

The Virtual Element Method for an Elliptic Hemivariational Inequality with Convex Constraint

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Abstract. An abstract framework of numerical method is devised for solving an elliptic hemivariational inequality with convex constraint. Convergence of the method is explored under the minimal solution regularity available from the well-posedness of the hemivariational inequality. A Céa-type inequality is derived for error estimation. As a typical example, a virtual element method is proposed to solve a frictionless unilateral contact problem and its optimal error estimates are obtained as well. Numerical results are reported to show the performance of the proposed method.

AMS subject classifications: 65N30

Key words: Virtual element method, hemivariational inequality, error estimate, multiobjective double bundle method.

1. Introduction

Hemivariational inequalities (HVIs) arise in the study of various industrial processes and engineering applications. In the past three decades, many researchers have contributed mathematical theories for such models (cf. [18, 32, 37, 39–41]). The finite element method has been used to solve them with systematic theoretical analysis (cf. [6, 27, 29–32]). We refer the reader to the survey paper [28] for details along this line. In this paper, we intend to propose and analyze the virtual element method for an elliptic hemivariational inequality with convex constraint, which can be viewed as an extension of our earlier work in [23].

We first introduce some notation about function spaces for later uses. Throughout

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this paper, we will use the standard notation for Sobolev spaces and their norms and seminorms (cf. [1]). Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded domain with Lipschitz boundary $\partial\Omega$. Given an integer $m \geq 1$, let X be a closed subspace of $H^1(\Omega; \mathbb{R}^m)$ and K a closed and convex subset of X with $0_X \in K$. Moreover, let X_j be another Banach space and $\gamma_j \in \mathcal{L}(X, X_j)$. Then, the mathematical problem to be studied can be described as follows.

Problem (P). Find an element $u \in K$ such that

$$a(u, v - u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v - u \rangle, \quad \forall v \in K, \quad (1.1)$$

where $a(\cdot, \cdot)$ is a bilinear form over X , $j : X_j \rightarrow \mathbb{R}$ is a locally Lipschitz function, f is a bounded linear functional over X , while $j^0(x; v)$ denotes the generalized Clarke directional derivative of j at x in a direction v defined by (cf. [21])

$$j^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{1}{\lambda} (j(y + \lambda v) - j(y)).$$

Recently, virtual element methods (VEMs) were developed and have gained popularity as a numerical approach for solving partial differential equations (PDEs), started with [2, 7, 9]. VEMs have some advantages over standard finite element methods. For example, they are more convenient to handle PDEs on complex geometric domains or the ones associated with high-regularity admissible spaces. The methods have been applied to solve many different kinds of mathematical physical problems, e.g., conforming and nonconforming VEMs for second-order elliptic problems [5, 17, 24, 35], fourth-order problems [3, 16, 44], elasticity problems [8, 45], and $(2m)$ -th-order elliptic problems in any dimensions (cf. [20]). Some systematic theoretical analyses were given in [10, 14, 19, 20] for conforming and nonconforming VEMs. In the reference [23], we introduced an abstract framework of numerical method and established an error analysis for the problem (1.1) without constraint, i.e. for the case $K = X$. We applied the VEM for solving two contact problems and derived optimal order error estimates of their numerical solutions under appropriate solution regularity assumptions.

In this paper, we first extend the ideas in [23] to devise naturally an abstract framework of numerical method for the problem (1.1). Then, we extend arguments presented in [25, 31] in a subtle way to show convergence of the numerical solutions. In addition, we derive a Céa-type inequality by using the techniques in [23, 25]. As a typical example, we apply the previous results to propose a VEM for a frictionless unilateral contact problem and derive its optimal error estimates. Under some assumptions, this discrete problem is equivalent to a minimization problem (cf. [26, 32]), from which we are able to compute the numerical solution by means of the multiobjective double bundle method (cf. [38]) effectively. Finally, we provide several numerical results to show the performance of the proposed method.

We end this section by introducing some basic results for later requirements. The following elementary inequality is simple but useful in the forthcoming analysis:

$$a, b, x \geq 0 \quad \text{and} \quad x^2 \leq ax + b \quad \implies \quad x^2 \leq a^2 + 2b. \quad (1.2)$$

If $\psi : X \rightarrow \mathbb{R}$ is a locally Lipschitz functional on the Banach space X , the generalized gradient (subdifferential) of ψ at x is defined by ([21])

$$\partial\psi(x) = \{\zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v \rangle, \forall v \in X\},$$

where X^* denotes the dual space of X . The following properties hold:

$$\psi^0(x; v) = \max \{\langle \zeta, v \rangle \mid \zeta \in \partial\psi(x)\}. \quad (1.3)$$

$$\psi^0(x; v_1 + v_2) \leq \psi^0(x; v_1) + \psi^0(x; v_2), \quad \forall v_1, v_2 \in X. \quad (1.4)$$

$$x_n \rightarrow x \quad \text{and} \quad v_n \rightarrow v \quad \text{in } X \quad \Rightarrow \quad \limsup_{n \rightarrow \infty} \psi^0(x_n; v_n) \leq \psi^0(x; v). \quad (1.5)$$

Here, $\langle \cdot, \cdot \rangle_{X^* \times X}$ stands for the duality pairing between X^* and X , and is simply written as $\langle \cdot, \cdot \rangle$ when there is no confusion caused.

2. A framework of numerical solution and its theoretical analysis

As usual, we make the following assumptions for Problem (P).

(H_a) $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ is bilinear, symmetric, continuous and X -elliptic; we will denote the X -ellipticity constant by $m_A > 0$:

$$a(v, v) \geq m_A \|v\|_X^2, \quad \forall v \in X. \quad (2.1)$$

(H_j) $j : X_j \rightarrow \mathbb{R}$ is locally Lipschitz, and there are constants $c_0, c_1, \alpha_j \geq 0$ such that

$$\|\partial j(z)\|_{X_j^*} \leq c_0 + c_1 \|z\|_{X_j}, \quad \forall z \in X_j, \quad (2.2)$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leq \alpha_j \|z_1 - z_2\|_{X_j}^2, \quad \forall z_1, z_2 \in X_j. \quad (2.3)$$

Moreover, denote by $c_j \geq 0$ an upper bound of the norm of the operator $\gamma_j \in \mathcal{L}(X, X_j)$:

$$\|\gamma_j v\|_{X_j} \leq c_j \|v\|_X, \quad \forall v \in X. \quad (2.4)$$

Define a linear operator $A : X \rightarrow X^*$ by

$$\langle Au, v \rangle = a(u, v), \quad \forall u, v \in X. \quad (2.5)$$

It is easy to check that $A \in \mathcal{L}(X, X^*)$ and it is monotone with a monotonicity constant m_A , which implies A is pseudomonotone ([43, Proposition 27.6]). Hence, following [29, Theorem 3.1], we have the next result.

Theorem 2.1. Assume (H_a), (H_j), and

$$\alpha_j c_j^2 < m_A, \quad (2.6)$$

where c_j is from (2.4). Then for any $f \in X^*$, Problem (P) has a unique solution $u \in K$.

2.1. A framework of numerical solution and its unique solvability

Let $\mathcal{T}_h = \{E\}_{E \in \mathcal{T}_h}$ be a polytopal mesh of Ω into polytopes, with E denoting a generic element; $h = \max_{E \in \mathcal{T}_h} h_E$ and $h_E = \text{diam}(E)$. With this mesh, we associate a finite dimensional subspace X_h of X . Let K_h be a non-empty, closed and convex subset of X_h and $0 \in K_h$. Moreover, for any domain D and a nonnegative integer k , denote by $\mathbb{P}_k(D)$ the set of all polynomials on D with the total degree no more than k . Assume that there exists a natural number k such that $\mathbb{P}_k(E; \mathbb{R}^m) \subset X_{h|E}$ for all $E \in \mathcal{T}_h$, and the bilinear form $a(\cdot, \cdot)$ can be decomposed as

$$a(v, w) = \sum_{E \in \mathcal{T}_h} a^E(v, w), \quad \forall v, w \in X,$$

where $a^E(\cdot, \cdot)$ is a bilinear, symmetric and nonnegative form over $X_E = X|_E$. We equip the space $X|_E$ with a norm or semi-norm $\|\cdot\|_{X,E}$ such that

$$\|v\|_X^2 = \sum_{E \in \mathcal{T}_h} \|v\|_{X,E}^2, \quad \forall v \in X, \tag{2.7}$$

and for all $E \in \mathcal{T}_h$, there holds

$$a^E(v, v) \lesssim \|v\|_{X,E}^2, \quad \forall v \in X_E. \tag{2.8}$$

Here and below, for any two quantities a and b , “ $a \lesssim b$ ” stands for “ $a \leq Cb$ ”, where the hidden constant C is independent of the mesh sizes but may take different values at different occurrences.

With the help of the above preparation, our abstract frame of numerical method for Problem (P) reads:

Problem (P^h). Find an element $u_h \in K_h$ such that

$$a_h(u_h, v_h - u_h) + j^0(\gamma_j u_h; \gamma_j v_h - \gamma_j u_h) \geq \langle f_h, v_h - u_h \rangle, \quad \forall v_h \in K_h, \tag{2.9}$$

where

$$f_h \in X_h^*, \tag{2.10}$$

and it satisfies the condition

$$\|f - f_h\|_{X_h^*} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad \text{and} \quad \langle f_h, v \rangle \leq c \|f\|_{X^*} \|v\|_X, \quad \forall v \in X_h. \tag{2.11}$$

Here,

$$\|f - f_h\|_{X_h^*} = \sup_{v_h \in K_h} \frac{\langle f - f_h, v_h \rangle}{\|v_h\|_X},$$

and the bilinear form is obtained by

$$a_h(u, v) = \sum_{E \in \mathcal{T}_h} a_h^E(u, v) \tag{2.12}$$

with the symmetric bilinear form $a_h^E(\cdot, \cdot)$ satisfying

- *k*-Consistency: For all $p \in \mathbb{P}_k(E; \mathbb{R}^m)$ and for all $v_h \in X_{h|E}$,

$$a_h^E(p, v_h) = a^E(p, v_h). \quad (2.13)$$

- Stability: There exist two positive constants α_* and α^* , independent of h_E and E , such that

$$\alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h), \quad v_h \in X_{h|E}. \quad (2.14)$$

For the study of the discrete problem, we assume that

$$m_A > \max \left\{ 1, \frac{1}{\alpha_*} \right\} \alpha_j c_j^2. \quad (2.15)$$

Let $\tilde{m}_A = \alpha_* m_A$. Then from (2.15),

$$\tilde{m}_A > \alpha_j c_j^2. \quad (2.16)$$

A routine computation yields the X_h -ellipticity of $a_h(\cdot, \cdot)$:

$$a_h(v, v) \geq \alpha_* a(v, v) \geq \tilde{m}_A \|v\|_X^2, \quad \forall v \in X_h. \quad (2.17)$$

The arguments of Theorem 2.1 can be applied in the setting of the finite dimensional set $u_h \in K_h$. Thus, we can easily obtain the following result.

Theorem 2.2. *Under the assumptions (H_a) , (H_j) , (2.4), (2.10), (2.15) and (2.17), Problem (P^h) has a unique solution.*

2.2. Convergence analysis

In this subsection, we will study the convergence of Problem (P^h) . Using the similar arguments in [23, 29], we can obtain the following result readily.

Lemma 2.1. *If the assumptions (H_a) , (H_j) , (2.11), (2.15) and (2.17) hold, then*

$$\|u_h\|_X \lesssim (\|f\|_{X^*} + 1).$$

To further our analysis, we make the following two assumptions for a certain natural number k .

Assumption B1. For every $v \in H^{k+1}(E; \mathbb{R}^m)$, there exists a function $v_E = \Pi_E v \in \mathbb{P}_k(E; \mathbb{R}^m)$ such that

$$\|v - v_E\|_{0,E} + h_E \|v - v_E\|_{X,E} \lesssim h_E^{k+1} |v|_{k+1,E}, \quad \forall v \in H^{k+1}(E; \mathbb{R}^m). \quad (2.18)$$

Assumption B2. There exists an interpolation operator $I_E : H^{k+1}(E; \mathbb{R}^m) \cap X_E \rightarrow X_{h|E}$ such that

$$\|v - I_E v\|_{0,E} + h_E \|v - I_E v\|_{X,E} \lesssim h_E^{k+1} |v|_{k+1,E}, \quad \forall v \in H^{k+1}(E; \mathbb{R}^m) \cap X_E. \quad (2.19)$$

Moreover, we write v_I as the global interpolant of v , i.e. $v_I(\mathbf{x})$ is equal to $I_E v(\mathbf{x})$ for $\mathbf{x} \in E$.

In order to get the convergence analysis of Problem (P^h) , we assume $\{K_h\}_h$ approximates K in the following sense

$$v_h \in K_h \quad \text{and} \quad v_h \rightharpoonup v \quad \text{in } X \quad \text{imply} \quad v \in K, \tag{2.20}$$

$$\forall v \in K, \exists v_h \in K_h \quad \text{such that} \quad v_h \rightharpoonup v \quad \text{in } X \quad \text{as} \quad h \rightarrow 0, \tag{2.21}$$

where the symbol “ \rightharpoonup ” stands for the weak convergence over the Banach space X .

To derive the convergence of Problem (P^h) , we require an auxiliary result in advance.

Lemma 2.2. *Let $\{w_h\}_h$ be a sequence in X_h . If $w_h \rightharpoonup w$ in X and $\|f - f_h\|_{X_h^*} \rightarrow 0$ as $h \rightarrow 0$, then for all $v \in X$,*

$$a_h(v, w_h) \rightarrow a(v, w), \quad \langle f_h, w_h \rangle \rightarrow \langle f, w \rangle \quad \text{as} \quad h \rightarrow 0. \tag{2.22}$$

Proof. Since Ω is a bounded domain with Lipschitz boundary, $C^\infty(\bar{\Omega}; \mathbb{R}^m)$ is dense in $H^1(\Omega; \mathbb{R}^m)$. Recall that X is assumed to be a closed subspace of $H^1(\Omega; \mathbb{R}^m)$. Hence, given any $v \in X$, for all $\varepsilon > 0$ there exists a certain $v_\varepsilon \in C^\infty(\bar{\Omega}; \mathbb{R}^m)$ such that

$$\|v - v_\varepsilon\|_X < \varepsilon. \tag{2.23}$$

For all $E \in \mathcal{T}_h$, according to assumption **B1**, we have $v_\varepsilon^E = \Pi_E v_\varepsilon \in \mathbb{P}_k(E; \mathbb{R}^m)$ such that

$$\|v_\varepsilon - v_\varepsilon^E\|_{1,E} \lesssim h|v_\varepsilon|_{2,E}, \tag{2.24}$$

leading to

$$\sum_{E \in \mathcal{T}_h} \|v_\varepsilon - v_\varepsilon^E\|_{1,E}^2 \lesssim h^2 \sum_{E \in \mathcal{T}_h} |v_\varepsilon|_{2,E}^2 \lesssim h^2 |v_\varepsilon|_{2,\Omega}^2. \tag{2.25}$$

Hence, by (2.23), (2.25) and the triangle inequality,

$$\begin{aligned} \sum_{E \in \mathcal{T}_h} \|v - v_\varepsilon^E\|_{1,E}^2 &\leq 2 \left(\|v - v_\varepsilon\|_X^2 + \sum_{E \in \mathcal{T}_h} \|v_\varepsilon - v_\varepsilon^E\|_{1,E}^2 \right) \\ &\lesssim \varepsilon^2 + h^2 |v_\varepsilon|_{2,\Omega}^2. \end{aligned} \tag{2.26}$$

Since X is reflexive, according to the Banach-Steinhaus theorem in functional analysis, the weak convergence of the sequence $\{w_h\}$ in X implies the boundedness of $\{\|w_h\|_X\}$. Therefore, in view of (2.13), (2.14), (2.26) and the Cauchy-Schwarz inequality,

$$\begin{aligned} &|a_h(v, w_h) - a(v, w)| \\ &= |a_h(v, w_h) - a(v, w_h) + a(v, w_h) - a(v, w)| \\ &\leq \left| \sum_{E \in \mathcal{T}_h} [a_h^E(v - v_\varepsilon^E, w_h) - a^E(v - v_\varepsilon^E, w_h)] \right| + |a(v, w_h - w)| \\ &\lesssim \left(\sum_{E \in \mathcal{T}_h} \|v - v_\varepsilon^E\|_{1,E}^2 \right)^{\frac{1}{2}} \|w_h\|_X + |a(v, w_h - w)| \\ &\lesssim (\varepsilon + h|v_\varepsilon|_{2,\Omega}) \|w_h\|_X + |a(v, w_h - w)|. \end{aligned} \tag{2.27}$$

Note that

$$a(v, w_h - w) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

So the estimate (2.27) yields

$$\limsup_{h \rightarrow 0} |a_h(v, w_h) - a(v, w)| \lesssim \varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$, and $|a_h(v, w_h) - a(v, w)| \geq 0$, we obtain

$$a_h(v, w_h) \rightarrow a(v, w) \quad \text{as } h \rightarrow 0.$$

Furthermore, by means of the weak convergence of $\{w_h\}$ in X ,

$$\begin{aligned} \langle f_h, w_h \rangle - \langle f, w \rangle &= \langle f_h - f, w_h \rangle + \langle f, w_h - w \rangle \\ &\leq \|f - f_h\|_{X_h^*} \|w_h\|_X + \langle f, w_h - w \rangle \rightarrow 0 \quad \text{as } h \rightarrow 0 \end{aligned} \quad (2.28)$$

as required. \square

Theorem 2.3. *If assumptions $(H_a), (H_j)$, (2.11), (2.13)-(2.15), (2.18), and (2.20)-(2.21) hold, then*

$$u_h \rightarrow u \quad \text{in } X \quad \text{as } h \rightarrow 0. \quad (2.29)$$

Proof. The proof is rather involved and is divided into three steps for clarity.

Step 1: By Lemma 2.1, $\{u_h\}$ is bounded in X . Since X is reflexive and $\gamma_j \in \mathcal{L}(X, X_j)$, there exists a subsequence $\{u_{h'}\} \subset \{u_h\}$ and an element $w \in X$ such that

$$u_{h'} \rightharpoonup w \quad \text{in } X, \quad \gamma_j u_{h'} \rightharpoonup \gamma_j w \quad \text{in } X_j. \quad (2.30)$$

By the assumption (2.20), we know that $w \in K$.

Step 2: Intend to show the strong convergence, $u_{h'} \rightarrow w$ in X . By (2.21), there exists a sequence $\{w_{h'}\} \subset X$ with $w_{h'} \in K_{h'}$, such that

$$w_{h'} \rightarrow w \quad \text{in } X, \quad \gamma_j w_{h'} \rightarrow \gamma_j w \quad \text{in } X_j \quad \text{as } h' \rightarrow 0. \quad (2.31)$$

Owing to (2.17), there holds

$$\begin{aligned} \tilde{m}_A \|w_{h'} - u_{h'}\|_X^2 &\leq a_{h'}(w_{h'} - u_{h'}, w_{h'} - u_{h'}) \\ &= a_{h'}(w_{h'}, w_{h'} - u_{h'}) - a_{h'}(u_{h'}, w_{h'} - u_{h'}) \\ &= a_{h'}(w_{h'} - w, w_{h'} - u_{h'}) + a_{h'}(w, w_{h'} - u_{h'}) \\ &\quad - a_{h'}(u_{h'}, w_{h'} - u_{h'}) \\ &= a_{h'}(w_{h'} - w, w_{h'} - u_{h'}) + a_{h'}(w, w_{h'} - w) \\ &\quad + a_{h'}(w, w - u_{h'}) - a_{h'}(u_{h'}, w_{h'} - u_{h'}). \end{aligned} \quad (2.32)$$

According to (2.31) and the boundedness of $\{w_{h'} - u_{h'}\}$ in X , we easily achieve

$$a_{h'}(w_{h'} - w, w_{h'} - u_{h'}) \rightarrow 0, \quad a_{h'}(w, w_{h'} - w) \rightarrow 0 \quad \text{as } h' \rightarrow 0. \quad (2.33)$$

Due to (2.30) and Lemma 2.2, we have

$$a_{h'}(w, w - u_{h'}) \rightarrow 0 \quad \text{as } h' \rightarrow 0. \tag{2.34}$$

Furthermore, it follows from (2.9), (1.4) and (2.3) that

$$\begin{aligned} -a_{h'}(u_{h'}, w_{h'} - u_{h'}) &\leq j^0(\gamma_j u_{h'}; \gamma_j w_{h'} - \gamma_j u_{h'}) - \langle f_{h'}, w_{h'} - u_{h'} \rangle \\ &\leq j^0(\gamma_j u_{h'}; \gamma_j w_{h'} - \gamma_j w) + j^0(\gamma_j u_{h'}; \gamma_j w - \gamma_j u_{h'}) \\ &\quad + j^0(\gamma_j w; \gamma_j u_{h'} - \gamma_j w) - j^0(\gamma_j w; \gamma_j u_{h'} - \gamma_j w) \\ &\quad - \langle f_{h'}, w_{h'} - u_{h'} \rangle \\ &\leq \alpha_j c_j^2 \|u_{h'} - w\|_X^2 + j^0(\gamma_j u_{h'}; \gamma_j w_{h'} - \gamma_j w) \\ &\quad - j^0(\gamma_j w; \gamma_j u_{h'} - \gamma_j w) - \langle f_{h'}, w_{h'} - u_{h'} \rangle. \end{aligned} \tag{2.35}$$

From the boundedness of $\{\gamma_j u_{h'}\}$ in X_j , (1.3) and (2.31), it follows that

$$\limsup_{h' \rightarrow 0} j^0(\gamma_j u_{h'}; \gamma_j w_{h'} - \gamma_j w) \rightarrow 0, \tag{2.36}$$

and by (2.30),

$$-j^0(\gamma_j w; \gamma_j u_{h'} - \gamma_j w) \leq -\langle \zeta_\tau, \gamma_j u_{h'} - \gamma_j w \rangle \rightarrow 0 \quad \text{as } h' \rightarrow 0, \tag{2.37}$$

where $\zeta_\tau \in \partial j(\gamma_j w)$. Note that

$$\begin{aligned} \|u_{h'} - w\|_X^2 &\leq \|u_{h'} - w_{h'}\|_X^2 + \|w_{h'} - w\|_X^2 \\ &\quad + 2\|u_{h'} - w_{h'}\|_X \|w_{h'} - w\|_X. \end{aligned} \tag{2.38}$$

Hence, the combination of (2.32) to (2.38) implies

$$\begin{aligned} &(\tilde{m}_A - \alpha_j c_j^2) \|w_{h'} - u_{h'}\|_X^2 \\ &\leq \alpha_j c_j^2 [\|w - w_{h'}\|_X^2 + 2\|w_{h'} - u_{h'}\|_X \|w_{h'} - w\|_X] - \langle f_{h'}, w_{h'} - u_{h'} \rangle, \end{aligned}$$

and in view of Young's inequality and (2.16),

$$\|w_{h'} - u_{h'}\|_X^2 \lesssim \|w - w_{h'}\|_X^2 - \langle f_{h'}, w_{h'} - u_{h'} \rangle. \tag{2.39}$$

On the other hand, we have

$$\langle f_{h'}, w_{h'} - u_{h'} \rangle = \langle f_{h'} - f, w_{h'} - u_{h'} \rangle + \langle f, w_{h'} - w \rangle + \langle f, w - u_{h'} \rangle.$$

It follows from (2.30) and (2.31) that

$$\langle f, w_{h'} - w \rangle + \langle f, w - u_{h'} \rangle \rightarrow 0 \quad \text{as } h' \rightarrow 0.$$

By (2.11),

$$\langle f_{h'} - f, w_{h'} - u_{h'} \rangle \leq \|f - f_{h'}\|_{X_{h'}^*} \|w_{h'} - u_{h'}\|_X \rightarrow 0 \quad \text{as } h' \rightarrow 0.$$

Therefore,

$$\langle f_{h'}, w_{h'} - u_{h'} \rangle \rightarrow 0 \quad \text{as } h' \rightarrow 0,$$

which together with (2.31) and (2.39) implies

$$\|w_{h'} - u_{h'}\|_X^2 \rightarrow 0 \quad \text{as } h' \rightarrow 0. \quad (2.40)$$

The strong convergence follows readily from (2.31) and (2.40), i.e.,

$$u_{h'} \rightarrow w, \quad h' \rightarrow 0. \quad (2.41)$$

Step 3: Intend to show that the strong limit w is the unique solution of Problem (P). For any $v \in K$, there exists a sequence $\{v_{h'}\} \subset X$ with $v_{h'} \in K_{h'}$, such that $v_{h'} \rightarrow v$ in X . Then $\gamma_j v_{h'} \rightarrow \gamma_j v$ in X_j . By definition,

$$a_{h'}(u_{h'}, v_{h'} - u_{h'}) + j^0(\gamma_j u_{h'}; \gamma_j v_{h'} - \gamma_j u_{h'}) \geq \langle f_{h'}, v_{h'} - u_{h'} \rangle, \quad \forall v_{h'} \in K_{h'}. \quad (2.42)$$

Write

$$a_{h'}(u_{h'}, v_{h'} - u_{h'}) = a_{h'}(u_{h'} - w, v_{h'} - u_{h'}) + a_{h'}(w, v_{h'} - u_{h'}). \quad (2.43)$$

Using (2.41) and the boundedness of $\{v_{h'} - u_{h'}\}$ in X gives

$$a_{h'}(u_{h'} - w, v_{h'} - u_{h'}) \rightarrow 0 \quad \text{as } h' \rightarrow 0.$$

Since $(v_{h'} - u_{h'}) \rightarrow (v - w)$ in X , an application of Lemma 2.2 immediately implies

$$a_{h'}(w, v_{h'} - u_{h'}) \rightarrow a(w, v - w), \quad \langle f_{h'}, v_{h'} - u_{h'} \rangle \rightarrow \langle f, v - w \rangle \quad \text{as } h' \rightarrow 0.$$

By (1.5),

$$\limsup_{h' \rightarrow 0} j^0(\gamma_j u_{h'}; \gamma_j v_{h'} - \gamma_j u_{h'}) \leq j^0(\gamma_j w; \gamma_j v - \gamma_j w).$$

Consequently, it follows from (2.42) that

$$a(w, v - w) + j^0(\gamma_j w; \gamma_j v - \gamma_j w) \geq \langle f, v - w \rangle, \quad \forall v \in K.$$

Thus, w is a solution of Problem (P). Note that the solution of Problem (P) is unique, so $w = u$, which, in conjunction with (2.41), yields the strong convergence $u_h \rightarrow u$ as $h \rightarrow 0$. \square

2.3. Error estimates

In this subsection, we are ready to state and prove a Céa's type inequality, which can be viewed as a starting point for further error estimates.

Theorem 2.4. Assume that $(H_a), (H_j), (2.4), (2.11), (2.13)-(2.15), (2.18)-(2.19)$ hold. If the exact solution u of (1.1) belongs to $H^{k+1}(\Omega)$ for a certain natural number k and $u_I \in K_h$, then for all $v \in K$ we have the error bound

$$\begin{aligned} \|u - u_h\|_X &\lesssim h^k |u|_{k+1, \Omega} + \|\gamma_j u - \gamma_j u_I\|_{X_j}^{\frac{1}{2}} + \|f - f_h\|_{X_h^*} \\ &\quad + |R_u(u_I, u)|^{\frac{1}{2}} + |R_u(v, u_h)|^{\frac{1}{2}}, \end{aligned} \tag{2.44}$$

where

$$R_u(v, w) = a(u, v - w) + j^0(\gamma_j u; \gamma_j v - \gamma_j w) - \langle f, v - w \rangle. \tag{2.45}$$

Proof. Let $w = u_I - u_h$. Then, due to (2.17),

$$\tilde{m}_A \|w\|_X^2 \leq a_h(u_I, w) - a_h(u_h, w). \tag{2.46}$$

In view of (2.13) and (2.18), there exists $u_E \in \mathbb{P}_k(E; \mathbb{R}^m)$ such that

$$a_h^E(u_E, v_h) = a^E(u_E, v_h), \quad \forall v_h \in X_{h|E},$$

so

$$\begin{aligned} a_h(u_I, w) &= \sum_{E \in \mathcal{T}_h} \left(a_h^E(I_E u - u_E, w) + a^E(u_E - u, w) \right) \\ &\quad + a(u, u_I - u) + a(u, u - v) + a(u, v - u_h). \end{aligned} \tag{2.47}$$

On the other hand, using (1.1) and (2.9) gives rise to

$$\begin{aligned} a(u, u - v) - a_h(u_h, w) &\leq j^0(\gamma_j u; \gamma_j v - \gamma_j u) + j^0(\gamma_j u_h; \gamma_j u_I - \gamma_j u_h) \\ &\quad - \langle f, v - u \rangle - \langle f_h, w \rangle. \end{aligned} \tag{2.48}$$

Plugging (2.47) and (2.48) into (2.46), we derive

$$\begin{aligned} \tilde{m}_A \|w\|_X^2 &\leq \sum_{E \in \mathcal{T}_h} \left(a_h^E(I_E u - u_E, w) + a^E(u_E - u, w) \right) \\ &\quad + \langle f - f_h, w \rangle + I_j(v, u_I) + R_u(u_I, u) + R_u(v, u_h), \end{aligned} \tag{2.49}$$

where

$$\begin{aligned} I_j(v, u_I) &= j^0(\gamma_j u; \gamma_j v - \gamma_j u) + j^0(\gamma_j u_h; \gamma_j u_I - \gamma_j u_h) \\ &\quad - j^0(\gamma_j u; \gamma_j u_I - \gamma_j u) - j^0(\gamma_j u; \gamma_j v - \gamma_j u_h), \\ R_u(v, w) &= a(u, v - w) + j^0(\gamma_j u; \gamma_j v - \gamma_j w) - \langle f, v - w \rangle. \end{aligned}$$

Applying the subadditivity of the generalized directional derivative, we have

$$\begin{aligned} j^0(\gamma_j u; \gamma_j v - \gamma_j u) &\leq j^0(\gamma_j u; \gamma_j v - \gamma_j u_h) + j^0(\gamma_j u; \gamma_j u_h - \gamma_j u), \\ j^0(\gamma_j u_h; \gamma_j u_I - \gamma_j u_h) &\leq j^0(\gamma_j u_h; \gamma_j u_I - \gamma_j u) + j^0(\gamma_j u_h; \gamma_j u - \gamma_j u_h). \end{aligned}$$

By (2.3),

$$j^0(\gamma_j u; \gamma_j u_h - \gamma_j u) + j^0(\gamma_j u_h; \gamma_j u - \gamma_j u_h) \leq \alpha_j \|\gamma_j u - \gamma_j u_h\|_{X_j}^2.$$

Moreover,

$$\begin{aligned} |j^0(\gamma_j u_h; \gamma_j u_I - \gamma_j u)| &\leq (c_0 + c_1 \|\gamma_j u_h\|_{X_j}) \|\gamma_j u_I - \gamma_j u\|_{X_j}, \\ |j^0(\gamma_j u; \gamma_j u_I - \gamma_j u)| &\leq (c_0 + c_1 \|\gamma_j u\|_{X_j}) \|\gamma_j u_I - \gamma_j u\|_{X_j}. \end{aligned}$$

Note that $\|\gamma_j u_h\|_{X_j}$ is uniformly bounded by a constant independent of h , we find by combing the above five inequalities that

$$I_j(v, u_I) \leq \alpha_j \|\gamma_j u - \gamma_j u_h\|_{X_j}^2 + C \|\gamma_j u_I - \gamma_j u\|_{X_j}. \tag{2.50}$$

On the other hand,

$$\begin{aligned} &\sum_{E \in \mathcal{T}_h} \left(a_h^E(I_E u - u_E, w) + a^E(u_E - u, w) \right) \\ &\lesssim \left(\sum_{E \in \mathcal{T}_h} (\|I_E u - u_E\|_{X,E}^2 + \|u - u_E\|_{X,E}^2) \right)^{\frac{1}{2}} \|w\|_X \end{aligned}$$

by using (2.14) and the Cauchy-Schwarz inequality. It is evident that

$$\|u - u_h\|_X^2 \leq \|u - u_I\|_X^2 + \|w\|_X^2 + 2\|u - u_I\|_X \|w\|_X.$$

Hence, we deduce from (2.4), (2.16), (2.49) and (2.50) that

$$\begin{aligned} &(\tilde{m}_A - a_j c_j^2) \|w\|_X^2 \\ &\lesssim \left[\left(\sum_{E \in \mathcal{T}_h} \|I_E u - u_E\|_{X,E}^2 + \sum_{E \in \mathcal{T}_h} \|u - u_E\|_{X,E}^2 \right)^{\frac{1}{2}} + \|f - f_h\|_{X_h^*} \right] \|w\|_X \\ &\quad + \|u - u_I\|_X^2 + \|\gamma_j u - \gamma_j u_I\|_{X_j} + R_u(u_I, u) + R_u(v, u_h). \end{aligned} \tag{2.51}$$

It follows from (2.18) and (2.19) that

$$\begin{aligned} &\left(\sum_{E \in \mathcal{T}_h} \|u - u_E\|_{X,E}^2 \right)^{\frac{1}{2}} \lesssim h^k |u|_{k+1, \Omega}, \\ \|u - u_I\|_X &= \left(\sum_{E \in \mathcal{T}_h} \|u - I_E u\|_{X,E}^2 \right)^{\frac{1}{2}} \lesssim h^k |u|_{k+1, \Omega}. \end{aligned} \tag{2.52}$$

Applying (1.2) in (2.51), we find

$$\|w\|_X^2 \lesssim h^{2k} |u|_{k+1, \Omega}^2 + \|f - f_h\|_{X_h^*}^2 + \|\gamma_j u - \gamma_j u_I\|_{X_j} + R_u(u_I, u) + R_u(v, u_h),$$

which readily leads to (2.44) by means of (2.52) and the triangle inequality. \square

3. A virtual element method and its error analysis for the contact problem with unilateral constraint

Based on the results developed in the preceding section, as a typical example, we will propose and analyze a virtual element method for a contact problem with unilateral constraint in [29]. In this example, Ω is the reference configuration of the linear elastic body, assumed to be an open, bounded, connected polygon in \mathbb{R}^2 . The boundary $\Gamma = \partial\Omega$ is made up of Γ_1, Γ_2 and Γ_3 , where $\text{meas}(\Gamma_1) > 0$. We assume that the body is clamped on Γ_1 . The surface traction of density \mathbf{f}_2 is applied to Γ_2, Γ_3 is the contact surface with a rigid foundation. Volume forces of density \mathbf{f}_0 act in Ω . Here, for a vector \mathbf{v} , denote by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ its normal component and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ the tangential component, respectively. We use \mathbb{S}^2 for the space of second order symmetric tensors which is equipped with the canonical inner product “:”. For a second order tensor $\boldsymbol{\sigma}$, define its normal component as $\sigma_\nu = \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\nu}$ and tangential component as $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. For the contact problems under consideration, we have the elastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \tag{3.1}$$

the equilibrium equation

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \tag{3.2}$$

the displacement boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{3.3}$$

the traction boundary condition

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \tag{3.4}$$

and the frictionless unilateral contact condition

$$u_\nu \leq g, \quad \sigma_\nu + \xi_\nu \leq 0, \quad (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0, \quad \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3, \tag{3.6}$$

where $-\sigma_\nu = \xi_\nu \in \partial j_\nu(u_\nu)$ and g represents the thickness of the elastic layer. The problem described by (3.1)-(3.4) and (3.5)-(3.6) represents the frictionless version of a nonlinear elastic contact model studied in [34]. In (3.1), $\mathcal{F} : \Omega \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the linear elasticity operators (cf. [29]) and the potential function in (3.5), $j_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ are satisfied with the conditions in [23]. The relation $u_\nu \leq g$ restricts the allowed penetration. We assume $g \in H^1(\Gamma_3)$ in advance.

Introduce a Hilbert space $Q = L^2(\Omega; \mathbb{S}^2)$, equipped with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}(\mathbf{x}) \tau_{ij}(\mathbf{x}) \, dx$$

and the induced norm $\|\cdot\|_Q$. For simplicity, write (\cdot, \cdot) for $(\cdot, \cdot)_Q$.

Let

$$X = \mathbf{V} = \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^2) \mid \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1 \}$$

equipped with the norm

$$\| \mathbf{v} \|_{\mathbf{V}} = (\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{3.7}$$

Since $\text{meas}(\Gamma_1) > 0$, by Korn’s inequality (see e.g. [11, Remark 1.1]), we find that

$$\| \mathbf{v} \|_{H^1(\Omega; \mathbb{R}^2)} \lesssim \| \mathbf{v} \|_{\mathbf{V}} \lesssim \| \mathbf{v} \|_{H^1(\Omega; \mathbb{R}^2)}, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{3.8}$$

Assume

$$\mathbf{f}_0 \in L^2(\Omega; \mathbb{R}^2), \quad \mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^2),$$

and define $\mathbf{f} \in \mathbf{V}^*$ by

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} = (\mathbf{f}_0, \mathbf{v})_{L^2(\Omega; \mathbb{R}^2)} + (\mathbf{f}_2, \mathbf{v})_{L^2(\Gamma_2; \mathbb{R}^2)}, \quad \forall \mathbf{v} \in \mathbf{V}.$$

Define

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{v} \rangle &= (\mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ j(z) &= \int_{\Gamma_3} j_\nu(\cdot, z(\cdot)) \, ds, \quad z \in X_j. \end{aligned}$$

Note that ([37, Theorem 3.1])

$$j^0(z; w) \leq \int_{\Gamma_3} j_\nu^0(\cdot, z(\cdot); w(\cdot)) \, ds, \quad z, w \in X_j. \tag{3.9}$$

Considering the constraint $u_\nu \leq g$ on Γ_3 , we introduce a subset of the space \mathbf{V}

$$U = \{ \mathbf{v} \in \mathbf{V} \mid v_\nu \leq g \text{ a.e. on } \Gamma_3 \}. \tag{3.10}$$

Letting $K = U$ in (1.1), $X_j = L^2(\Gamma_3)$ and $\gamma_j \mathbf{v} = v_\nu$ for $\mathbf{v} \in \mathbf{V}$, we then achieve the following weak formulation of this contact problem.

Problem (P₁). Find a displacement field $\mathbf{u} \in U$ such that

$$(\mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}))_Q + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu) \, ds \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{\mathbf{V}^* \times \mathbf{V}}, \quad \forall \mathbf{v} \in U. \tag{3.11}$$

Remark 3.1. Using the analogous notation in [23, 29], the assumption (H_a) is satisfied with $m_A = m_{\mathcal{F}}$ and (H_j) is satisfied with $\alpha_j = \alpha_{j\nu}$, where $\alpha_{j\nu}$ is the constant given in [29, Eq. (65)]. Applying (3.9) and Theorem 2.1, we know that Problem (P₁) has a solution $\mathbf{u} \in U$ under the stated assumptions, and (2.6) takes the form

$$\alpha_{j\nu} < \lambda_{1,V} m_{\mathcal{F}},$$

where $\lambda_{1,V} > 0$ is the smallest eigenvalue of the problem

$$\mathbf{u} \in \mathbf{V}, \quad \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx = \lambda \int_{\Gamma_3} u_\nu \cdot v_\nu \, ds, \quad \forall \mathbf{v} \in \mathbf{V}.$$

The uniqueness of a solution to Problem (P₁) can be proved in view of Theorem 2.1.

We make the following assumption on the family of meshes $\{\mathcal{T}_h\}_h$ (cf. [15, 19]).

Assumption B. For each $E \in \mathcal{T}_h$, there exists a “virtual triangulation” \mathcal{T}_E of E such that \mathcal{T}_E is uniformly shape regular and quasi-uniform. The corresponding mesh size of \mathcal{T}_E is bounded below by a constant multiple of h_E . Each edge of E is a side of a certain triangle in \mathcal{T}_E .

Furthermore, we express the three parts of the boundary Γ as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma}_k = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \leq k \leq 3.$$

Then, we construct virtual linear element spaces corresponding to \mathcal{T}_h . Let

$$\begin{aligned} V_1(E) &= \left\{ v \in H^1(E) \mid \Delta v = 0 \text{ in } E, v|_{\partial E} \in C(\partial E), \right. \\ &\quad \left. v|_e \in \mathbb{P}_1(e) \text{ for each edge } e \subset \partial E \right\}, \\ W_h &= \left\{ v \in C(\overline{\Omega}) \mid v|_E \in V_1(E) \text{ for all } E \in \mathcal{T}_h \right\}. \end{aligned}$$

The displacement fields will be sought in the space

$$X_h = \mathbf{V}_h := (W_h)^2 \cap \mathbf{V}.$$

Let $\mathbf{\Pi}_E$ be a projection operator from $\mathbf{V}_h(E)$ into $\mathbb{P}_0(E)_{sym}^{2 \times 2}$ such that for any given $\mathbf{v}_h \in \mathbf{V}_h(E)$,

$$\int_E \mathbf{\Pi}_E(\mathbf{v}_h) : \boldsymbol{\varepsilon}^P dx = \int_E \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}^P dx, \quad \forall \boldsymbol{\varepsilon}^P \in \mathbb{P}_0(E)_{sym}^{2 \times 2},$$

where $\mathbb{P}_0(E)_{sym}^{2 \times 2}$ stands for the set of all second order symmetric tensor fields with each entry being constant. Then, following [4, Eq. (12)], define

$$a_h^E(\mathbf{v}_h, \mathbf{w}_h) = \int_E \mathcal{F} \mathbf{\Pi}_E(\mathbf{v}_h) : \mathbf{\Pi}_E(\mathbf{w}_h) dx + b_h^E(\mathbf{v}_h, \mathbf{w}_h), \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h(E), \quad (3.12)$$

where the first term on the right of (3.12) is essentially equivalent to the first term given in [8, Eq. (4.1)] and the second term is a stabilization term. We refer to [4, 8] for details.

Next, define a local projection $\Pi_1^\nabla : H^1(E) \rightarrow \mathbb{P}_1(E)$ as the solution of

$$\begin{cases} (\nabla \Pi_1^\nabla v, \nabla p)_E = (\nabla v, \nabla p)_E, & \forall p \in \mathbb{P}_1(E), \\ \overline{\Pi_1^\nabla v} = \bar{v} \end{cases}$$

for all $v \in H^1(E)$. Here, $(\cdot, \cdot)_E$ means the $L^2(E)$ inner product, and \bar{v} is the integral average of v on the boundary ∂E of E . For the convenience, we also use Π_1^∇ to represent the related element-wise defined global operator.

For the right-hand side \mathbf{f} , define the approximation \mathbf{f}_h such that

$$\langle \mathbf{f}_h, \mathbf{v}_h \rangle = \sum_{E \in \mathcal{T}_h} \int_E \mathbf{f}_0 \cdot \mathbf{\Pi}_1^\nabla \mathbf{v}_h \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v}_h \, ds, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (3.13)$$

where $\mathbf{\Pi}_1^\nabla$ is the vectorized analogue of Π_1^∇ . Using the same arguments in [22, 23] shows

$$\langle \mathbf{f}_h, \mathbf{v}_h \rangle \lesssim (\|\mathbf{f}_0\|_{L^2(\Omega; \mathbb{R}^2)}^2 + \|\mathbf{f}_2\|_{L^2(\Gamma_2; \mathbb{R}^2)}^2)^{\frac{1}{2}} \|\mathbf{v}_h\|_{\mathbf{V}}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (3.14)$$

Moreover, for any $\mathbf{v}_h \in \mathbf{V}_h$, we deduce from [19, Lemma 2.2] and the H^1 -boundedness of $\mathbf{\Pi}_1^\nabla$ that

$$\begin{aligned} |\langle \mathbf{f} - \mathbf{f}_h, \mathbf{v}_h \rangle| &= \left| \sum_{E \in \mathcal{T}_h} \int_E \mathbf{f}_0 (\mathbf{v}_h - \mathbf{\Pi}_1^\nabla \mathbf{v}_h) \, dx \right| \\ &\leq \sum_{E \in \mathcal{T}_h} \|\mathbf{f}_0\|_{L^2(E; \mathbb{R}^2)} \|\mathbf{v}_h - \mathbf{\Pi}_1^\nabla \mathbf{v}_h\|_{L^2(E; \mathbb{R}^2)} \\ &\lesssim \sum_{E \in \mathcal{T}_h} \|\mathbf{f}_0\|_{L^2(E; \mathbb{R}^2)} h_E |\mathbf{v}_h|_{H^1(E; \mathbb{R}^2)} \\ &\lesssim h \|\mathbf{f}_0\|_{L^2(\Omega; \mathbb{R}^2)} \|\mathbf{v}_h\|_{\mathbf{V}}. \end{aligned} \quad (3.15)$$

It is evident that $\|\mathbf{f} - \mathbf{f}_h\|_{\mathbf{V}_h^*} \rightarrow 0$ as $h \rightarrow 0$. In view of (3.14) and (3.15), we can verify the condition (2.11).

Furthermore, we define

$$K_h = \mathbf{U}_h = \{ \mathbf{v}_h \in \mathbf{V}_h \mid v_\nu^h \leq g \text{ at node points on } \Gamma_3 \}.$$

Note that $\mathbf{0} \in \mathbf{U}_h$ and in general $\mathbf{U}_h \not\subseteq \mathbf{U}$ unless g is concave. Introduce the following virtual element method for Problem (P₁).

Problem (P₁^h). Find an element $\mathbf{u}_h \in \mathbf{U}_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \int_{\Gamma_3} j_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h) \, ds \geq \langle \mathbf{f}_h, \mathbf{v}_h - \mathbf{u}_h \rangle_{\mathbf{V}^* \times \mathbf{V}}, \quad \forall \mathbf{v}_h \in \mathbf{U}_h. \quad (3.16)$$

According to [4, 8, 11, 19], the bilinear form $a_h^E(\cdot, \cdot)$ in (3.12) satisfies conditions (2.13) and (2.14). Since the mesh satisfies the assumption **B**, assumption **B1** holds by the classical Scott-Dupont theory (cf. [13]). According to [12, 19], there exists a nodal interpolation operator $\mathbf{I}_E : H^2(E; \mathbb{R}^2) \rightarrow \mathbf{V}(E)$ such that

$$\|\mathbf{v} - \mathbf{I}_E \mathbf{v}\|_{0,E} + h_E |\mathbf{v} - \mathbf{I}_E \mathbf{v}|_{1,E} \lesssim h_E^2 \|\mathbf{v}\|_{2,E}, \quad \forall \mathbf{v} \in H^2(E; \mathbb{R}^2).$$

For all $\mathbf{v} \in H^2(\Omega; \mathbb{R}^2)$ write its global interpolant as \mathbf{v}_I . Hence, the assumption **B2** holds for $k = 1$ by using the interpolation operator \mathbf{I}_E . Moreover, it is easy to check that $\mathbf{u}_I \in K_h$.

On the other hand, (3.15) implies that

$$\|\mathbf{f} - \mathbf{f}_h\|_{\mathbf{V}_h^*} \lesssim h. \quad (3.17)$$

Assume the solution regularity

$$\mathbf{u} \in H^2(\Omega; \mathbb{R}^2), \quad \boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_3; \mathbb{R}^2). \quad (3.18)$$

By an argument similar to that for proving [25, Theorem 5], we know that for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$,

$$|R_{\mathbf{u}}(\mathbf{v}, \mathbf{w})| \lesssim \|\mathbf{v} - \mathbf{w}\|_{L^2(\Gamma_3; \mathbb{R}^2)}.$$

So it follows from Theorem 2.4 that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} \lesssim h + \|\mathbf{u}_I - \mathbf{u}\|_{L^2(\Gamma_3; \mathbb{R}^2)}^{\frac{1}{2}} + \inf_{\mathbf{v} \in \mathbf{U}} \|\mathbf{v} - \mathbf{u}_h\|_{L^2(\Gamma_3; \mathbb{R}^2)}^{\frac{1}{2}}. \quad (3.19)$$

Additionally, assume Γ_3 is a flat component of the boundary $\partial\Omega$ and

$$\mathbf{u}|_{\Gamma_3} \in H^2(\Gamma_3; \mathbb{R}^2). \quad (3.20)$$

According to [25, Theorem 6] and using the error estimate for the interpolation operator for Lagrange elements (cf. [13]), we have

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(\Gamma_3; \mathbb{R}^2)} \lesssim h^2.$$

The last term on the right-hand side of inequality (3.19) can be bounded as follows. Assume Γ_3 is a smooth portion of the boundary Γ . Then the unit outward normal $\boldsymbol{\nu}(\mathbf{x})$, $\mathbf{x} \in \Gamma_3$, can be extended to an H^1 function $\bar{\boldsymbol{\nu}}$ in a small neighborhood of Γ_3 and $\bar{\boldsymbol{\nu}}|_{\Gamma_3} = \boldsymbol{\nu}$ ([36, Section 2.1]). By multiplying $\bar{\boldsymbol{\nu}}$ with a smooth cut-off function which is zero outside the small neighborhood of Γ_3 and is 1 near Γ_3 , we get a function $\tilde{\boldsymbol{\nu}} \in H^1(\Omega; \mathbb{R}^2)$ with $\tilde{\boldsymbol{\nu}}|_{\Gamma_3} = \boldsymbol{\nu}$. Assume

$$g|_{\Gamma_3} \in H^2(\Gamma_3). \quad (3.21)$$

Then, g is continuous and is the restriction of an $H^1(\Omega)$ function \tilde{g} on Γ_3 . Define

$$\mathbf{v} = \mathbf{u}^h + (\min\{\tilde{g}, \mathbf{u}^h \cdot \tilde{\boldsymbol{\nu}}\} - \mathbf{u}^h \cdot \tilde{\boldsymbol{\nu}})\tilde{\boldsymbol{\nu}}.$$

Then,

$$\mathbf{v}_\tau = \mathbf{u}_\tau^h, \quad v_\nu = \min\{g, u_\nu^h\} \quad \text{on } \Gamma_3$$

and consequently, $\mathbf{v} \in \mathbf{U}$. Note that

$$\|\mathbf{v} - \mathbf{u}^h\|_{L^2(\Gamma_3; \mathbb{R}^2)} = \|v_\nu - u_\nu^h\|_{L^2(\Gamma_3)}.$$

Now

$$0 \leq u_\nu^h(\mathbf{x}) - v_\nu(\mathbf{x}), \quad \mathbf{x} \in \Gamma_3,$$

and since $u_\nu^h(\mathbf{x}) \leq g(\mathbf{x})$ at all nodes \mathbf{x} on $\overline{\Gamma_3}$, we have

$$u_\nu^h(\mathbf{x}) \leq \Pi^h g(\mathbf{x}), \quad \mathbf{x} \in \Gamma_3,$$

where $\Pi^h g$ is the continuous piecewise linear interpolant of g on $\overline{\Gamma_3}$. Thus, on the part of Γ_3 where $u_\nu^h > g$, we have $v_\nu = g$ and

$$0 \leq u_\nu^h(\mathbf{x}) - v_\nu(\mathbf{x}) \leq \Pi^h g(\mathbf{x}) - g(\mathbf{x}).$$

On the remaining part of Γ_3 , $u_\nu^h \leq g$ and $v_\nu = u_\nu^h$. Thus,

$$\|v_\nu - u_\nu^h\|_{L^2(\Gamma_3)} \leq \|\Pi^h g - g\|_{L^2(\Gamma_3)}.$$

So we get

$$\inf_{\mathbf{v} \in \mathbf{U}} \|\mathbf{v} - \mathbf{u}_h\|_{L^2(\Gamma_3; \mathbb{R}^2)} \lesssim h^2.$$

Therefore, under the smoothness assumptions (3.18), (3.20) and (3.21), we obtain from (3.19) that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} \lesssim h.$$

Without assuming the solution regularities (3.18), (3.20) and in the case $g = 0$, we can apply Theorem 2.3 to show the convergence of the virtual element solution:

$$\mathbf{u}_h \rightarrow \mathbf{u} \quad \text{in } \mathbf{V}. \tag{3.22}$$

For this purpose, let us verify (2.20) (with a general g) and (2.21) (with $g = 0$). Note that \mathbf{U}_h can be equivalently defined as

$$\mathbf{U}_h = \{ \mathbf{v}_h \in \mathbf{V}_h \mid v_\nu^h \leq \Pi^h g \text{ on } \Gamma_3 \}.$$

Suppose $\mathbf{v}_h \in \mathbf{U}_h$ and $\mathbf{v}_h \rightharpoonup \mathbf{v}$ in \mathbf{V} . Due to the compactness of the trace operator $H^1(\Omega) \subset L^2(\Gamma)$ ([1]), we have a sub-sequence $\{\mathbf{v}_{h'}\} \subset \{\mathbf{v}_h\}$ such that

$$\mathbf{v}_{h'} \rightarrow \mathbf{v} \quad \text{in } L^2(\Gamma_3; \mathbb{R}^2) \quad \text{and a.e. on } \Gamma_3.$$

Then from

$$v_\nu^{h'} \leq \Pi^{h'} g \quad \text{on } \Gamma_3,$$

we find that

$$v_\nu \leq g \quad \text{a.e. on } \Gamma_3.$$

It is then obviously true that $\mathbf{v} \in \mathbf{U}$, i.e., (2.20) is valid.

To verify (2.21) for the case $g = 0$, we note that $\mathbf{U} \cap C^\infty(\overline{\Omega}; \mathbb{R}^2)$ is dense in \mathbf{U} . This result is proved in [33]. Thus, for an arbitrarily by fixed element $\mathbf{v} \in \mathbf{U}$ and any $\epsilon > 0$, there exists an element $\mathbf{v}_\epsilon \in \mathbf{U} \cap C^\infty(\overline{\Omega}; \mathbb{R}^2)$ such that

$$\|\mathbf{v}_\epsilon - \mathbf{v}\|_{\mathbf{V}} < \frac{\epsilon}{2}.$$

For $h > 0$ sufficiently small, we have

$$\|\mathbf{v}_\epsilon - \mathbf{v}_{\epsilon,I}^h\|_{\mathbf{V}} \leq Ch\|\mathbf{v}_\epsilon\|_{H^2(\Omega;\mathbb{R}^2)} < \frac{\epsilon}{2},$$

where $\mathbf{v}_{\epsilon,I}^h \in \mathbf{U}_h$ is the global interpolant of \mathbf{v}_ϵ . Then,

$$\|\mathbf{v}_{\epsilon,I}^h - \mathbf{v}\|_{\mathbf{V}} \leq \|\mathbf{v}_{\epsilon,I}^h - \mathbf{v}_\epsilon\|_{\mathbf{V}} + \|\mathbf{v}_\epsilon - \mathbf{v}\|_{\mathbf{V}} < \epsilon.$$

Hence, $\mathbf{v} \in \mathbf{U}$ can be approximated by a sequence of virtual element functions.

4. Numerical experiments

In order to do numerical simulation, we first rewrite Problem (P_1^h) in matrix/vector notation. Let N_0 be the number of nodal points, and let $\{\phi_i\}_{i=1}^{2N_0}$ be the shape basis functions of \mathbf{V}_h . Then a function $\mathbf{v} \in \mathbf{V}_h$ can be expressed as $\mathbf{v} = \sum_{k=1}^{2N_0} \alpha_k \phi_k$, such that the coefficients form a vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_{2N_0}]^T \in \mathbb{R}^{2N_0}$. Define $\ell : \mathbb{R}^{2N_0} \rightarrow \mathbb{R}$ by

$$\ell(\boldsymbol{\alpha}) = \int_{\Gamma_3} j_\nu(\mathbf{v}) ds = \int_{\Gamma_3} j_\nu \left(\sum_{k=1}^{2N_0} \alpha_k \phi_k(\mathbf{x}) \right) ds, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{2N_0}. \tag{4.1}$$

Then, we require to find $\boldsymbol{\alpha}^* = [\alpha_k^*]_{k=1}^{2N_0}$ such that it satisfies the conditions

$$\sum_{k=1}^{2N_0} \alpha_k \phi_k(\mathbf{x}) \in \mathbf{U}_h, \tag{4.2}$$

$$\mathbf{b} - \mathbf{A}\boldsymbol{\alpha} \in \partial\ell(\boldsymbol{\alpha}), \tag{4.3}$$

where

$$\mathbf{b} = [b_i]_{i=1}^{2N_0}, \quad b_i = \langle \mathbf{f}_h, \phi_i \rangle, \quad \mathbf{A} = [A_{ik}]_{i,k=1}^{2N_0}, \quad A_{ik} = a_h(\phi_i, \phi_k).$$

The numerical solution is given by

$$\mathbf{u}_h = \sum_{k=1}^{2N_0} \alpha_k^* \phi_k(\mathbf{x}) \in \mathbf{U}_h.$$

To numerically evaluate $\ell(\cdot)$, we approximate it through numerical integration:

$$\int_{\Gamma_3} j \left(\sum \alpha_k \phi_k(\mathbf{x}) \right) ds \approx \sum_{i \in I} w_i j(\alpha_k \phi_k(\mathbf{x}_i)) =: \mathbf{w}^T \mathbf{j}(\boldsymbol{\alpha}).$$

In addition, we assume that N_1 components corresponding to the index set I of Γ_3 are listed first, and write the vector $\boldsymbol{\alpha}$ in block form as $\boldsymbol{\alpha} = [(\boldsymbol{\alpha}_1)^T, (\boldsymbol{\alpha}_2)^T]^T$ with $\boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}$. Similarly,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{0} \end{bmatrix}. \tag{4.4}$$

As shown in [26], Problem (P_1^h) is equivalent to a minimization problem. Following arguments similar to that in [23], we can reformulate it as

$$\begin{cases} \min & F(\boldsymbol{\alpha}_1), \\ \text{s.t.} & \boldsymbol{\alpha}_1 \in \mathcal{C}, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} F(\boldsymbol{\alpha}_1) &= \frac{1}{2} \boldsymbol{\alpha}_1^T \widetilde{\mathbf{A}}_1 \boldsymbol{\alpha}_1 - \widetilde{\mathbf{b}}_1^T \boldsymbol{\alpha}_1 + \mathbf{w}_1^T \mathbf{j}(\boldsymbol{\alpha}_1), \quad \forall \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}, \\ \mathcal{C} &= \{ \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1} \mid \max\{|\boldsymbol{\alpha}_1| - \mathbf{g}\} \leq 0 \} \end{aligned} \tag{4.6}$$

with

$$\widetilde{\mathbf{A}}_1 = \mathbf{A}_{11} - \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \mathbf{A}_{12}^T, \quad \widetilde{\mathbf{b}}_1 = \mathbf{b}_1 - \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \mathbf{b}_2.$$

Here, $|\boldsymbol{\alpha}_1|$ denotes a new vector formed by taking the absolute value for each entry of $\boldsymbol{\alpha}_1$ and \mathbf{g} is a vector formed by the values of $g(\boldsymbol{x})$ at node points on Γ_3 .

Finally, the problem (4.5) is recast as a DC programming problem where the objective function is the difference of two convex functions, which can be solved efficiently by using the multiobjective double bundle method developed in [38]. Now, let us consider the numerical simulation of an example of Problem (P_1) to investigate the computational performance of our method proposed.

Example 4.1. The domain $\Omega = (0, 1) \times (0, 1)$ is the cross section of a three-dimensional linearly elastic body and the plane stress condition is imposed. The boundary $\partial\Omega$ is decomposed into three parts: $\Gamma_1 = (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$ where the body is clamped, $\Gamma_3 = (0, 1) \times \{0\}$ where frictional contact takes place, and the remaining part $\Gamma_2 = (0, 1) \times \{1\}$ for traction boundary condition. The elasticity tensor \mathcal{F} is given by

$$(\mathcal{F}\boldsymbol{\varepsilon})_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}, \quad 1 \leq i, j \leq 2,$$

where E is the Young modulus, ν is the Poisson ratio of the material and δ_{ij} is the Kronecker delta. We use the following data:

$$\begin{aligned} E &= 65 \text{ daN/mm}^2, \quad \nu = 0.29, \\ \mathbf{f}_1 &= (0, 0) \text{ GPa m}, \quad \mathbf{f}_2 = (0, -8) \text{ GPa m}. \end{aligned}$$

For the numerical simulations we choose $g = 0.02$. In addition, we choose

$$k_v(r) = 40 (0.5r^+ + p(r)), \quad r \in \mathbb{R},$$

where $r^+ = \max\{r, 0\}$ and

$$p(r) = \begin{cases} 0, & \text{if } r < 0, \\ r, & \text{if } r \in [0, 0.01], \\ 0.02 - r, & \text{if } r \in (0.01, 0.02), \\ r - 0.02, & \text{if } r \geq 0.02. \end{cases}$$

Choosing $j_\nu(u_\nu) = \int_0^{u_\nu} k_\nu(r) dr$, we obtain

$$j_\nu(u_\nu) = \begin{cases} 0, & \text{if } u_\nu \leq 0, \\ 30u_\nu^2, & \text{if } u_\nu \in (0, 0.01], \\ 0.8u_\nu - 10u_\nu^2 - 0.004, & \text{if } u_\nu \in (0.01, 0.02), \\ 30u_\nu^2 - 0.8u_\nu + 0.012, & \text{if } u_\nu \geq 0.02. \end{cases} \quad (4.7)$$

Let $j_\nu(u_\nu) = j_\nu^1(u_\nu) - j_\nu^2(u_\nu)$, where

$$j_\nu^1(u_\nu) = \begin{cases} 10u_\nu^2, & \text{if } u_\nu \leq 0, \\ 40u_\nu^2, & \text{if } u_\nu \in (0, 0.01], \\ 0.8u_\nu - 0.004, & \text{if } u_\nu \in (0.01, 0.02), \\ 40u_\nu^2 - 0.8u_\nu + 0.012, & \text{if } u_\nu \geq 0.02, \end{cases} \quad (4.8)$$

and

$$j_\nu^2(u_\nu) = 10u_\nu^2.$$

For the numerical solution u_ν^h , we choose

$$j_\nu(u_\nu^h) = j_\nu^1(u_\nu^h) - j_\nu^2(u_\nu^h),$$

$$\int_{\Gamma_3} j_\nu(u_\nu^h) ds = \int_{\Gamma_3} (j_\nu^1(u_\nu^h) - j_\nu^2(u_\nu^h)) ds \approx \mathbf{w}_1^T \mathbf{j}(\boldsymbol{\alpha}_1) =: j(\boldsymbol{\alpha}_1)$$

with

$$j(\boldsymbol{\alpha}_1) = j_\nu^1(\boldsymbol{\alpha}_1) - j_\nu^2(\boldsymbol{\alpha}_1),$$

and the corresponding constraint is

$$\mathcal{C} = \{\boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1} \mid \max\{|\boldsymbol{\alpha}_1| - \mathbf{g}\} \leq 0\}.$$

Then, we use the multiobjective double bundle method to obtain the numerical results.

The numerical solutions in normal direction, corresponding to four different types polygonal meshes (cf. [42]) are displayed in Fig. 1, respectively.

We compute the numerical solutions for different N , which is the element number of the mesh. According to the numerical solutions in normal direction on the boundary $[0, 1] \times \{0\}$, a similar convergence trend is clearly observed (cf. Fig. 2).

In Table 1 and Fig. 3, we report relative errors $\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|_E / \|\mathbf{u}_{\text{ref}}\|_E$ of the numerical solutions in the energy norm on square meshes, where the energy norm is given by

$$\|\mathbf{v}\|_E = \frac{1}{\sqrt{2}} (\mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{v})), \boldsymbol{\varepsilon}(\mathbf{v}))_Q^{\frac{1}{2}}.$$

Table 1: Numerical errors on square meshes for lowest-order VEM.

h	1/4	1/8	1/16	1/32	1/64
error	52.07%	30.67%	17.89%	10.31%	5.602%

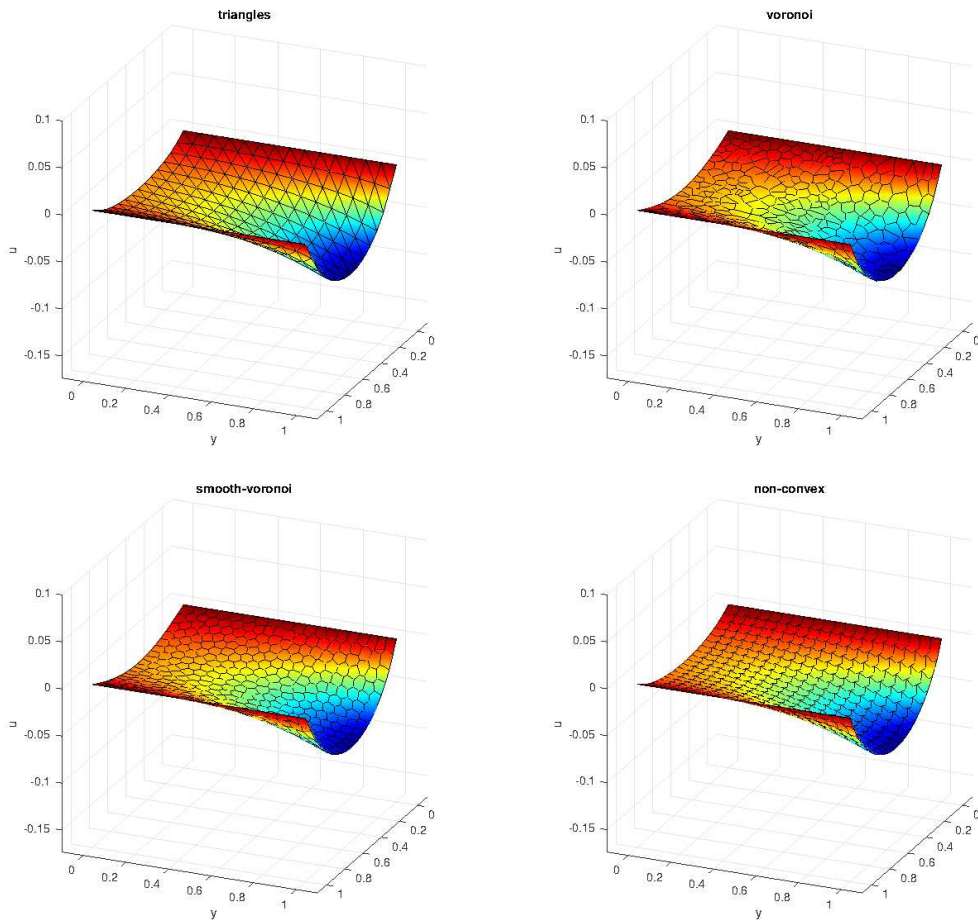


Figure 1: The numerical solution in normal direction related to different polygonal meshes.

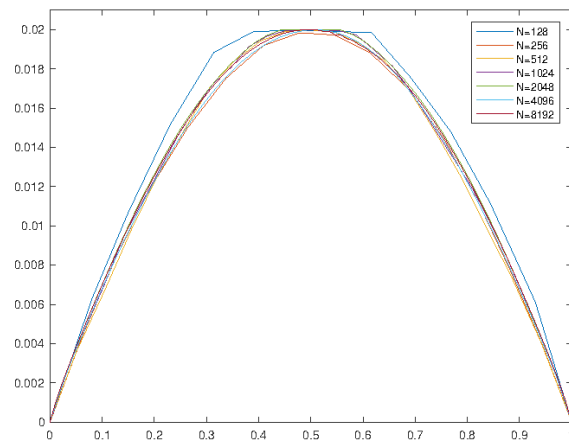


Figure 2: The numerical solutions in normal direction on $[0, 1] \times \{0\}$ for different N .

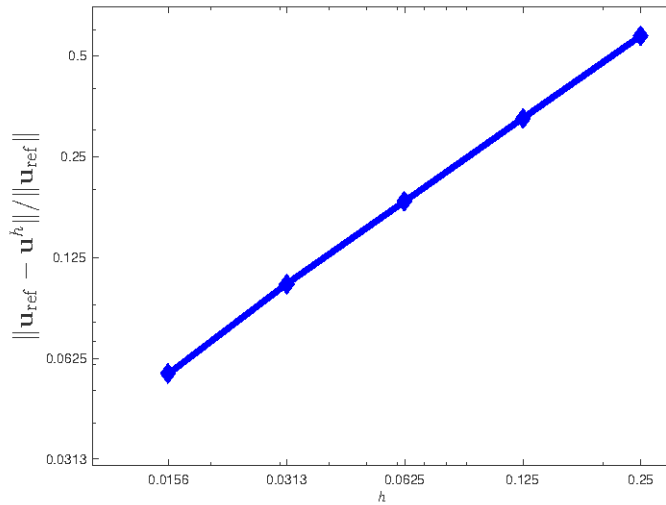


Figure 3: Relative errors in energy norm.

Note that the error bound predicts an optimal first order convergence of the numerical solutions measured in the energy norm, under the suitable regularity assumptions. Since the true solution u is not available, we use the numerical solution with a fine mesh as the “reference” solution u_{ref} in computing the solution errors. Specifically, the “reference” solution u_{ref} is set as the numerical solution with $h = \frac{1}{256}$.

The relative errors in energy norm are shown in Fig. 3.

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