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Low Regularity Error Analysis for Weak Galerkin Finite Element Methods for Second Order Elliptic Problems

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Abstract. This paper presents error estimates in both an energy norm and the L^2-norm for the weak Galerkin (WG) finite element methods for elliptic problems with low regularity solutions. The error analysis for the continuous Galerkin finite element remains same regardless of regularity. A totally different analysis is needed for discontinuous finite element methods if the elliptic regularity is lower than H-1.5. Numerical results confirm the theoretical analysis.

AMS subject classifications: Primary: 65N15, 65N30; Secondary: 35J50

Key words: Weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, low regularity.

1. Introduction

The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. The WG method was first introduced in [16] and then has been applied to solve various partial differential equations such as second order elliptic equations, biharmonic equations, Stokes equations, convection dominant problems, two-phase flow problems and Maxwell's equations [1,2,4–8,10–14,17–19]. However, the standard a priori error analysis of weak Galerkin finite element methods requires additional regularity on solutions. For second order elliptic problems, it is usually assumed that the solutions are in H^{1+s} with $s > \frac{1}{2}$. It is desirable to develop a new type of error estimates for the problems with low regularity solutions such as elliptic interface problems.

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In [3], a new error estimate in energy norm with low regularity assumption has been developed for the interior penalty discontinuous Galerkin methods. The purpose of this work is to provide an error estimate in an energy norm for elliptic problems with low regularity solutions for the WG methods following the ideas in [3]. In addition, we also derive a convergence analysis in the L^2 norm with low regularity assumption for the weak Galerkin finite element method.

We consider the following elliptic problem that seeks an unknown function u satisfying

$$-\nabla\cdot(a\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$.

2. Weak Galerkin finite element schemes

We adopt standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s\geq 0$. When $D=\Omega$, we drop the subscript D in the norm and inner product notation.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of triangles and tetrahedrons. Denote by $\mathcal{E}_h$ the set of all edges and faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0=\mathcal{E}_h\backslash\partial\Omega$ be the set of all interior edges and faces. For every element $T\in\mathcal{T}_h$, we denote by h_T its diameter and mesh size $h=\max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$. We adopt the following notations,

$$(v,w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}(v,w)_T = \sum_{T\in\mathcal{T}_h}\int_T vw dx,$$

$$\langle v,w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\langle v,w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\int_{\partial T} vw ds.$$

For a given integer $k\geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

$$V_h = \big\{v=\{v_0,v_b\} : v_0|_T\in P_k(T),\ v_b|_e\in P_k(e),\ e\subset\partial T, T\in\mathcal{T}_h\big\} \tag{2.1}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \big\{v : v\in V_h,\ v_b = 0 \text{ on } \partial\Omega\big\}. \tag{2.2}$$

We would like to emphasize that any function $v\in V_h$ has a single value v_b on each edge $e\in\mathcal{E}_h$.

For $v=\{v_0,v_b\}\in V_h$ or $v\in H_0^1(\Omega)$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T\in\mathcal{T}_h$, $\nabla_w v\in[P_{k-1}(T)]^d$ satisfies

$$(\nabla_w v,\mathbf{q})_T = -(v_0,\nabla\cdot\mathbf{q})_T + \langle v_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T},\quad \forall\mathbf{q}\in[P_{k-1}(T)]^d. \tag{2.3}$$

In the above equation, we let $v_0 = v$ and $v_b = v$ if $v \in H^1(\Omega)$. Now we introduce two bilinear forms on V_h as follows:

$$
\begin{aligned}
s(v,w) &= \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle v_0 - v_b, w_0 - w_b\rangle_{\partial T},\\
a(v,w) &= (a\nabla_w v, \nabla_w w) + s(v,w).
\end{aligned}
$$

A weak Galerkin approximation for (1.1)-(1.2) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h^0$ satisfying the following equation:

$$a(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_h^0. \tag{2.4}$$

3. Error estimates in energy norm

We begin this section by defining two norms. For any $v \in V_h$ or $v \in H^1(\Omega)$, define two semi-norms

$$|||v|||^2 = a(v,v), \tag{3.1}$$

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 + s(v,v). \tag{3.2}$$

It has been proved in [9] that for any $v \in V_h$ one has

$$C_1\|v\|_h \le |||v||| \le C_2\|v\|_h. \tag{3.3}$$

It is obvious that $\|\cdot\|_h$ defines a norm. The norm equivalence (3.3) implies that $|||\cdot|||$ also defines a norm. The data oscillation is defined as

$$osc(f)^2 = \sum_{T\in\mathcal{T}_h} h_T^2\|f - f_T\|_T^2, \tag{3.4}$$

where f_T is the L^2 projection of f onto $P_k(T)$ for $T \in \mathcal{T}_h$. Let T_1 and T_2 be two triangles/tetrahedrons sharing e if $e \in \mathcal{E}_h^0$. For $e \in \mathcal{E}_h$ and $v \in V_h$ or $v \in H_0^1(\Omega)$, the jump $[v]$ is defined as

$$[v] = v, \quad \text{if } e \subset \partial\Omega, \quad [v] = v|_{T_1} - v|_{T_2}, \quad \text{if } e \in \mathcal{E}_h^0. \tag{3.5}$$

The order of T_1 and T_2 is not essential. The average $\{v\}$ is defined as

$$\{v\} = v, \quad \text{if } e \subset \partial\Omega, \quad \{v\} = \frac{1}{2}\big(v|_{T_1} + v|_{T_2}\big), \quad \text{if } e \in \mathcal{E}_h^0. \tag{3.6}$$

Lemma 3.1. *Let $\phi \in H^1(\Omega)$, then for $T \in \mathcal{T}_h$,*

$$(\nabla_w\phi, \mathbf{q})_T = (\nabla\phi, \mathbf{q})_T, \quad \forall \mathbf{q} \in [P_{k-1}(T)]^d. \tag{3.7}$$

Furthermore, if $\phi \in V_h \cap H^1(\Omega)$, one has

$$\nabla_w\phi = \nabla\phi. \tag{3.8}$$

Proof. Using (2.3) and integration by parts, we have that for any $\mathbf{q} \in [P_{k-1}(T)]^d$

$$(\nabla_w \phi, \mathbf{q})_T = -(\phi, \nabla \cdot \mathbf{q})_T + \langle \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} = (\nabla \phi, \mathbf{q})_T,$$

which implies (3.7). The Eq. (3.8) is a direct result of (3.7). □

For a $v = \{v_0, v_b\} \in V_h$, there exists $v_I \in V_h \cap H_0^1(\Omega)$ such that

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|v_0 - v_I\|_T^2 + \|v_I\|_h^2 \leq \|v\|_h^2. \tag{3.9}$$

The above estimate can be found as the estimate [3, (3.12)] although the norm $\|\cdot\|_h$ is defined slightly different in the two papers.

Since $v_I \in H_0^1(\Omega)$, we have

$$(\nabla u, \nabla v_I) = (f, v_I). \tag{3.10}$$

It follows from (3.7) and (3.8),

$$(\nabla u, \nabla v_I) = (\nabla u, \nabla_w v_I) = (\nabla_w u, \nabla_w v_I), \tag{3.11}$$

$$s(u, v_I) = 0, \tag{3.12}$$

which imply

$$a(u, v_I) = (f, v_I). \tag{3.13}$$

The following lemma has been proved in [15].

Lemma 3.2. *Let $v \in V_h$, then*

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|f + \Delta v\|_T^2 \leq C\Big(\sum_{T \in \mathcal{T}_h} \|\nabla(u - v)\|_T^2 + osc(f)^2 \Big), \tag{3.14}$$

$$\sum_{e \in \mathcal{E}_h^0} h_e \|[\nabla v]\|_e^2 \leq C\Big(\sum_{T \in \mathcal{T}_h} \|\nabla(u - v)\|_T^2 + osc(f)^2 \Big). \tag{3.15}$$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true:

$$\|\varphi\|_e^2 \leq C \left(h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2 \right). \tag{3.16}$$

Theorem 3.1. *Let $u_h \in V_h^0$ be the weak Galerkin finite element solution of* (2.4). *Assume the exact solution $u \in H^1(\Omega)$. Then, there exists a constant C such that*

$$\|u - u_h\|_h \leq C\big(\|u - u_I\|_h + osc(f) \big), \tag{3.17}$$

where $u_I \in V_h \cap H_0^1(\Omega)$ is the standard interpolation of u.

Proof. It follows from (3.3), (2.4) and (3.13) that for $\phi = u_h - u_I$,

$$\begin{aligned} C\|u_h - u_I\|_h^2 &\le a(u_h - u_I, \phi) = (f, \phi) - a(u_I, \phi) \\ &= a(u, \phi_I) - a(u_I, \phi_I) + (f, \phi_0 - \phi_I) - a(u_I, \phi - \phi_I) \\ &\le \big|a(u, \phi_I) - a(u_I, \phi_I)\big| + \big|(f, \phi_0 - \phi_I) - a(u_I, \phi - \phi_I)\big| \\ &=: T_1 + T_2. \end{aligned} \tag{3.18}$$

Here $\phi_I \in V_h \cap H_0^1(\Omega)$ is an averaging function of $\phi \in V_h$.

Next we will estimate the terms T_1 and T_2. The Cauchy-Schwarz inequality and (3.11) imply

$$\begin{aligned} T_1 &= \big|a(u, \phi_I) - a(u_I, \phi_I)\big| \\ &= \big|(\nabla u, \nabla \phi_I) - (\nabla u_I, \nabla \phi_I)\big| \\ &\le C\|u - u_I\|_h \|\phi\|_h. \end{aligned} \tag{3.19}$$

To bound T_2, we define $\psi = \phi - \phi_I$ and apply (3.8) and (2.3) to have

$$\begin{aligned} &(f, \phi - \phi_I) - a(u_I, \phi - \phi_I) \\ =& (f, \psi_0) - a(u_I, \psi) \\ =& (f, \psi_0) - (\nabla u_I, \nabla_w \psi)_{\mathcal{T}_h} \\ =& (f + \Delta u_I, \psi_0)_{\mathcal{T}_h} - \langle \nabla u_I \cdot \mathbf{n}, \psi_b \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{3.20}$$

To obtain (3.20), we also use the fact $s(u_I, \psi) = 0$. The estimate (3.14) and (3.9) imply

$$\begin{aligned} &\big|(f + \Delta u_I, \psi_0)_{\mathcal{T}_h}\big| \\ \le& \Big(\sum_{T \in \mathcal{T}_h} h_T^2 \|f + \Delta u_I\|_T^2\Big)^{\frac{1}{2}} \Big(\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\psi_0\|_T^2\Big)^{\frac{1}{2}} \\ \le& C\Big(\sum_{T \in \mathcal{T}_h} \|\nabla(u - u_I)\|_T^2 + osc(f)^2\Big)^{\frac{1}{2}} \Big(\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\phi_0 - \phi_I\|_T^2\Big)^{\frac{1}{2}} \\ \le& C\big(\|u - u_I\|_h + osc(f)\big)\|\phi\|_h. \end{aligned} \tag{3.21}$$

Applying (3.15), (3.16) and (3.9), we have

$$\begin{aligned} &\big|\langle \nabla u_I \cdot \mathbf{n}, \psi_b \rangle_{\partial \mathcal{T}_h}\big| = \Big|\sum_{e \in \mathcal{E}_h^0} \int_e [\nabla u_I]\{\psi_b\}\Big| \\ =& \Big|\sum_{e \in \mathcal{E}_h^0} \int_e \big([\nabla u_I]\{\psi_b - \psi_0\} + [\nabla u_I]\{\psi_0\}\big)\Big| \\ \le& C\Big(\sum_{e \in \mathcal{E}_h^0} h_e \|[\nabla u_I]\|_e^2\Big)^{\frac{1}{2}} \bigg(\Big(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\psi_0 - \psi_b\|_e^2\Big)^{\frac{1}{2}} + \Big(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\phi_0 - \phi_I\|_e^2\Big)^{\frac{1}{2}}\bigg) \\ \le& C\big(\|\nabla(u - u_I)\|_h + osc(f)\big)\big(\|\psi\|_h + \|\phi\|_h\big) \\ \le& C\big(\|\nabla(u - u_I)\|_h + osc(f)\big)\|\phi\|_h. \end{aligned} \tag{3.22}$$

Using (3.21) and (3.22), we have

$$\begin{aligned} T_2 &= \big|(f, \phi - \phi_I) - a(u_I, \phi - \phi_I)\big| \\ &\le C\big(\|\nabla(u - u_I)\|_h + osc(f)\big)\|\phi\|_h. \end{aligned} \tag{3.23}$$

Combining (3.19) and (3.23) with (3.18) gives

$$C\|u_h - u_I\|_h \le C\big(\|\nabla(u - u_I)\|_h + osc(f)\big). \tag{3.24}$$

The above estimate and the triangle inequality imply (3.17) and we have proved the theorem. □

If $s \in [0,1]$ and $osc(f) = 0$, the estimate (3.17) implies

$$\|u - u_h\|_h \le Ch^s|u|_{1+s}. \tag{3.25}$$

4. Error estimates in L^2 norm

In this section, we provide an error estimate in the L^2-norm with low regularity assumption for the weak Galerkin finite element solution. Consider a dual problem that seeks $w \in H_0^1(\Omega)$ satisfying

$$(\nabla w, \nabla v) = (\phi_I, v), \quad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Assume that the following regularity holds for $s \in [0,1]$,

$$\|w\|_{1+s} \le C\|\phi_I\|. \tag{4.2}$$

Theorem 4.1. *Let $u_h = \{u_0, u_b\} \in V_h^0$ be the weak Galerkin finite element solution of* (2.4). *Assume that the exact solution $u \in H^{1+s}(\Omega)$ for $s \in [0,1]$ and* (4.2) *holds. Then, there exists a constant C such that*

$$\|u - u_0\| \le C\big(h^{2s}|u|_{1+s} + h^{1+s}|u|_{1+s}\big). \tag{4.3}$$

Proof. Recall $\phi = u_h - u_I$. It follows from (4.1) that

$$\|\phi_I\|^2 = (\nabla w, \nabla\phi_I) = (\nabla w - \nabla w_I, \nabla\phi_I) + (\nabla w_I, \nabla\phi_I). \tag{4.4}$$

The estimates (3.24) and (4.2) imply

$$\begin{aligned} \big|(\nabla w - \nabla w_I, \nabla\phi_I)\big| &\le \|\nabla(w - w_I)\|\|\phi\|_h \\ &\le Ch^{2s}\|w\|_{1+s}|u|_{1+s} \\ &\le Ch^{2s}\|\phi_I\||u|_{1+s}. \end{aligned} \tag{4.5}$$

Now we estimate the second term in (4.4). Recall $\psi = \phi - \phi_I$. It follows from (3.8),

$$\begin{aligned} (\nabla w_I, \nabla\phi_I) &= (\nabla_w w_I, \nabla_w\phi_I - \nabla_w\phi) - (\nabla w_I, \nabla_w\phi) \\ &= (\nabla w_I, \nabla_w\psi) - (\nabla_w w_I, \nabla_w\phi). \end{aligned} \tag{4.6}$$

Next we estimate the two terms on the right hand side of (4.6). It follows from (2.3)

$$(\nabla w_I, \nabla_w \psi) = -(\Delta w_I, \psi_0)_{\mathcal{T}_h} + \langle \nabla w_I \cdot \mathbf{n}, \psi_b \rangle_{\partial \mathcal{T}_h}. \tag{4.7}$$

Using (3.14), (3.9), (3.24), (4.2) and the fact $osc(\phi_I) = 0$, one has

$$\begin{aligned}
&\big|(\Delta w_I, \psi_0)_{\mathcal{T}_h}\big| = \big|(\Delta w_I + \phi_I, \psi_0)_{\mathcal{T}_h} - (\phi_I, \psi_0)_{\mathcal{T}_h}\big| \\
\leq & \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|\Delta w_I + \phi_I\|_T^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2} \|\phi_0 - \phi_I\|_T^2\Big)^{\frac{1}{2}} + \|\phi_I\| \|\phi_0 - \phi_I\| \\
\leq & C\big(\|\nabla(w - w_I)\| \|\phi\|_h + h\|\phi_I\| \|\phi\|_h\big) \\
\leq & C\big(h^{2s} + h^{1+s}\big)|u|_{1+s} \|\phi_I\|.
\end{aligned} \tag{4.8}$$

The estimates (3.15), (3.9), (3.24), (3.16) and (4.2) yield

$$\begin{aligned}
&\big|\langle \nabla w_I \cdot \mathbf{n}, \psi_b \rangle_{\partial \mathcal{T}_h}\big| = \Big|\sum_{e\in\mathcal{E}_h^0} \int_e [\nabla w_I]\{\psi_b\} ds\Big| \\
= & \Big|\sum_{e\in\mathcal{E}_h^0} \int_e \big([\nabla w_I]\{\psi_b - \psi_0\} + [\nabla w_I]\{\psi_0\}\big) ds\Big| \\
\leq & C\Big(\sum_{e\in\mathcal{E}_h^0} h_e \|[\nabla w_I]\|_e^2\Big)^{\frac{1}{2}} \bigg(\Big(\sum_{e\in\mathcal{E}_h^0} h_e^{-1} \|\psi_0 - \psi_b\|_e^2\Big)^{\frac{1}{2}} + \Big(\sum_{e\in\mathcal{E}_h^0} h_e^{-1} \|\phi_0 - \phi_I\|_e^2\Big)^{\frac{1}{2}}\bigg) \\
\leq & C\big(\|\nabla(w - w_I)\|\big)\big(\|\psi\|_h + \|\phi\|_h\big) \\
\leq & C\big(\|\nabla(w - w_I)\|\big)\|\phi\|_h \\
\leq & Ch^{2s}|u|_{1+s} \|\phi_I\|.
\end{aligned} \tag{4.9}$$

Combining (4.7)-(4.9), we obtain the estimate for the first term on the right hand side of (4.6),

$$|(\nabla w_I, \nabla_w \psi)| \leq C\big(h^{2s} + h^{1+s}\big)|u|_{1+s} \|\phi_I\|. \tag{4.10}$$

Now we estimate the second term on the right hand side of (4.6). It follows from (3.7) and (3.8)

$$\begin{aligned}
(\nabla_w w_I, \nabla_w \phi) &= \big(\nabla_w (u_I - u_h), \nabla_w w_I\big) \\
&= \big(\nabla_w (u_I - u), \nabla_w w_I\big) + \big(\nabla_w (u - u_h), \nabla_w w_I\big) \\
&= \big(\nabla (u_I - u), \nabla w_I\big) + \big(\nabla_w (u - u_h), \nabla_w w_I\big).
\end{aligned} \tag{4.11}$$

The Cauchy-Schwarz inequality and (4.1) imply

$$\begin{aligned}
\big|(\nabla(u_I - u), \nabla w_I)\big| &= \big|(\nabla(u_I - u), \nabla(w_I - w)) + (\nabla(u_I - u), \nabla w)\big| \\
&\leq \|\nabla(u - u_I)\| \|\nabla(w_I - w)\| + |(\phi_I, u - u_I)| \\
&\leq C\big(h^{2s} + h^{1+s}\big)|u|_{1+s} \|\phi_I\|.
\end{aligned} \tag{4.12}$$

Applying (3.13) and (2.4), we have

$$\begin{aligned}\big(\nabla_w(u-u_h),\nabla_w w_I\big) &= a(u,w_I)-a(u_h,w_I)\\ &= (f,w_I)-(f,w_I)=0.\end{aligned} \tag{4.13}$$

It follows from (4.11)-(4.13)

$$\big|(\nabla w_I,\nabla_w\phi)\big| \le C\big(h^{2s}+h^{1+s}\big)|u|_{1+s}\|\phi_I\|. \tag{4.14}$$

It follows from (4.6), (4.10) and (4.14)

$$\big|(\nabla w_I,\nabla_w\phi_I)\big| \le C\big(h^{2s}+h^{1+s}\big)|u|_{1+s}\|\phi_I\|. \tag{4.15}$$

Combining (4.5), (4.15) with (4.4) gives

$$\|\phi_I\| \le C\big(h^{2s}+h^{1+s}\big)|u|_{1+s}. \tag{4.16}$$

It follows from (3.9), (3.24) and (4.16)

$$\begin{aligned}\|\phi_0\| &\le \|\phi_0-\phi_I\|+\|\phi_I\|\\ &\le Ch^{1+s}|u|_{1+s}+C\big(h^{2s}+h^{1+s}\big)|u|_{1+s}\\ &\le C\big(h^{2s}+h^{1+s}\big)|u|_{1+s}.\end{aligned} \tag{4.17}$$

The above estimate and the triangle inequality imply (4.3). □

5. Numerical experiments

5.1. Example 1

We solve the following Laplace equation:

$$\begin{cases}-\Delta u=0 & \text{in } \Omega=(0,1)^2\setminus\big\{\big(\tfrac12,1\big)\times\big\{\tfrac12\big\}\big\},\\ u=f & \text{on } \partial\Omega,\end{cases} \tag{5.1}$$

where

$$u=r^{\frac12}\sin\left(\frac{\theta}{2}\right) \quad \text{with} \quad \begin{cases}x=\dfrac12+r\cos\theta,\\ y=\dfrac12+r\sin\theta.\end{cases} \tag{5.2}$$

We use a special type of triangular grids as shown in Fig. 1. We note that we intentionally make the grids symmetric at the singular point $(\frac12,\frac12)$. The error and the order of convergence are listed in Table 1, for P_1 and P_2 WG finite element methods, i.e. $k=1$ and $k=2$ in (2.1). As the solution $u\in H^{3/2-\epsilon}(\Omega)$, as proved in the two theorems, we have an half order ($s=\frac12$) convergence in the H^1-like norm, and an one order ($2s=1$) convergence in the L^2 norm.

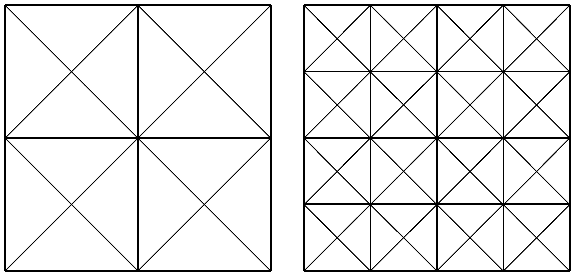

Figure 1: The first two levels of triangular grids, used in Examples 1 and 2.

Table 1: Example 1. Error profiles and convergence rates on Fig. 1 grids.

level	$\|\|\|Q_hu-u_h\|\|\|$	order	$\|Q_hu-u_h\|$	order
	P_1 WG finite element			
4	0.101E+00	0.47	0.513E-02	0.98
5	0.722E-01	0.49	0.259E-02	0.99
6	0.513E-01	0.49	0.130E-02	0.99
7	0.364E-01	0.50	0.651E-03	1.00
	P_2 WG finite element			
4	0.536E-01	0.49	0.168E-02	1.00
5	0.380E-01	0.50	0.841E-03	1.00
6	0.269E-01	0.50	0.420E-03	1.00
7	0.190E-01	0.50	0.210E-03	1.00

5.2. Example 2

We solve the Laplace equation (5.1) on same domain with the boundary condition and the exact solution

$$u = r^{\frac{1}{5}} \sin\left(\frac{\theta}{5}\right) \quad \text{with} \quad \begin{cases} x = \dfrac{1}{2} + r\cos\theta, \\ y = \dfrac{1}{2} + r\sin\theta. \end{cases} \tag{5.3}$$

We use the triangular grids shown in Fig. 1 with $h = \frac{1}{2}^l$ on the l-th level. The solution is in space $H^{1+\frac{1}{5}}(\Omega)$. As it is proved, the H^1-convergence is of order $s = \frac{1}{5} = 0.2$, verified by our computational results in Table 2. The proved order of convergence in L^2-norm is $2s = 0.4$, i.e., $\|u - u_h\|_0 = \mathcal{O}(h^{0.4})$. However, for this example but not for the first example, the numerical order of convergence is higher than $2s$ (seems to be $s + \frac{1}{2}$), by Table 2. So it is likely we have a superconvergence for $\|Q_h u - u_h\|_0$. To see well the solution behaviors, we plot the solution, the P_1 error and the P_2 in Fig. 2.

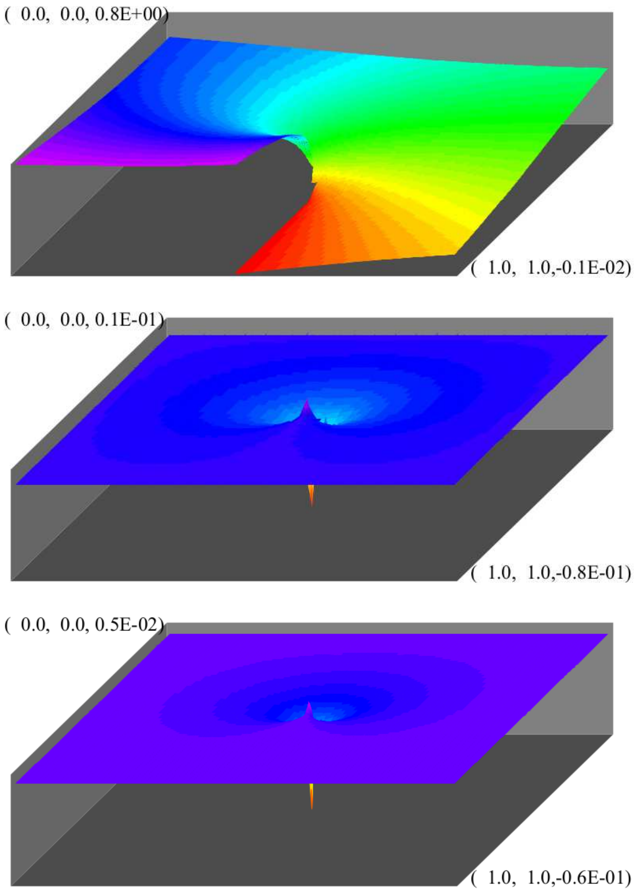

Figure 2: Example 2. The solution u_h of the P_1 element on the 5-th level grid (top), the error $Q_hu - u_h$ of the P_1 element (middle), and the error of the P_2 element (bottom).

Table 2: Example 2. Error profiles and convergence rates on Fig. 1 grids.

level	$\|\|\|Q_hu - u_h\|\|\|$	order	$\|\|Q_hu - u_h\|\|$	order
	P_1 WG finite element			
4	0.149E+00	0.18	0.694E-02	0.69
5	0.131E+00	0.19	0.428E-02	0.69
6	0.114E+00	0.20	0.264E-02	0.70
7	0.995E-01	0.20	0.163E-02	0.70
	P_2 WG finite element			
4	0.110E+00	0.19	0.346E-02	0.70
5	0.962E-01	0.20	0.213E-02	0.70
6	0.838E-01	0.20	0.131E-02	0.70
7	0.730E-01	0.20	0.809E-03	0.70

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