lemma 4.2. *Assume that Assumptions 2.1, 2.2 and (3.2) hold. For any $R > |x_0|$ and $\Delta \in (0, 1]$, we have*

$$\mathbb{P}\{\rho_R \leq T\} \leq \frac{C}{R^2}, \quad (4.7)$$

where C is a constant independent of R and Δ .

Proof. By the Itô formula, we have

$$\begin{aligned} & \mathbb{E}|x_\Delta(t \wedge \rho_R)|^2 \\ &= |x_0|^2 + 2\mathbb{E} \int_0^{t \wedge \rho_R} \left(x_\Delta^\top(s) f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) + \frac{1}{2} |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right) ds \\ &= |x_0|^2 + 2\mathbb{E} \int_0^{t \wedge \rho_R} \left(\bar{x}_\Delta^\top(s) f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) + \frac{1}{2} |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right) ds \\ & \quad + 2\mathbb{E} \int_0^{t \wedge \rho_R} \left((x_\Delta(s) - \bar{x}_\Delta(s))^\top f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) \right) ds. \end{aligned} \quad (4.8)$$

Using Lemma 3.1, we have

$$\begin{aligned} & 2\mathbb{E} \int_0^{t \wedge \rho_R} \left(\bar{x}_\Delta^\top(s) f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) + \frac{1}{2} |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right) ds \\ & \leq 6K\mathbb{E} \int_0^{t \wedge \rho_R} \left(1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta([s])|^2 \right) ds \\ & \leq 6KT + 12K \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|\bar{x}_\Delta(u \wedge \rho_R)|^2 ds. \end{aligned} \quad (4.9)$$

By (3.4) and Lemma 4.1, the third term of (4.8) yields

$$2\mathbb{E} \int_0^{t \wedge \rho_R} \left((x_\Delta(s) - \bar{x}_\Delta(s))^\top f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) \right) ds \leq 2TC\hat{h}^2. \quad (4.10)$$

Combining (4.9) and (4.10), we have

$$\begin{aligned} & \mathbb{E}|x_\Delta(t \wedge \rho_R)|^2 \\ & \leq |x_0|^2 + 6KT + 2TC\hat{h}^2 + 12K \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|\bar{x}_\Delta(u \wedge \rho_R)|^2 ds. \end{aligned} \quad (4.11)$$

For any $0 \leq t \leq T$, applying the Gronwall inequality, we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t \wedge \rho_R)|^2 \leq C. \quad (4.12)$$

Then $\mathbb{E}|x_\Delta(T \wedge \rho_R)|^2 \leq C$, which implies that $R^2\mathbb{P}\{\rho_R \leq T\} \leq C$. We complete the proof. \square

Lemma 4.3. *Assume that Assumptions 2.1 and 2.2 hold, then there exists a positive constant C related to p, T, x_0 but independent of Δ such that*

$$\sup_{0 < \Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^p \leq C. \quad (4.13)$$

Proof. For any fixed $\Delta \in (0, 1]$, by the Itô formula, Hölder inequality, Lemma 4.1, (3.4) and (3.2), we derive

$$\begin{aligned} \mathbb{E}|x_\Delta(t)|^p &\leq |x_0|^p + p\mathbb{E} \int_0^t |x_\Delta(s)|^{p-2} \left[x_\Delta^\top(s) f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) \right. \\ &\quad \left. + \frac{p-1}{2} |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right] ds \\ &\leq |x_0|^p + 2pK\mathbb{E} \int_0^t |x_\Delta(s)|^{p-2} \left(1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta([s])|^2 \right) ds \\ &\quad + p\mathbb{E} \int_0^t |x_\Delta(s)|^{p-2} |x_\Delta(s) - \bar{x}_\Delta(s)| |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))| ds \\ &\leq |x_0|^p + T \left(C\hat{h} + 2^{\frac{p}{2}+1}K \right) \\ &\quad + \left((p-2)(1+2K) + 2^{\frac{p}{2}+2}K \right) \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^p ds. \end{aligned} \quad (4.14)$$

Applying the Gronwall inequality, we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^p \leq C. \quad (4.15)$$

Since (4.15) holds for any $\Delta \in (0, 1]$ and C is independent of Δ , we obtain the required assertion (4.13). \square

Theorem 4.1. *Let Assumptions 2.1 and 2.2 hold. Then for any $q \in [2, p)$, we obtain*

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|x_\Delta(T) - x(T)|^q = 0. \quad (4.16)$$

Proof. Let τ_R and ρ_R be the same as before. Set

$$\theta_R = \tau_R \wedge \rho_R, \quad e_\Delta(T) = x(T) - x_\Delta(T).$$

Using the Young inequality, we derive that for any $\delta > 0$,

$$\begin{aligned} \mathbb{E}|e_\Delta(T)|^q &= \mathbb{E}(|e_\Delta(T)|^q I_{\{\theta_R > T\}}) + \mathbb{E}(|e_\Delta(T)|^q I_{\{\theta_R \leq T\}}) \\ &\leq \mathbb{E}(|e_\Delta(T)|^q I_{\{\theta_R > T\}}) + \frac{q\delta}{p} \mathbb{E}|e_\Delta(T)|^p + \frac{p-q}{p\delta^{(p-q)}} \mathbb{P}\{\theta_R \leq T\}. \end{aligned} \quad (4.17)$$

By Lemmas 2.1 and 4.3, we obtain

$$\mathbb{E}|e_\Delta(T)|^p \leq C. \quad (4.18)$$

Applying Lemmas 2.2 and 4.2, we have

$$\mathbb{P}\{\theta_R \leq T\} \leq \mathbb{P}\{\tau_R \leq T\} + \mathbb{P}\{\rho_R \leq T\} \leq \frac{C}{R^2}. \quad (4.19)$$

By basic inequalities, we have

$$\begin{aligned} &\mathbb{E}|e_\Delta(t \wedge \theta_R)|^q \\ &= \mathbb{E} \left| \int_0^{t \wedge \theta_R} \left(f(x(s), x([s])) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) \right) ds \right. \\ &\quad \left. + \int_0^{t \wedge \theta_R} \left(g(x(s), x([s])) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s])) \right) dB(s) \right|^q \\ &\leq J_1 + J_2, \end{aligned} \quad (4.20)$$

where

$$\begin{aligned} J_1 &= (2T)^{q-1} \mathbb{E} \int_0^t |f(x(s \wedge \theta_R), x([s \wedge \theta_R])) \\ &\quad - f_\Delta(\bar{x}_\Delta(s \wedge \theta_R), \bar{x}_\Delta([s \wedge \theta_R]))|^q ds, \\ J_2 &= 2^{q-1} T^{\frac{q}{2}-1} \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} \mathbb{E} \int_0^t \left[|g(x(s \wedge \theta_R), x([s \wedge \theta_R])) \right. \\ &\quad \left. - g_\Delta(\bar{x}_\Delta(s \wedge \theta_R), \bar{x}_\Delta([s \wedge \theta_R])) \right|^q ds. \end{aligned} \quad (4.21)$$

Applying basic inequalities, we have

$$J_1 \leq (6T)^{q-1} (J_{11} + J_{12} + J_{13}), \quad (4.22)$$

where

$$\begin{aligned} J_{11} &= \mathbb{E} \int_0^t |f(x(s \wedge \theta_R), x([s \wedge \theta_R])) - f(x_\Delta(s \wedge \theta_R), x_\Delta([s \wedge \theta_R]))|^q ds, \\ J_{12} &= \mathbb{E} \int_0^t |f(x_\Delta(s \wedge \theta_R), x_\Delta([s \wedge \theta_R])) - f(\bar{x}_\Delta(s \wedge \theta_R), \bar{x}_\Delta([s \wedge \theta_R]))|^q ds, \\ J_{13} &= \mathbb{E} \int_0^t |f(\bar{x}_\Delta(s \wedge \theta_R), \bar{x}_\Delta([s \wedge \theta_R])) - f_\Delta(\bar{x}_\Delta(s \wedge \theta_R), \bar{x}_\Delta([s \wedge \theta_R]))|^q ds. \end{aligned} \quad (4.23)$$

By the local Lipschitz condition and basic inequalities, we have

$$\begin{aligned} J_{11} &\leq 2^{q-1} L_R^q \int_0^t \left[\mathbb{E} |x(s \wedge \theta_R) - x_\Delta(s \wedge \theta_R)|^q + \mathbb{E} |x([s \wedge \theta_R]) - x_\Delta([s \wedge \theta_R])|^q \right] ds \\ &\leq (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x(u \wedge \theta_R) - x_\Delta(u \wedge \theta_R)|^q ds. \end{aligned} \quad (4.24)$$

Similarly to J_{11} , applying Lemma 4.1, we derive

$$\begin{aligned} J_{12} &\leq 2^{q-1} L_R^q \int_0^t \mathbb{E} \left[|x_\Delta(s \wedge \theta_R) - \bar{x}_\Delta(s \wedge \theta_R)|^q + \mathbb{E} |x_\Delta([s \wedge \theta_R]) - \bar{x}_\Delta([s \wedge \theta_R])|^q \right] ds \\ &\leq (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u \wedge \theta_R) - \bar{x}_\Delta(u \wedge \theta_R)|^q ds \\ &\leq (2L_R)^q C T \Delta^{\frac{q}{2}} (h(\Delta))^q \leq C \hat{h}^q L_R^q \Delta^{\frac{q}{4}}. \end{aligned} \quad (4.25)$$

By local Lipschitz condition, Hölder inequality, Chebyshev's inequality and Lemma 4.3, we have

$$\begin{aligned} J_{13} &\leq (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |\bar{x}_\Delta(u \wedge \theta_R) - \pi_\Delta(\bar{x}_\Delta(u \wedge \theta_R))|^q ds \\ &\leq (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} (|\bar{x}_\Delta(u \wedge \theta_R)|^q I_{\{\bar{x}_\Delta(u \wedge \theta_R) > \mu^{-1}(h(\Delta))\}}) ds \\ &\leq (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \left(\mathbb{E} |\bar{x}_\Delta(u \wedge \theta_R)|^p \right)^{\frac{q}{p}} \left(\mathbb{P} \{ \bar{x}_\Delta(u \wedge \theta_R) > \mu^{-1}(h(\Delta)) \} \right)^{\frac{p-q}{p}} ds \\ &\leq C (2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \left(\frac{\mathbb{E} |\bar{x}_\Delta(u \wedge \theta_R)|^p}{|\mu^{-1}(h(\Delta))|^{p-q}} \right) ds \\ &\leq \frac{C (2L_R)^q}{|\mu^{-1}(h(\Delta))|^{p-q}}. \end{aligned} \quad (4.26)$$

Combining the Eqs. (4.22) and (4.24)-(4.26), we obtain

$$\begin{aligned} J_1 &\leq (6T)^{q-1} \left[(2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x(u \wedge \theta_R) - x_\Delta(u \wedge \theta_R)|^q ds \right. \\ &\quad \left. + C \hat{h}^q L_R^q \Delta^{\frac{q}{4}} + \frac{C (2L_R)^q}{|\mu^{-1}(h(\Delta))|^{p-q}} \right]. \end{aligned} \quad (4.27)$$

Similarly to J_1 , we have

$$\begin{aligned} J_2 &\leq 6^{q-1} T^{\frac{q}{2}-1} \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} \left[(2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x(u \wedge \theta_R) - x_\Delta(u \wedge \theta_R)|^q ds \right. \\ &\quad \left. + C \hat{h}^q L_R^q \Delta^{\frac{q}{4}} + \frac{C (2L_R)^q}{|\mu^{-1}(h(\Delta))|^{p-q}} \right]. \end{aligned} \quad (4.28)$$

Hence,

$$\begin{aligned} & \mathbb{E}|e_{\Delta}(t \wedge \theta_R)|^q \\ & \leq \left((6T)^{q-1} + 6^{q-1}T^{\frac{q}{2}-1} \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} \right) \\ & \quad \times \left((2L_R)^q \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \theta_R)|^q ds + C\hat{h}^q L_R^q \Delta^{\frac{q}{4}} + \frac{C(2L_R)^q}{|\mu^{-1}(h(\Delta))|^{p-q}} \right). \end{aligned} \quad (4.29)$$

By the Gronwall inequality, we have

$$\mathbb{E}|e_{\Delta}(t \wedge \theta_R)|^q \leq CL_R^q \left(\Delta^{\frac{q}{4}} + \frac{1}{|\mu^{-1}(h(\Delta))|^{p-q}} \right) e^{CL_R^q}. \quad (4.30)$$

Submitting Eq. (4.30) into Eq. (4.17), we obtain

$$\mathbb{E}|e_{\Delta}(T)|^q \leq CL_R^q \left(\Delta^{\frac{q}{4}} + \frac{1}{|\mu^{-1}(h(\Delta))|^{p-q}} \right) e^{CL_R^q} + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^2\delta^{\frac{q}{p-q}}}. \quad (4.31)$$

For any $\varepsilon > 0$, we can choose $\delta > 0$ such that $\frac{Cq\delta}{p} < \frac{\varepsilon}{3}$. Then we choose R sufficiently large for

$$\frac{C(p-q)}{pR^2\delta^{\frac{q}{p-q}}} < \frac{\varepsilon}{3}. \quad (4.32)$$

Finally, we can choose $\Delta > 0$ sufficiently small for

$$CL_R^q \left(\Delta^{\frac{q}{4}} + \frac{1}{|\mu^{-1}(h(\Delta))|^{p-q}} \right) e^{CL_R^q} < \frac{\varepsilon}{3}. \quad (4.33)$$

This implies (4.16) as desired. □

5. Convergence rate

Assumption 5.1. Assume that there is a pair of constants $q > 2$ and $K_1 > 0$ such that

$$\begin{aligned} & (x_1 - x_2)^T (f(x_1, y_1) - f(x_2, y_2)) + \frac{q-1}{2} |g(x_1, y_1) - g(x_2, y_2)|^2 \\ & \leq K_1 (|x_1 - x_2|^2 + |y_1 - y_2|^2) \end{aligned} \quad (5.1)$$

for all $x_1, x_2, y_1, y_2 \in \mathbf{R}^n$.

Assumption 5.2. Assume that there is a pair of positive constants r and K_2 such that

$$\begin{aligned} & |f(x_1, y_1) - f(x_2, y_2)|^2 \vee |g(x_1, y_1) - g(x_2, y_2)|^2 \\ & \leq K_2 (1 + |x_1|^r + |x_2|^r + |y_1|^r + |y_2|^r) (|x_1 - x_2|^2 + |y_1 - y_2|^2) \end{aligned} \quad (5.2)$$

for all $x_1, x_2, y_1, y_2 \in \mathbf{R}^n$.

Remark 5.1. According to Assumption 5.2, we can define the function μ as

$$\mu(R) = CR^{1+\frac{r}{2}}, \quad (5.3)$$

where the constant C is related to K_2 but independent of R .

Theorem 5.1. Let Assumptions 2.1, 2.2, 5.1 and 5.2 hold. Assume that $\bar{q} \in [2, q)$ and $q(2+r) \leq 2p$, then, for any $\Delta \in (0, 1]$, we have

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \leq C \left((\mu^{-1}(h(\Delta)))^{-(p-\bar{q})+\frac{\bar{q}r}{2}} + \Delta^{\frac{\bar{q}}{2}}(h(\Delta))^{\bar{q}} \right). \quad (5.4)$$

Proof. Let $e(t) = x(t) - x_\Delta(t)$ for $t \geq 0$. For each integer $R > |x_0|$, define the stopping time $\theta_R = \tau_R \wedge \rho_R$. By the Itô formula, for any $0 \leq t \leq T$, we have

$$\begin{aligned} & \mathbb{E}|e(t \wedge \theta_R)|^{\bar{q}} \\ & \leq \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q}|e(s)|^{\bar{q}-2} \left[e^T(s) (f(x(s), x([s])) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))) \right. \\ & \quad \left. + \frac{\bar{q}-1}{2} |g(x(s), x([s])) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right] ds \\ & \leq \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q}|e(s)|^{\bar{q}-2} \left[e^T(s) (f(x(s), x([s])) - f(x_\Delta(s), x_\Delta([s]))) \right. \\ & \quad \left. + \frac{\bar{q}-1}{2} \left(1 + \frac{q-\bar{q}}{q-1} \right) |g(x(s), x([s])) - g(x_\Delta(s), x_\Delta([s]))|^2 \right] ds \\ & \quad + \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q}|e(s)|^{\bar{q}-2} \left[e^T(s) (f(x_\Delta(s), x_\Delta([s])) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))) \right. \\ & \quad \left. + \frac{\bar{q}-1}{2} \left(1 + \frac{\bar{q}-1}{q-\bar{q}} \right) |g(x_\Delta(s), x_\Delta([s])) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right] ds. \quad (5.5) \end{aligned}$$

By Assumption 5.1, we have

$$\mathbb{E}|e(t \wedge \theta_R)|^{\bar{q}} \leq \bar{q}K_1 \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \theta_R)|^{\bar{q}} ds + J_3, \quad (5.6)$$

where

$$\begin{aligned} J_3 = & \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q}|e(s)|^{\bar{q}-2} \left[e^T(s) (f(x_\Delta(s), x_\Delta([s])) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))) \right. \\ & \left. + \frac{\bar{q}-1}{2} \left(1 + \frac{\bar{q}-1}{q-\bar{q}} \right) |g(x_\Delta(s), x_\Delta([s])) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right] ds. \quad (5.7) \end{aligned}$$

Rearranging J_3 , we obtain

$$J_3 \leq J_{31} + J_{32}, \quad (5.8)$$

where

$$J_{31} = \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q} |e(s)|^{\bar{q}-2} \left[e^T(s) (f(x_\Delta(s), x_\Delta([s])) - f_\Delta(x_\Delta(s), x_\Delta([s]))) \right. \\ \left. + \frac{(\bar{q}-1)(q-1)}{q-\bar{q}} |g(x_\Delta(s), x_\Delta([s])) - g_\Delta(x_\Delta(s), x_\Delta([s]))|^2 \right] ds, \quad (5.9)$$

$$J_{32} = \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q} |e(s)|^{\bar{q}-2} \left[e^T(s) (f_\Delta(x_\Delta(s), x_\Delta([s])) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))) \right. \\ \left. + \frac{(\bar{q}-1)(q-1)}{q-\bar{q}} |g_\Delta(x_\Delta(s), x_\Delta([s])) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta([s]))|^2 \right] ds. \quad (5.10)$$

By the Young inequality $a^{\bar{q}-2}b \leq \frac{\bar{q}-2}{\bar{q}}a^{\bar{q}} + \frac{2}{\bar{q}}b^{\frac{\bar{q}}{2}}$, $\forall a, b \geq 0$, we obtain

$$J_{31} \leq \mathbb{E} \int_0^{t \wedge \theta_R} \bar{q} |e(s)|^{\bar{q}-2} \left[\frac{1}{2} |e^T(s)|^2 + \frac{1}{2} |f(x_\Delta(s), x_\Delta([s])) - f_\Delta(x_\Delta(s), x_\Delta([s]))|^2 \right. \\ \left. + \frac{(\bar{q}-1)(q-1)}{q-\bar{q}} |g(x_\Delta(s), x_\Delta([s])) - g_\Delta(x_\Delta(s), x_\Delta([s]))|^2 \right] ds \\ \leq \mathbb{E} \int_0^{t \wedge \theta_R} \left[\frac{(q-2)(\bar{q}-1)^2}{q-\bar{q}} |e(s)|^{\bar{q}} + |f(x_\Delta(s), x_\Delta([s])) - f_\Delta(x_\Delta(s), x_\Delta([s]))|^{\bar{q}} \right. \\ \left. + \frac{2(\bar{q}-1)(q-1)}{q-\bar{q}} |g_\Delta(x_\Delta(s), x_\Delta([s])) - g_\Delta(x_\Delta(s), x_\Delta([s]))|^{\bar{q}} \right] ds \\ \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |e(u \wedge \theta_R)|^{\bar{q}} ds + J_{311}, \quad (5.11)$$

where

$$C = \max \left\{ \frac{(q-2)(\bar{q}-1)^2}{q-\bar{q}}, 1, \frac{2(\bar{q}-1)(q-1)}{q-\bar{q}} \right\},$$

and

$$J_{311} = C \mathbb{E} \int_0^{t \wedge \theta_R} \left[|f(x_\Delta(s), x_\Delta([s])) - f_\Delta(x_\Delta(s), x_\Delta([s]))|^{\bar{q}} \right. \\ \left. + |g(x_\Delta(s), x_\Delta([s])) - g_\Delta(x_\Delta(s), x_\Delta([s]))|^{\bar{q}} \right] ds. \quad (5.12)$$

Applying Assumption 5.2, we get

$$J_{311} \leq 2CK_2 \mathbb{E} \int_0^{t \wedge \theta_R} \left(1 + |x_\Delta(s)|^r + |x_\Delta([s])|^r + |\pi_\Delta(x_\Delta(s))|^r + |\pi_\Delta(x_\Delta([s]))|^r \right)^{\frac{\bar{q}}{2}} \\ \times \left(|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |x_\Delta([s]) - \pi_\Delta(x_\Delta([s]))|^2 \right)^{\frac{\bar{q}}{2}} ds \\ \leq 10^{\frac{\bar{q}}{2}} CK_2 \mathbb{E} \int_0^{t \wedge \theta_R} \left(1 + |x_\Delta(s)|^{\frac{r\bar{q}}{2}} + |x_\Delta([s])|^{\frac{r\bar{q}}{2}} + |\pi_\Delta(x_\Delta(s))|^{\frac{r\bar{q}}{2}} + |\pi_\Delta(x_\Delta([s]))|^{\frac{r\bar{q}}{2}} \right) \\ \times \left(|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\bar{q}} + |x_\Delta([s]) - \pi_\Delta(x_\Delta([s]))|^{\bar{q}} \right) ds$$

$$\begin{aligned}
&\leq 10^{\frac{\bar{q}}{2}+1}CK_2 \int_0^t \mathbb{E} \left(1 + |x_\Delta(s \wedge \theta_R)|^{\frac{r\bar{q}}{2}} + |x_\Delta([s \wedge \theta_R])|^{\frac{r\bar{q}}{2}} \right) \\
&\quad \times \left(|x_\Delta(s \wedge \theta_R) - \pi_\Delta(x_\Delta(s \wedge \theta_R))|^{\bar{q}} \right. \\
&\quad \left. + |x_\Delta([s \wedge \theta_R]) - \pi_\Delta(x_\Delta([s \wedge \theta_R]))|^{\bar{q}} \right) ds. \tag{5.13}
\end{aligned}$$

Applying Hölder inequality, we have

$$\begin{aligned}
J_{311} &\leq C \int_0^t \left(\mathbb{E} (1 + |x_\Delta(s \wedge \theta_R)|^p + |x_\Delta([s \wedge \theta_R])|^p) \right)^{\frac{\bar{q}r}{2p}} \\
&\quad \times \left(\mathbb{E} \left(|x_\Delta(s \wedge \theta_R) - \pi_\Delta(x_\Delta(s \wedge \theta_R))|^{\frac{2p\bar{q}}{2p-\bar{q}r}} \right. \right. \\
&\quad \left. \left. + |x_\Delta([s \wedge \theta_R]) - \pi_\Delta(x_\Delta([s \wedge \theta_R]))|^{\frac{2p\bar{q}}{2p-\bar{q}r}} \right) \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \int_0^t \left(1 + \sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u \wedge \theta_R)|^p \right)^{\frac{\bar{q}r}{2p}} \\
&\quad \times \left(\sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u \wedge \theta_R) - \pi_\Delta(x_\Delta(u \wedge \theta_R))|^{\frac{2p\bar{q}}{2p-\bar{q}r}} \right)^{\frac{2p-\bar{q}r}{2p}} ds. \tag{5.14}
\end{aligned}$$

By Lemma 4.3 and Hölder inequality, we derive

$$\begin{aligned}
J_{311} &\leq C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u \wedge \theta_R) - \pi_\Delta(x_\Delta(u \wedge \theta_R))|^{\frac{2p\bar{q}}{2p-\bar{q}r}} \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E} \left(|x_\Delta(u \wedge \theta_R)|^{\frac{2p\bar{q}}{2p-\bar{q}r}} I_{\{|x_\Delta(u \wedge \theta_R)| > \mu^{-1}(h(\Delta))\}} \right) \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \int_0^t \left(\sup_{0 \leq u \leq s} \left(\mathbb{E} |x_\Delta(u \wedge \theta_R)|^p \right)^{\frac{2\bar{q}}{2p-\bar{q}r}} \right. \\
&\quad \left. \times \left(\mathbb{E} (I_{\{|x_\Delta(u \wedge \theta_R)| > \mu^{-1}(h(\Delta))\}})^{\frac{2(p-\bar{q})-\bar{q}r}{2(p-\bar{q})-\bar{q}r}} \right)^{\frac{2(p-\bar{q})-\bar{q}r}{2p}} \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \int_0^t \left(\sup_{0 \leq u \leq s} \left(\mathbb{P} \{ |x_\Delta(u \wedge \theta_R)| > \mu^{-1}(h(\Delta)) \} \right)^{\frac{2(p-\bar{q})-\bar{q}r}{2p-\bar{q}r}} \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \int_0^t \left(\sup_{0 \leq u \leq s} \left(\frac{\mathbb{E} |x_\Delta(u \wedge \theta_R)|^p}{(\mu^{-1}(h(\Delta)))^p} \right)^{\frac{2(p-\bar{q})-\bar{q}r}{2p-\bar{q}r}} \right)^{\frac{2p-\bar{q}r}{2p}} ds \\
&\leq C \left(\mu^{-1}(h(\Delta)) \right)^{\bar{q}(\frac{r}{2}+1)-p}. \tag{5.15}
\end{aligned}$$

Substituting (5.15) into (5.11), we get

$$J_{31} \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \theta_R)|^{\bar{q}} ds + C \left(\mu^{-1}(h(\Delta)) \right)^{\bar{q}(\frac{r}{2}+1)-p}. \quad (5.16)$$

Similarly, we can show

$$J_{32} \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \theta_R)|^{\bar{q}} ds + C \Delta^{\frac{\bar{q}}{2}} (h(\Delta))^{\bar{q}}. \quad (5.17)$$

Combining (5.6), (5.8), (5.16) and (5.17), we obtain

$$\begin{aligned} & \mathbb{E}|e(t \wedge \theta_R)|^{\bar{q}} \\ & \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \theta_R)|^{\bar{q}} ds + C \left(\left(\mu^{-1}(h(\Delta)) \right)^{\bar{q}(\frac{r}{2}+1)-p} + \Delta^{\frac{\bar{q}}{2}} (h(\Delta))^{\bar{q}} \right). \end{aligned} \quad (5.18)$$

By Gronwall inequality, we have

$$\mathbb{E}|e(T \wedge \theta_R)|^{\bar{q}} \leq C \left(\left(\mu^{-1}(h(\Delta)) \right)^{\bar{q}(\frac{r}{2}+1)-p} + \Delta^{\frac{\bar{q}}{2}} (h(\Delta))^{\bar{q}} \right). \quad (5.19)$$

Using the well-known Fatou lemma, the required assertion (5.4) follows from $R \rightarrow \infty$. \square

Remark 5.2. (1) Assumption 5.1 implies Assumption 2.2 for any $p > 2$. In fact,

$$\begin{aligned} & x^T f(x, y) + \frac{p-1}{2} |g(x, y)|^2 \\ & \leq x^T (f(x, y) - f(0, 0)) + x^T f(0, 0) \\ & \quad + \frac{p-1}{2} |g(x, y) - g(0, 0)|^2 + \frac{p-1}{2} |g(0, 0)|^2 \\ & \leq K_1 (|x|^2 + |y|^2) + \frac{1}{2} |x|^2 + \frac{1}{2} |f(0, 0)|^2 + \frac{p-1}{2} |g(0, 0)|^2 \\ & \leq K (1 + |x|^2 + |y|^2), \end{aligned} \quad (5.20)$$

where

$$K = \max \left\{ K_1 + \frac{1}{2}, \frac{1}{2} |f(0, 0)|^2 + \frac{p-1}{2} |g(0, 0)|^2 \right\}.$$

(2) We may define

$$h(\Delta) = \Delta^{-\varepsilon} \quad (5.21)$$

for some $\varepsilon \in (0, \frac{1}{4}]$. Hence, (5.4) can be rewritten as

$$\begin{aligned} & \mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \\ & \leq C \left(\Delta^{\frac{\varepsilon[2(p-\bar{q})-\bar{q}r]}{2+r}} + \Delta^{\bar{q}(\frac{1}{2}-\varepsilon)} \right) \leq C \Delta^{\left(\frac{\varepsilon(2(p-\bar{q})-\bar{q}r)}{2+r} \wedge (\bar{q}(\frac{1}{2}-\varepsilon)) \right)}. \end{aligned} \quad (5.22)$$

If we choose p sufficiently large for

$$\frac{\varepsilon(2(p - \bar{q}) - \bar{q}r)}{2 + r} \geq \bar{q} \left(\frac{1}{2} - \varepsilon \right), \quad (5.23)$$

we can get

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \leq C\Delta^{\bar{q}(\frac{1}{2} - \varepsilon)}. \quad (5.24)$$

This remark shows that the order of convergence could be close to $\frac{1}{2}$ arbitrarily.

6. Stability analysis

In this section, we set $f(0, 0) = 0$ and $g(0, 0) = 0$. By Assumption 2.1, we have

$$\sup_{|x| \vee |y| \leq r} |f(x, y)| \leq 2rL_r. \quad (6.1)$$

In the following, we define

$$\mu(r) = 2rL_r, \quad (6.2)$$

where L_r is the local Lipschitz coefficient in Assumption 2.1. This operation links the local Lipschitz coefficients L_r to the truncation function $\mu(r)$, ensuring that the upper bound of the step size can be finally obtained, which is clear and easy to verify.

Definition 6.1. The SDEPCAs (2.1) is said to be exponentially stable in mean square if there exists a pair of positive constants λ and M such that for any initial value x_0

$$\mathbb{E}|x(t)|^2 \leq M|x_0|^2 e^{-\lambda t}, \quad \forall t > 0, \quad (6.3)$$

where λ is the rate constant and M is the growth constant.

Definition 6.2. For a given step size $\Delta > 0$, the truncated Euler-Maruyama method is said to be exponentially stable in mean square if there exists a pair of positive constants γ and H such that for any initial value x_0

$$\mathbb{E}|X_\Delta(t_{sm+l})|^2 \leq H|x_0|^2 e^{-\gamma(sm+l)\Delta} \quad (6.4)$$

for all $s \in \mathbb{N}, l = 0, \dots, m-1$, where γ is the rate constant and H is the growth constant.

Assumption 6.1. Assume that there are positive constants $\lambda_1 > \lambda_2 > 0$, such that f and g satisfy

$$x^T f(x, y) + \frac{1}{2}|g(x, y)|^2 \leq -\lambda_1|x|^2 + \lambda_2|y|^2, \quad \forall x, y \in \mathbf{R}^n. \quad (6.5)$$

Under Assumption 6.1, the exponential stability of SDEPCAs (2.1) has been obtained in [13].

Lemma 6.1. *Assume that Assumption 6.1 holds. Then, for all $\Delta \in (0, 1)$, we have*

$$x^T f_\Delta(x, y) + \frac{1}{2}|g_\Delta(x, y)|^2 \leq \left(-\lambda_1 + \frac{\lambda_2}{2}\right)|x|^2 + \frac{\lambda_2}{2}|y|^2, \quad \forall x, y \in \mathbf{R}^n, \quad (6.6)$$

where λ_1 and λ_2 are the same as Assumption 6.1.

The proof is similar to Lemma 3.1, so we omit it. In the following, we firstly state two conclusions obtained in [13] as lemmas for the use of this paper.

Lemma 6.2. *Let z_{sm+l} be a sequence of numbers, $s \in \mathbb{N}, l = 0, \dots, m - 1$. If there are constants $\alpha > \beta > 0$ such that $1 - \alpha\Delta > 0$ and*

$$z_{sm+l+1} \leq (1 - \alpha\Delta)z_{sm+l} + \beta\Delta z_{sm}, \quad (6.7)$$

then

$$z_{sm+l+1} \leq \left(\frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right)e^{-\alpha(l+1)\Delta}\right)z_{sm}. \quad (6.8)$$

Lemma 6.3. *Assume that α, β are two positive constants. If $\alpha > \beta$, then for all $t > 0$ we have*

$$0 < \frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right)e^{-\alpha t} < 1. \quad (6.9)$$

For simplicity of presentation, we set

$$\begin{aligned} \alpha &= 2\lambda_1 - \beta, & \beta &= 2L_{\mu^{-1}(h(\Delta))}^2\Delta + \lambda_2, \\ r(l+1) &= \frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right)e^{-\alpha(l+1)\Delta}, \end{aligned}$$

where $l = 0, \dots, m - 1$. The main theorem of this section is given below.

Theorem 6.1. *Let Assumptions 2.1 and 6.1 hold. Then for every $0 < \Delta < \Delta^* = \min\left\{\frac{1}{2\lambda_1}, \frac{4(\lambda_1 - \lambda_2)^2}{\hat{h}^4}\right\}$ and any initial value $x_0 \in \mathbf{R}^n$, the truncated Euler-Maruyama method (3.6) is exponentially stable in mean square with rate constant $\gamma = -\log r(m)$ and growth constant $H = \frac{1}{r(m)}$.*

Proof. By Assumption 2.1, Lemma 6.1 and the property of Brownian motion, we have

$$\begin{aligned} &\mathbb{E}|X_\Delta(t_{sm+l+1})|^2 \\ &\leq \mathbb{E}|X_\Delta(t_{sm+l})|^2 + 2L_{\mu^{-1}(h(\Delta))}^2\Delta^2\left(\mathbb{E}|X_\Delta(t_{sm+l})|^2 + \mathbb{E}|X_\Delta(t_{sm})|^2\right) \\ &\quad + 2\Delta\left(-\lambda_1 + \frac{\lambda_2}{2}\right)\mathbb{E}|X_\Delta(t_{sm+l})|^2 + \lambda_2\Delta\mathbb{E}|X_\Delta(t_{sm})|^2 \\ &= (1 - \alpha\Delta)\mathbb{E}|X_\Delta(t_{sm+l})|^2 + \beta\Delta\mathbb{E}|X_\Delta(t_{sm})|^2. \end{aligned} \quad (6.10)$$

By (6.2), we have

$$L_{\mu^{-1}(h(\Delta))} = \frac{h(\Delta)}{2\mu^{-1}(h(\Delta))}. \tag{6.11}$$

Recalling that $h(\Delta) > \mu(1)$, we have $\mu^{-1}(h(\Delta)) > 1$. Applying (3.2) and $\Delta < \frac{4(\lambda_1 - \lambda_2)^2}{h^4}$, we obtain

$$\beta = 2 \left(\frac{h(\Delta)}{2\mu^{-1}(h(\Delta))} \right)^2 \Delta + \lambda_2 < \lambda_1. \tag{6.12}$$

Hence $\alpha > \beta > 0$. By $\Delta < \frac{1}{2\lambda_1}$, we have

$$\alpha\Delta = (2\lambda_1 - \beta)\Delta < \frac{2\lambda_1 - \beta}{2\lambda_1} < 1. \tag{6.13}$$

Applying Lemma 6.2, we have

$$\begin{aligned} \mathbb{E}|X_\Delta(t_{sm+l+1})|^2 &\leq (1 - \alpha\Delta)\mathbb{E}|X_\Delta(t_{sm+l})|^2 + \beta\Delta\mathbb{E}|X_\Delta(t_{sm})|^2 \\ &\leq \left(\frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha} \right) e^{-\alpha(l+1)\Delta} \right) \mathbb{E}|X_\Delta(t_{sm})|^2 \\ &= r(l+1)\mathbb{E}|X_\Delta(t_{sm})|^2. \end{aligned} \tag{6.14}$$

By Lemma 6.3, we have $0 < r(l+1) < 1$. If $l = m - 1$, the above equation can be written as

$$\mathbb{E}|X_\Delta(t_{(s+1)m})|^2 \leq r(m)\mathbb{E}|X_\Delta(t_{sm})|^2 \leq \dots \leq (r(m))^{s+1}|x_0|^2. \tag{6.15}$$

Submitting (6.15) into (6.14), we can obtain

$$\begin{aligned} &\mathbb{E}|X_\Delta(t_{sm+l+1})|^2 \\ &\leq r(l+1)r(m)^s|x_0|^2 = \frac{r(l+1)}{r(m)^{(l+1)\Delta}}|x_0|^2 e^{(sm+l+1)\Delta \log(r(m))} \\ &\leq \frac{1}{r(m)} e^{(sm+l+1)\Delta \log(r(m))} |x_0|^2 = H e^{-\gamma(sm+l+1)\Delta} |x_0|^2, \end{aligned} \tag{6.16}$$

where $H = \frac{1}{r(m)}$ and $\gamma = -\log(r(m)) > 0$. □

7. Numerical simulation

This section gives some examples to illustrate the results above. The convergence theory is tested by the first two examples. The third example verifies the stability.

Example 7.1. We consider the following scalar SDEPCAs

$$dx(t) = \left(-x^3(t) + x([t]) \right) dt + \left(|x(t)|^{\frac{3}{2}} + x([t]) \right) dB(t) \tag{7.1}$$

with initial value $x(0) = 1, t \in [0, 2]$.

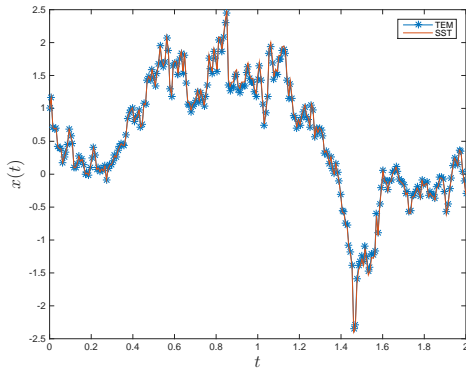


Figure 1: One path of the exact solution and the numerical solution for one dimension.

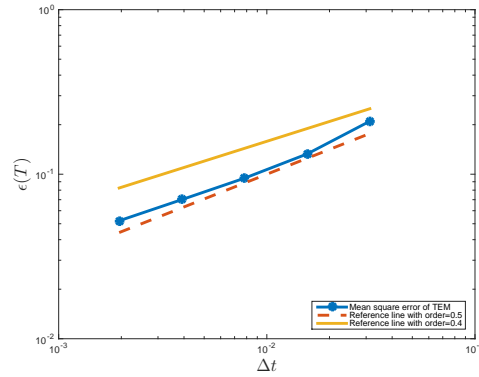


Figure 2: Convergence order at $T = 2$.

Let $\Delta = \frac{1}{m}$, then the split-step theta (SST) method for (7.1) is as follows.

$$\hat{x}_{sm+l} = x_{sm+l} + \theta \Delta \left(-\hat{x}_{sm+l}^3 + x_{sm} \right), \tag{7.2}$$

$$x_{sm+l+1} = x_{sm+l} + \Delta \left(-\hat{x}_{sm+l}^3 + x_{sm} \right) + \left(|\hat{x}_{sm+l}|^{\frac{3}{2}} + x_{sm} \right) \Delta B_{sm+l}. \tag{7.3}$$

According to [14], the SST method with $\theta \in [\frac{1}{2}, 1]$ is strongly convergent to SDEPCAs. We use the SST method with step size $\Delta = 2^{-14}$ and $\theta = 1$ as the "exact solution". The coefficients of (7.1) satisfy Assumptions 2.1, 2.2, 5.1 and 5.2 with $q = 3, \bar{q} = 2, r = 4$ and $p = 30$. Moreover, let $\mu(r) = 2r^3$ and $h(\Delta) = (\frac{125}{4})\Delta^{-\frac{1}{10}}$. Fig. 1 shows that the truncated EM method is convergent to the exact solution. We take 100 samples and choose $\Delta = 2^{-8}$. Table 1 shows the number of truncation applied in each sample. We also compute the numerical solutions using different step sizes $\Delta = 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}, 2^{-5}$ on the same Brownian path and obtain corresponding errors. We take $M = 5000$.

Let ϵ denote the error in mean square, then by the law of large numbers, at the final

Table 1: The number of truncation applied in each sample.

512	448	512	509	512	508	507	496	512	499
490	512	490	507	431	499	493	511	512	512
465	511	455	512	475	512	510	506	508	512
499	492	512	512	504	462	511	512	510	485
512	397	459	480	507	512	499	489	475	482
477	512	512	512	500	512	486	504	512	512
512	487	512	499	491	512	512	512	512	509
511	498	509	512	512	507	511	512	493	495
510	486	512	489	512	509	511	512	502	512
512	492	511	511	509	512	512	512	512	508

Table 2: The Monte Carlo errors for (7.1).

$\Delta t \backslash M$	1000	2000	3000	4000	5000
2^{-9}	0.00310153	0.00248716	0.00193821	0.00160219	0.00144283
2^{-8}	0.00380181	0.00312375	0.00244000	0.00205428	0.00193796
2^{-7}	0.00504908	0.00403356	0.00322514	0.00274647	0.00261020
2^{-6}	0.00783951	0.00554820	0.00464228	0.00391444	0.00364825
2^{-5}	0.01303230	0.00953383	0.00758430	0.00639328	0.00568892

integer time T , ϵ satisfies

$$\epsilon(T) = \left(\mathbb{E} |x(T) - X_\Delta(T)|^2 \right)^{\frac{1}{2}} \approx \left(\frac{1}{M} \sum_{i=1}^M |x(T, w_i) - X_\Delta(T, w_i)|^2 \right)^{\frac{1}{2}}. \tag{7.4}$$

The log-log plot of the strong errors against the step sizes is shown in Fig. 2. Compared with the reference line with order 0.5, it can be verified that the order of the strong error is close to 0.5.

The Monte Carlo errors with confidence level $1 - \alpha$ is defined as follows.

$$\epsilon = \frac{\lambda_\alpha \sigma}{\sqrt{M}}, \tag{7.5}$$

where λ_α and α are one-to-one correspondence, σ is the standard deviation of $|x(T) - X_\Delta(T)|^2$. Since σ can not be obtained exactly, we use the estimation $\hat{\sigma}$.

$$\hat{\sigma} = \sqrt{\frac{1}{M} \sum_{i=1}^M |x(T, w_i) - X_\Delta(T, w_i)|^4 - \left(\frac{1}{M} \sum_{i=1}^M |x(T, w_i) - X_\Delta(T, w_i)|^2 \right)^2}. \tag{7.6}$$

Table 2 shows the Monte Carlo error with $\alpha = 0.05$ and $\lambda_\alpha = 1.96$.

Example 7.2. Consider the following two dimensional SDEPCA:

$$\begin{cases} dx_1(t) = (-x_1(t) - 6x_1^3(t)) dt + |x_2(t)|^{\frac{3}{2}} dB(t), \\ dx_2(t) = (-x_2(t) - 6x_2^3(t) + x_1^2(t)) dt + |x_1(t)| dB(t) \end{cases} \tag{7.7}$$

with initial data $x_1(0) = 1.5$ and $x_2(0) = 1$.

Let $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (x_1([t]), x_2([t]))^T$ and

$$\begin{aligned} f_1(x, y) &= -x_1 - 6x_1^3, & f_2(x, y) &= -x_2 - 6x_2^3 + x_1^2, \\ g_1(x, y) &= |x_2|^{\frac{3}{2}}, & g_2(x, y) &= |y_1|, \\ f(x, y) &= (f_1(x, y), f_2(x, y))^T, & g(x, y) &= (g_1(x, y), g_2(x, y))^T. \end{aligned}$$

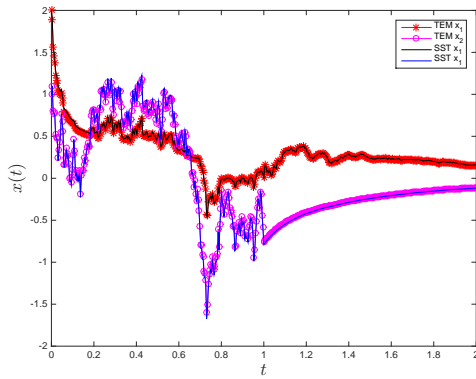


Figure 3: One path of the exact solution and the numerical solution for multi dimension.

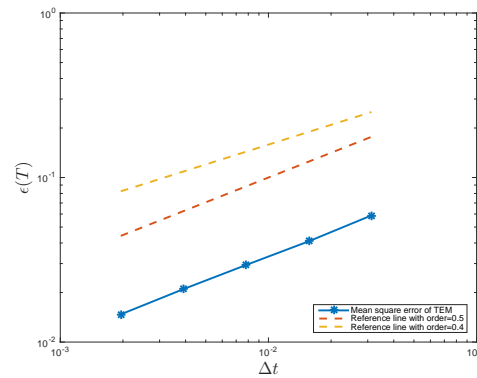


Figure 4: Convergence order at $T = 2$.

The coefficients of (7.7) satisfy Assumptions 2.1, 2.2, 5.1 and 5.2. Clearly,

$$\sup_{|x| \vee |y| \leq r} (|f(x, y)| \vee |g(x, y)|) \leq 14r^3, \quad \forall r \geq 1. \tag{7.8}$$

Then we choose $\mu(r) = 14r^3$. Moreover, choose $h(\Delta) = 56\Delta^{-\frac{1}{10}}$. We use the split-step theta method with step size $\Delta = 2^{-14}$ as the "exact solution". We also compute the numerical solutions using different step sizes $\Delta = 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}, 2^{-5}$ on the same Brownian path and obtain the corresponding errors. We take 5000 sample paths. Fig. 3 shows that the truncated EM method converges to the exact solution. The log-log plot of the strong errors against the step sizes is shown in Fig. 4. Table 3 shows the Monte Carlo error with confidence level 0.95.

Table 3: The Monte Carlo errors for (7.7).

$\Delta t \backslash M$	1000	2000	3000	4000	5000
2^{-9}	0.00003079	0.00002650	0.00002014	0.00001752	0.00001544
2^{-8}	0.00007564	0.00007700	0.00005478	0.00004534	0.00003861
2^{-7}	0.00010589	0.00010254	0.00008480	0.00007772	0.00006680
2^{-6}	0.00015667	0.00016543	0.00020037	0.00016448	0.00014073
2^{-5}	0.00046301	0.00032523	0.00027995	0.00024947	0.00022006

Example 7.3. Consider the following SDEPCAs

$$dx(t) = (-x^3(t)x^2([t]) - x(t)) dt + (x^2(t)x([t]) + 0.4x([t])) dB(t) \tag{7.9}$$

with initial data $x(0) = 8$.

The coefficients satisfy both Assumptions 2.1 and 6.1 with $\lambda_1 = 1, \lambda_2 = 0.16$. We test the stability of the truncated Euler-Maruyama method with three different

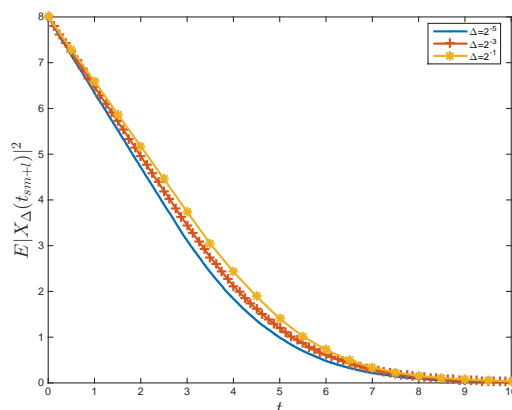


Figure 5: The mean square exponential stability of the numerical solutions with different step sizes.

step sizes $\Delta = 2^{-5}, 2^{-3}$ and 2^{-1} . We take 5000 sample paths. The truncated Euler-Maruyama method is stable if $\Delta < \frac{1}{2}$. The mean square of the numerical solutions is plotted in Fig. 5.

8. Conclusions

In this paper, we have analyzed the convergence and stability of the truncated Euler-Maruyama method for SDEPCAs. We obtain that the convergence rate is closed to 0.5 arbitrarily without any restrictions for the step size. Moreover, we have shown that the truncated Euler-Maruyama method preserves the exponential mean square stability of the SDEPCAs under some restrictions on the step size.

Acknowledgments

The authors thank the referees and the editor for their valuable comments and helpful suggestions. This work is supported by the National Natural Science Foundation of China (No. 11671113) and the National Postdoctoral Program for Innovative Talents (No. BX20180347).

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