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A Robust Hybrid Spectral Method for Nonlocal Problems with Weakly Singular Kernels

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Abstract. In this paper, we propose a hybrid spectral method for a type of nonlocal problems, nonlinear Volterra integral equations (VIEs) of the second kind. The main idea is to use the shifted generalized Log orthogonal functions (GLOFs) as the basis for the first interval and employ the classical shifted Legendre polynomials for other subintervals. This method is robust for VIEs with weakly singular kernel due to the GLOFs can efficiently approximate one-point singular functions as well as smooth functions. The well-posedness and the related error estimates will be provided. Abundant numerical experiments will verify the theoretical results and show the high-efficiency of the new hybrid spectral method.

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1. Introduction

In model reduction, nonlocality arises naturally. Based on the wide application of multiscale and stochastic modeling in various fields such as materials science, thermodynamics, image analysis fluid dynamics and fracture mechanics [1, 2, 17], nonlocal modeling plays an important role and develops rapidly. Therefore, many scholars have done more in-depth research on nonlocal problems, including the application of various nonlocal equations [11, 13–16, 31]. As a well known case, the integral equation is

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a very important branch of mathematics. According to the number of unknown functions, integral equations can be divided into first-kind, second-type and third-type. This paper is concerned with the following nonlinear second-kind VIEs:

$$y(t) = f(t) + \int_0^t (t-s)^{-\mu} K(t,s) G\big(s, y(s)\big) \mathrm{d}s, \quad t \in U = [0,T], \tag{1.1}$$

where $0 < \mu < 1$, $K \in C(D)$, $D := \{(t,s) : 0 \leq s \leq t \leq T\}$, $f \in C(U)$ and G is a continuous function of s and y.

There are a wide range of applications modeled by VIEs arising in physics, biology and other fields. Originally, Volterra oneself considered the numerical solution of integral equation in his book [35] and references therein. Afterwards, Chen *et al.* [6] obtained the error estimation by using the finite element method in space and the backward Euler scheme in time direction. Besides, piecewise polynomial collocation methods and Runge-Kutta methods [3, 4, 12, 34, 37] also have many analysis on VIEs. However, most local methods may be unsuitable for the nonlocal problems. In recent decades, spectral methods have been developed rapidly and widely used in various fields [5, 18, 20, 21, 26] and references therein. Spectral methods provide exceedingly accurate numerical results with relatively fewer degrees of freedom for smooth solutions. Therefore it has been widely used for VIEs with smooth kernel and solutions [19, 29, 30, 32, 36, 38]. However, the regularity of the solutions is low for VIEs with weakly singular kernel. The traditional spectral methods based on polynomials cannot obtain the efficient approximation to the corresponding solution.

In order to overcome the difficulties caused by the weakly singular kernel and enhance the convergence rate, abundant researchers made a lot of efforts for handling the singularity. Feldstem *et al.* [23,33] analyzed the most fundamental problem. Pedas and Vainikko [24] used piecewise polynomial collocation method through smooth transformation to deal with weakly singular kernels. After that, Chen and Tang [10] analyzed the convergence of the Jacobi spectral-collocation methods for VIEs with weakly singular kernel. Hou *et al.* [22] use Müntz spectral methods to deal with the weakly singular kernel. Since the solution of the weakly singular VIEs (1.1) is singular only at $t = 0$ but smooth for $t \neq 0$, we need to deal with the singularity in the first interval. Sheng and Shen proposed a hybrid spectral element method with mixed generalized Jacobi function [8,25] and Legendre polynomial to solve weakly kernel VIEs [28] where they divided the original domain into some subintervals and use the generalized Jacobi functions to approximate the solution in the first subinterval. The method proposed by Sheng and Shen performed excellent for a special case but not for the more general cases.

In this paper, we propose a universal method by replacing the basis in the first interval by GLOFs [7,9] and use a new hybrid spectral element method to solve VIEs (1.1). The main contributions and advantages of the new method highlighted as follows:

- Our hybrid spectral method can exponentially approximate the singular solutions caused by the weakly singular kernel in (1.1) as well as the smooth solutions.

- The hp-version error for VIEs with weakly singular kernel is established and the existence, uniqueness of the new method are provided.
- Comparing with the existing work [28], our new hybrid spectral method no longer limit to the special solution $t^r u(t)$. The theoretical and numerical results show that our method can efficiently approximate the more general one-point singular solution such as multi-term singular solution $\sum_i t^{r_i} u(t)$.

The paper is organized as follows. In next section, some preparations for GLOFs and Legendre polynomials. In Section 3, the hybrid spectral method for VIEs is established. The existence, uniqueness and hp-version error analysis are provided in Section 4. Some numerical results will be shown in Section 5 to verify the theoretical results. At last, some conclusions and remarks are given in Section 6.

2. Preliminaries

For a given interval Ω and a certain weight function ω, define

$$L^2_\omega(\Omega) := \left\{ v | v \text{ is measurable and } \int_\Omega v^2 \omega \, \mathrm{d}x < \infty \right\}$$

equipped with the inner product $(\cdot,\cdot)_{\Omega,\omega}$ and norm $\|\cdot\|_{\Omega,\omega}$ as follows:

$$(u,v)_{\Omega,\omega} = \int_\Omega uv\omega \mathrm{d}x, \quad \|v\|_{\Omega,\omega} = \big((v,v)_{\Omega,\omega}\big)^{\frac{1}{2}}.$$

In particular, we omit the weight whenever $\omega \equiv 1$.

As a starting point, we shall recast the original VIEs into an equivalent form based on the mesh in the following part. Then, in order to solve the weak singularity arising near $t = 0$, we review the GLOFs [7] and show their approximation results on the first element $I_1 = (0, t_1]$. For the general elements, we shall use the classical polynomials as the basis, so the shifted Legendre polynomials and their basic properties will be discussed in the last part of this section.

2.1. Reformulation of the VIEs

Define a mesh

$$\mathscr{I}_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

Set $h_n := t_n - t_{n-1}$, $h_{\max} = \max_{1\le n\le N} h_n$, $I_n = (t_{n-1}, t_n]$, $y^n(t)$ is the solution of (1.1) on the n-th element, namely

$$y^n(t) = y(t), \quad \forall t \in I_n, \quad 1 \le n \le N.$$

For any $t \in I_n$, Eq. (1.1) can get

$$y(t) = f(t) + \int_0^{t_{n-1}} (t-s)^{-\mu} K(t,s) G\big(s, y(s)\big) \mathrm{d}s$$

$$+\int_{t_{n-1}}^{t}(t-s)^{-\mu}K(t,s)G\big(s,y(s)\big)\mathrm{d}s,$$

which is equivalent to

$$\begin{aligned}y^n(t)=f(t)+\sum_{k=1}^{n-1}\int_{I_k}(t-\xi)^{-\mu}K(t,\xi)G\big(\xi,y^k(\xi)\big)\mathrm{d}\xi\\+\int_{t_{n-1}}^{t}(t-s)^{-\mu}K(t,s)G\big(s,y^n(s)\big)\mathrm{d}s.\end{aligned}$$

Furthermore, employing the linear mapping

$$s=s(t,\tau):=t_{n-1}+\frac{1}{h_n}(\tau-t_{n-1})(t-t_{n-1}),\quad \tau\in I_n, \tag{2.1}$$

we write the original integral equation (1.1) as

$$\begin{aligned}y^n(t)&=f(t)+\mathcal{V}_1^n y(t)+\mathcal{V}_2^n y^n(t)\\&:=f(t)+\sum_{k=1}^{n-1}\int_{I_k}(t-\xi)^{-\mu}K(t,\xi)G\big(\xi,y^k(\xi)\big)\mathrm{d}\xi+\Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\mu}\\&\quad\times\int_{I_n}(t_n-\tau)^{-\mu}K\big(t,s(t,\tau)\big)G\big(s(t,\tau),y^n(s(t,\tau))\big)\mathrm{d}\tau.\end{aligned}\tag{2.2}$$

2.2. Basis for the first element $I_1=(0,t_1]$

2.2.1. The standard GLOFs

Let $\alpha,\beta>-1$. We first consider the GLOFs on the canonical interval $I=(0,1)$ below

$$\mathcal{S}_n^{(\alpha,\beta,\lambda)}(x)=x^{\frac{\beta-\lambda}{2}}\mathcal{S}_n^{(\alpha,\beta)}(x):=x^{\frac{\beta-\lambda}{2}}\mathscr{L}_n^{(\alpha)}\big(-(\beta+1)\log x\big),\quad n=0,1,\ldots,$$

where $\mathscr{L}_n^{(\alpha)}(y),\alpha>-1$ are the Laguerre polynomials of $y\in\mathbb{R}^+$. The GLOFs are mutually orthogonal and satisfy

$$\int_0^1\mathcal{S}_n^{(\alpha,\beta,\lambda)}(x)\mathcal{S}_m^{(\alpha,\beta,\lambda)}(x)\chi^{\alpha,\lambda}(x)\mathrm{d}x=\gamma_n^{(\alpha,\beta)}\delta_{mn}, \tag{2.3}$$

where

$$\gamma_n^{(\alpha,\beta)}=\frac{\Gamma(n+\alpha+1)}{(\beta+1)^{\alpha+1}\Gamma(n+1)}$$

and δ_{mn} is the Kronecker symbol and the weight $\chi^{\alpha,\lambda}(x)=(-\log x)^{\alpha}x^{\lambda}$.

Let $\{y_j^{(\alpha)},\omega_j^{(\alpha)}\}_{j=0}^N$ be the Gauss-node and -weight of $\mathscr{L}_{N+1}^{(\alpha)}(y)$. Denote

$$\Big\{x_j^{(\alpha,\beta,\lambda)}:=e^{-(\beta+1)^{-1}y_j^{(\alpha)}},\ \chi_j^{(\alpha,\beta,\lambda)}=\big(x_j^{(\alpha,\beta.\lambda)}\big)^{\lambda-\beta}(\beta+1)^{-\alpha-1}\omega_j^{(\alpha)}\Big\}_{j=0}^N. \tag{2.4}$$

Then,

$$\int_0^1 f(x)(-\log x)^\alpha x^\lambda \mathrm{d}x = \sum_{j=0}^N f\big(x_j^{(\alpha,\beta,\lambda)}\big)\chi_j^{(\alpha,\beta,\lambda)}, \quad \forall f \in \mathcal{P}_{2N+1}^{\beta-\lambda,\log t}(I),$$

where

$$\mathcal{P}_K^{\beta-\lambda,\log x} := \operatorname{span}\left\{x^{\beta-\lambda},\ x^{\beta-\lambda}\log x,\ x^{\beta-\lambda}(\log x)^2,\ldots,x^{\beta-\lambda}(\log x)^K\right\}.$$

In particular, for clarity, we denote a special case (with $\alpha = 0$ and $\lambda = 0$) by

$$\mathcal{S}_n^{(\beta)}(x) := \mathcal{S}_n^{(0,\beta,0)}(x) = x^{\frac{\beta}{2}}\mathscr{L}_n(-(\beta+1)\log x), \quad n = 0,1,\ldots. \tag{2.5}$$

Similarly, denote by $x_j^{(\beta)} := x_j^{(0,\beta,0)}$, $\chi_j^{(\beta)} := \chi_j^{(0,\beta,0)}$ the corresponding nodes and weights.

Define the projection operator [7] $\tilde{\Pi}_N^\beta : L^2(I) \to \mathcal{P}_N^{\beta/2,\log t}(I)$ by

$$\left(u - \tilde{\Pi}_N^\beta u, v\right) = 0, \quad \forall v \in \mathcal{P}_N^{\beta/2,\log t}(I).$$

Thanks to the orthogonality of the basis $\{\mathcal{S}_n^{(\beta)}\}_{n=0}^\infty$, we have

$$\tilde{\Pi}_N^\beta u = \sum_{n=0}^N \hat{u}_n^\beta \mathcal{S}_n^{(\beta)}(x) \quad \text{with} \quad \hat{u}_n^\beta = (\beta+1)\int_0^1 u(x)\mathcal{S}_n^{(\beta)}(x)\mathrm{d}x.$$

Moreover, for describing the approximability of the projection $\tilde{\Pi}_N^\beta u$, define a pseudo-derivative

$$\hat{\partial}_{\gamma,t} u = t^{1+\gamma}\partial_t\{t^{-\gamma}u\}. \tag{2.6}$$

According to [9, Lemma 3.1], we have

Lemma 2.1. *Let $m, M_1 \in \mathbb{N}$, and $\beta > -1$. For any $u \in L^2(I)$ and $\hat{\partial}_{\beta/2,t}^m u \in L^2_{\chi^m}(I)$, we have*

$$\big\|\tilde{\Pi}_{M_1}^\beta u - u\big\|_I \le cM_1^{-\frac{m}{2}}\big\|\hat{\partial}_{\beta/2,x}^m u\big\|_{I,\chi^m}, \quad \chi^m = |\log(x)|^m. \tag{2.7}$$

2.2.2. The shifted GLOFs

The shifted Log orthogonal functions of degree p on I_1 is defined by

$$\mathcal{S}_{1,p}^{(\beta)}(t) = \mathcal{S}_p^{(\beta)}\left(\frac{t}{t_1}\right) = \left(\frac{t}{t_1}\right)^{\frac{\beta}{2}}\mathscr{L}_p^{(0)}\left(-(\beta+1)\log\frac{t}{t_1}\right), \quad t \in I_1 = (0,t_1].$$

It is easy to verify that

$$\int_{I_1}\mathcal{S}_{1,n}^{(\beta)}(t)\mathcal{S}_{1,m}^{(\beta)}(t)\mathrm{d}t = t_1(\beta+1)^{-1}\delta_{mn}.$$

Thus, for any $\phi(t) \in \mathcal{P}^{\beta,\log t}_{2M_1+1}(I_1)$,

$$\int_{I_1} \phi(\tau)\mathrm{d}\tau = \int_0^1 \phi(t_1 x)t_1 \mathrm{d}x = \sum_{j=0}^{M_1} \phi\big(t_1 x_j^{(\beta)}\big)t_1 \chi_j^{(\beta)}. \tag{2.8}$$

We define the following discrete inner product and norm:

$$\langle u, v\rangle_{I_1} = \sum_{j=0}^{M_1} u\big(t_1 x_j^{(\beta)}\big)v\big(t_1 x_j^{(\beta)}\big)t_1 \chi_j^{(\beta)}, \quad \|v\|_{M_1,I_1} = \langle v, v\rangle_{I_1}^{\frac{1}{2}}. \tag{2.9}$$

According to Eq. (2.8), for any $\phi, \psi \in \mathcal{P}^{\beta/2,\log t}_{M_1}(I_1)$

$$(\phi, \psi)_{I_1} = \langle \phi, \psi\rangle_{I_1}, \quad \|\phi\|_{I_1} = \|\phi\|_{M_1,I_1}.$$

Let $\Pi^{\beta}_{I_1,M_1}$ be the $L^2(I_1)$ orthogonal projection based on space $\mathcal{P}^{\beta/2,\log t}_{M_1}(I_1)$ on interval I_1

$$\Big(\Pi^{\beta}_{I_1,M_1} v - v, \psi\Big)_{I_1} = 0, \quad \forall \psi \in \mathcal{P}^{\beta/2,\log t}_{M_1}(I_1). \tag{2.10}$$

Then, we have the following error estimate for the projection

Lemma 2.2. *Let $\beta > -1$, for any $v \in L^2(I_1)$ and $\hat{\partial}^m_{\beta/2,t} v \in L^2_{\chi_1^m}(I_1)$, we have*

$$\big\|\Pi^{\beta}_{I_1,M_1} v - v\big\|_{I_1} \le c h_1^m M_1^{-\frac{m}{2}} \|\mathcal{L}^m v\|_{I_1,\chi_1^m}, \tag{2.11}$$

where

$$\chi_1^m = \left|\log\left(\frac{t}{t_1}\right)\right|^m, \quad \mathcal{L}v := \left(\frac{t}{t_1}\right)^{1+\frac{\beta}{2}} \partial_t \left(\left(\frac{t}{t_1}\right)^{-\frac{\beta}{2}} v\right).$$

Proof. Denote $u(x) = v(t)|_{t=t_1 x}, x \in (0,1)$. Since $\Pi^{\beta}_{I_1,M_1} v(t)|_{t=t_1 x}$, $\tilde{\Pi}^{\beta}_{M_1} u(x) \in \mathcal{P}^{\beta/2,\log t}_{M_1}(I)$, it holds that

$$\Pi^{\beta}_{I_1,M_1} v(t)\big|_{t=t_1 x} = \tilde{\Pi}^{\beta}_{M_1} u(x).$$

It is easy to derive from the estimate (2.7) and relation $\partial_x = t_1 \partial_t$ that

$$\begin{aligned}\big\|\Pi^{\beta}_{I_1,M_1} v - v\big\|^2_{I_1} &= \int_{I_1} \Big(\Pi^{\beta}_{I_1,M_1} v(t) - v(t)\Big)^2 \mathrm{d}t = h_1 \big\|\tilde{\Pi}^{\beta}_{M_1} u - u\big\|^2_I \\ &\le c h_1 M_1^{-m} \big\|\hat{\partial}^m_{\beta/2,x} u\big\|^2_{I,\chi^m} \le c h_1^{2m} M_1^{-m} \|\mathcal{L}^m v\|^2_{I_1,\chi_1^m}.\end{aligned} \tag{2.12}$$

The proof is complete. ∎

2.3. The shifted Legendre polynomial on I_n

Let $L_p(x)$ be the standard Legendre polynomial of degree p defined on $\Lambda=(-1,1)$. The shifted Legendre polynomial $L_{n,p}(t)$ is defined by

$$L_{n,p}(t)=L_p\left(\frac{2t-t_{n-1}-t_n}{h_n}\right), \quad t\in I_n=(t_{n-1},t_n],$$

where $L_{n,p}(t), p=0,1,2,\dots$ are mutually orthogonal (cf. [27]) and satisfy

$$\int_{I_n} L_{n,p}(t)L_{n,q}(t)\mathrm{d}t=\frac{h_n}{2p+1}\delta_{pq}, \tag{2.13}$$

where δ_{pq} is the Kronecker symbol.

For any given integer $M\ge 0$, we denote by $\{x_j,\omega_j\}_{j=0}^{M}$ the nodes and the corresponding Christoffel numbers of the standard Legendre-Gauss interpolation. Let $\mathcal{P}_{M_n}(I_n)$ be the set of polynomials of degree at most M_n on the interval I_n. Denote by $t_{n,j}$ the shifted Legendre-Gauss nodes on the interval I_n as follows:

$$t_{n,j}=\frac{h_nx_j+t_{n-1}+t_n}{2}\in I_n, \quad 1\le n\le N, \quad 0\le j\le M_n. \tag{2.14}$$

Combining with the Legendre-Gauss quadrature, it holds that

$$\int_{I_n}\phi(t)\mathrm{d}t=\frac{h_n}{2}\sum_{j=0}^{M_n}\phi(t_{n,j})\omega_{n,j}, \quad \forall\phi(t)\in\mathcal{P}_{2M_n+1}(I_n), \tag{2.15}$$

where $\omega_{n,j}=\omega_j, j=0,1,\dots,M_n$.

In particular, we denote by $\{x_j^{(\alpha,\beta)},\omega_j^{(\alpha,\beta)}\}_{j=0}^{M}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation. Similarly, we can obtained that

$$\int_{I_n}\phi(t)\omega_n^{\alpha,\beta}(t)\mathrm{d}t=\left(\frac{h_n}{2}\right)^{\alpha+\beta+1}\sum_{j=0}^{M_n}\phi\big(t_{n,j}^{(\alpha,\beta)}\big)\omega_{n,j}^{(\alpha,\beta)}, \quad \forall\phi(t)\in\mathcal{P}_{2M_n+1}(I_n), \tag{2.16}$$

where

$$\begin{aligned}&\omega_n^{\alpha,\beta}(t)=(t_n-t)^{\alpha}(t-t_{n-1})^{\beta},\\ &t_{n,j}^{(\alpha,\beta)}=\frac{1}{2}\left(h_nx_j^{(\alpha,\beta)}+t_{n-1}+t_n\right), \quad \omega_{n,j}^{(\alpha,\beta)}=\omega_j^{(\alpha,\beta)}, \quad j=0,1,\dots,M_n.\end{aligned}$$

Let $(u,v)_{I_n}$ and $\|v\|_{I_n}$ be the inner product and the norm of space $L^2(I_n)$ respectively. Denote by the corresponding discrete inner product and norm

$$\langle u,v\rangle_{I_n}=\frac{h_n}{2}\sum_{j=0}^{M_n}u(t_{n,j})v(t_{n,j})\omega_{n,j}, \quad \|v\|_{M_n,I_n}=\langle v,v\rangle_{I_n}^{1/2}. \tag{2.17}$$

For any $\phi, \psi \in \mathcal{P}_{2M_n+1}(I_n)$, it holds that

$$(\phi, \psi)_{I_n} = \langle \phi, \psi \rangle_{I_n}, \quad \|\psi\|_{I_n} = \|\psi\|_{M_n, I_n}. \tag{2.18}$$

Next, in order to estimate orthogonal projection, for any integer $m \geq 0$, we define the following non-uniformly Jacobi-weighted Sobolev space:

$$H^m_{\omega^{\alpha,\beta}}(\Lambda) = \big\{ v : \|v\|_{H^m_{\omega^{\alpha,\beta}}} < \infty \big\}, \quad \omega^{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta$$

with the corresponding norm and semi-norm

$$\|v\|_{H^m_{\omega^{\alpha,\beta}}} = \left(\sum_{k=0}^{m} |v|^2_{H^k_{\omega^{\alpha,\beta}}} \right)^{\frac{1}{2}}, \quad |v|_{H^k_{\omega^{\alpha,\beta}}} = \big\| \partial_t^k v \big\|_{\omega^{\alpha+k,\beta+k}}.$$

According to [27, Theorem 3.35], we have

Lemma 2.3. *For any $u \in H^m_{\omega^{0,0}}(\Lambda)$ and integer $1 \leq m \leq M_n + 1$, the following formula holds:*

$$\big\| u - \tilde{\Pi}_M u \big\|_{L^2_{\omega^{0,0}}} \leq c M^{-m} \big\| \partial_x^m u \big\|_{L^2_{\omega^{m,m}}}, \tag{2.19}$$

where $\tilde{\Pi}_M$ is the $L^2(\Lambda)$-orthogonal projection based on polynomial space $\mathcal{P}_{M_n}(\Lambda)$ satisfying

$$\big(\tilde{\Pi}_M u - u, \psi \big)_{L^2(\Lambda)} = 0, \quad \forall \psi \in \mathcal{P}_M(\Lambda).$$

Furthermore, denote by Π_{I_n,M_n} the $L^2(I_n)$-orthogonal projection based on polynomial space $\mathcal{P}_{M_n}(I_n)$ as follows:

$$\big(\Pi_{I_n,M_n} v - v, \psi \big)_{I_n} = 0, \quad \forall \psi \in \mathcal{P}_{M_n}(I_n).$$

Lemma 2.4. *For any $v \in H^m_{\omega^{0,0}}(I_n)$ and integer $1 \leq m \leq M_n + 1$, the following formula holds:*

$$\|v - \Pi_{I_n,M_n} v\|_{I_n} \leq c \left(\frac{h_n}{2} \right)^m M_n^{-m} \big\| \partial_t^m v \big\|_{I_n, \omega_n^{m,m}}, \tag{2.20}$$

where $H^m_{\omega^{0,0}}(I_n)$ is the usual Sobolev space, and

$$\omega_n^{\alpha,\beta}(t) = \left(\frac{2t_n - 2t}{h_n} \right)^\alpha \left(\frac{2t - 2t_{n-1}}{h_n} \right)^\beta.$$

Proof. Obviously, we have

$$u(x) = v(t)\big|_{t = \frac{h_n x + t_{n-1} + t_n}{2}},$$

while $\Pi_{I_n,M_n} v(t)\big|_{t = \frac{h_n x + t_{n-1} + t_n}{2}}$ and $\tilde{\Pi}_{M_n} u(x)$ belong to $\mathcal{P}_{M_n}(\Lambda)$, thus

$$\Pi_{I_n,M_n} v(t)\big|_{t = \frac{h_n x + t_{n-1} + t_n}{2}} = \tilde{\Pi}_{M_n} u(x).$$

We can deduce from (2.19) and relation $\partial_x = \frac{h_n}{2}\partial_t$ that

$$\begin{aligned}
\|v-\Pi_{I_n,M_n}v\|_{I_n}^2 &= \frac{h_n}{2}\int_{-1}^{1}\big(u(x)-\tilde{\Pi}_{M_n}u(x)\big)^2\mathrm{d}x \\
&\le \frac{ch_n}{2}M_n^{-2m}\int_{-1}^{1}\big(\partial_x^m u(x)\big)^2\big(1-x^2\big)^m\mathrm{d}x \\
&\le c\left(\frac{h_n}{2}\right)^{2m}M_n^{-2m}\int_{I_n}\big(\partial_t^m v(t)\big)^2\left(\frac{2t_n-2t}{h_n}\right)^m\left(\frac{2t-2t_{n-1}}{h_n}\right)^m\mathrm{d}t \\
&= c\left(\frac{h_n}{2}\right)^{2m}M_n^{-2m}\int_{I_n}\big(\partial_t^m v(t)\big)^2\omega_n^{m,m}(t)\mathrm{d}t.
\end{aligned}$$

The proof is complete. □

3. The hybrid spectral method for VIEs

The hybrid spectral method for (2.2) is: Find $Y^1(t)\in\mathcal{P}_{M_1}^{\beta/2,\log t}(I_1)$ and $Y^n(t)\in\mathcal{P}_{M_n}(I_n), n\ge 2$ such that

$$\begin{cases}(Y^1,\varphi)_{I_1}=(f,\varphi)_{I_1}+\big(\mathcal{V}_2^1Y^1,\varphi\big)_{I_1}, & \forall\varphi\in\mathcal{P}_{M_1}^{\beta/2,\log t}(I_1),\\ (Y^n,\psi)_{I_n}=(f,\psi)_{I_n}+\big(\mathcal{V}_1^nY,\psi\big)_{I_n}+\big(\mathcal{V}_2^nY^n,\psi\big)_{I_n}, & \forall\psi\in\mathcal{P}_{M_n}(I_n).\end{cases} \tag{3.1}$$

First of all, we expand the numerical solution by the related basis as follows:

$$\begin{cases}Y^1(t)=\sum\limits_{p=0}^{M_1}y_p^1\mathcal{S}_{1,p}^{(\beta)}(t), & t\in I_1,\\ Y^n(t)=\sum\limits_{p=0}^{M_n}y_p^nL_{n,p}(t), & t\in I_n,\quad n\ge 2.\end{cases}$$

By taking $\varphi=\mathcal{S}_{1,q}^{(\beta)}(t), 0\le q\le M_1$ and $\psi=L_{n,q}(t), 0\le q\le M_n$, it holds that

$$\begin{cases}\sum\limits_{p=0}^{M_1}y_p^1\left(\mathcal{S}_{1,p}^{(\beta)},\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1}-\left(\mathcal{V}_2^1Y^1,\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1}=\left(f,\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1},\\ \sum\limits_{p=0}^{M_n}y_p^n(L_{n,p},L_{n,q})_{I_n}-(\mathcal{V}_2^nY^n,L_{n,q})_{I_n}=(f,L_{n,q})_{I_n}+\big(\mathcal{V}_1^nY,L_{n,q}\big)_{I_n}.\end{cases}$$

For clarity, denote by

$$\mathbf{y}^n=\big(y_0^n,y_1^n,\dots,y_{M_n}^n\big)^T, \tag{3.2a}$$

$$\mathbf{f}^n=\big(f_0^n,\dots,f_{M_n}^n\big)^T,\quad f_q^1=\big(f,\mathcal{S}_{1,q}^{(\beta)}\big)_{I_1},\quad f_q^n=(f,L_{n,q})_{I_n},\quad n\ge 2, \tag{3.2b}$$

$$\mathbf{v}^n=\big(v_0^n,\dots,v_{M_n}^n\big)^T,\quad v_q^n=\big(\mathcal{V}_1^nY,L_{n,q}\big)_{I_n},\qquad n\ge 2, \tag{3.2c}$$

$$\mathbf{w}^n=\big(\omega_0^n,\ldots,\omega_{M_n}^n\big)^T,\;\; w_q^1=\big(\mathcal{V}_2^1Y^1,\mathcal{S}_{1,q}^{(\beta)}\big)_{I_1},\;\; w_q^n=\big(\mathcal{V}_2^nY^n,L_{n,q}\big)_{I_n},\qquad n\ge 2, \tag{3.2d}$$

$$\mathbf{A}^n=\big(a_{qp}^n\big)_{0\le p,q\le M_n}, \tag{3.2e}$$

$$a_{qp}^1=\big(\mathcal{S}_{1,p}^{(\beta)},\mathcal{S}_{1,q}^{(\beta)}\big)_{I_1}=t_1(\beta+1)^{-1}\delta_{pq},\;a_{qp}^n=(L_{n,p},L_{n,q})_{I_n}=\frac{h_n}{2p+1}\delta_{pq},\;n\ge 2. \tag{3.2f}$$

Then the numerical solution can be derived from the following linear system:

$$\begin{cases}\mathbf{A}^1\mathbf{y}^1-\mathbf{w}^1(\mathbf{y}^1)=\mathbf{f}^1,\\ \mathbf{A}^n\mathbf{y}^n-\mathbf{w}^n(\mathbf{y}^n)=\mathbf{f}^n+\mathbf{v}^n,\quad n\ge 2.\end{cases} \tag{3.3}$$

In order to obtain the accurate numerical solution $\mathbf{y}^n$ from the above system, we need to compute the entries of vectors and matrix as accurately as possible. In practice, for vector $\mathbf{f}^n$, with the aid of (2.9) and (2.18),

$$f_q^1=\left(f,\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1}=\int_0^1 f(t_1x)\mathcal{S}_{1,q}^{(\beta)}(t_1x)t_1\mathrm{d}x\approx h_1\sum_{j=0}^{M_1}f\left(t_1x_j^{(\beta)}\right)\mathcal{S}_q^{(\beta)}(x_j^{(\beta)})\chi_j^{(\beta)},$$

and

$$f_q^n\approx\langle f,L_{n,q}\rangle_{I_n}=\frac{h_n}{2}\sum_{i=0}^{M_n}f(t_{n,i})L_{n,q}(t_{n,i})\omega_{n,i}.$$

Owing to (2.2), (2.18), the vector $\mathbf{v}^n$ can be computed by

$$\begin{aligned}v_q^n\approx\langle\mathcal{V}_1^nY,L_{n,q}\rangle_{I_n}&=\sum_{k=1}^{n-1}\left\langle\int_{I_k}(t-\xi)^{-\mu}K(t,\xi)G\big(Y^k(\xi)\big)\mathrm{d}\xi,L_{n,q}(t)\right\rangle_{I_n}\\ &=\frac{h_n}{2}\sum_{i=0}^{M_n}\sum_{k=1}^{n-1}\left(\int_{I_k}(t_{n,i}-\xi)^{-\mu}K(t_{n,i},\xi)G\big(Y^k(\xi)\big)\mathrm{d}\xi\right)L_{n,q}(t_{n,i})\omega_{n,i},\end{aligned}$$

where $G(y)=G(\cdot,y)$ for notational simplicity. Similarly, combining with (2.5), (2.9), (2.16) and (2.18), we have that

$$\begin{aligned}&w_q^1=\left(\mathcal{V}_2^1Y^1,\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1}\\ &=\left(\left(\frac{t}{h_1}\right)^{1-\mu}\int_{I_1}(t_1-\tau)^{-\mu}K\big(t,s(t,\tau)\big)G\big(Y^1(s(t,\tau))\big)\mathrm{d}\tau,\mathcal{S}_{1,q}^{(\beta)}(t)\right)_{I_1}\\ &=\left(\left(\frac{t}{t_1}\right)^{1-\mu}\int_0^1(t_1-t_1x)^{-\mu}\mathcal{K}_1\mathcal{G}_1t_1\mathrm{d}x,\mathcal{S}_{1,q}^{(\beta)}(t)\right)_{I_1}\\ &=\left(\left(\frac{t}{t_1}\right)^{1-\mu}\int_0^1(t_1)^{-\mu}\frac{(1-x)^{-\mu}}{(-x\log x)^{-\mu}}\mathcal{K}_1\mathcal{G}_1t_1(-x\log x)^{-\mu}\mathrm{d}x,\mathcal{S}_{1,q}^{(\beta)}(t)\right)_{I_1}\\ &=t_1^{1-\mu}\int_{I_1}\left(\frac{t}{t_1}\right)^{1-\mu}\left(\int_0^1\frac{(1-x)^{-\mu}}{(-x\log x)^{-\mu}}\mathcal{K}_1\mathcal{G}_1(-x\log x)^{-\mu}\mathrm{d}x\right)\mathcal{S}_{1,q}^{(\beta)}(t)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= t_1^{2-\mu}\int_0^1 y^{1-\mu}\int_0^1 \frac{(1-x)^{-\mu}}{(-x\log x)^{-\mu}}\mathcal{K}_2\mathcal{G}_2(-x\log x)^{-\mu}\mathrm{d}x\,\mathcal{S}_{1,q}^{(\beta)}(t_1y)\mathrm{d}y\\
&\approx h_1^{2-\mu}\sum_{i,j=0}^{M_1}\frac{\big(1-x_i^{(-\mu,\beta,-\mu)}\big)^{-\mu}}{\big(-x_i^{(-\mu,\beta,-\mu)}\log x_i^{(-\mu,\beta,-\mu)}\big)^{-\mu}}\\
&\quad\times K\left(t_1x_j^{(0,\beta,1-\mu)}, s\left(t_1x_j^{(0,\beta,1-\mu)}, t_1x_i^{(-\mu,\beta,-\mu)}\right)\right)\\
&\quad\times G\left(Y^1\left(s(t_1x_j^{(0,\beta,1-\mu)}, t_1x_i^{(-\mu,\beta,-\mu)})\right)\right)\chi_i^{(-\mu,\beta,-\mu)}\mathcal{S}_q^{(\beta)}\left(x_j^{(0,\beta,1-\mu)}\right)\chi_j^{(0,\beta,1-\mu)},\\
&\ w_q^n = \big(\mathcal{V}_2^nY^n, L_{n,q}\big)_{I_n}\\
&= \left(\left(\frac{t-t_{n-1}}{h_n}\right)^{1-\mu}\int_{I_n}(t_n-\tau)^{-\mu}K\big(t,s(t,\tau)\big)G\big(Y^n(s(t,\tau))\big)\mathrm{d}\tau, L_{n,q}(t)\right)_{I_n}\\
&\approx \frac{h_n}{2^{2-\mu}}\sum_{i,j=0}^{M_n}(t_{n,i}-t_{n-1})^{1-\mu}K\left(t_{n,i}, s\big(t_{n,i}, t_{n,j}^{(-\mu,0)}\big)\right)G\left(Y^n\big(s(t_{n,i}, t_{n,j}^{(-\mu,0)})\big)\right)\\
&\quad\times L_{n,q}(t_{n,i})\omega_{n,j}^{(-\mu,0)}\omega_{n,i},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{K}_1 = K\big(t, s(t,t_1x)\big), && \mathcal{G}_1 = G\big(Y^1(s(t,t_1x))\big),\\
&\mathcal{K}_2 = K\big(t_1y, s(t_1y,t_1x)\big), && \mathcal{G}_2 = G\big(Y^1(s(t_1y,t_1x))\big).
\end{aligned}
$$

The remainder is to compute expansion coefficients $\{y_p^n\}_{p=0}^{M_n}$ that can be derived by a simple iterative algorithm.

Algorithm 3.1 A simple iterative algorithm.

---

for $n=1:N$ **do**
  Give the initial guess $\mathbf{y}^{n,(0)}=(1,\cdots,1)^T$.
  Proceed to the following process:
    $A^1\mathbf{y}^{1,(k)}=\mathbf{f}^1+\mathbf{w}^1(\mathbf{y}^{1,(k-1)})$,
    $A^n\mathbf{y}^{n,(k)}=\mathbf{f}^n+\mathbf{v}^n+\mathbf{w}^n(\mathbf{y}^{n,(k-1)}),\quad k=1,2,\dots.$
end for

Remark 3.1. For linear weakly singular VIEs ($G(s,y)=y$), the numerical solution can be derived directly as follows:

$$
\begin{cases}
\mathbf{A}^1\mathbf{y}^1-\mathbf{B}^1\mathbf{y}^1=\mathbf{f}^1 & \Rightarrow\quad \mathbf{y}^1=(\mathbf{A}^1-\mathbf{B}^1)^{-1}\mathbf{f}^1,\\
\mathbf{A}^n\mathbf{y}^n-\mathbf{B}^n\mathbf{y}^n=\mathbf{f}^n+\mathbf{v}^n & \Rightarrow\quad \mathbf{y}^n=(\mathbf{A}^n-\mathbf{B}^n)^{-1}(\mathbf{f}^n+\mathbf{v}^n),\quad n\ge 2,
\end{cases}
\tag{3.4}
$$

where $\mathbf{y}^n$, $\mathbf{f}^n$, $\mathbf{v}^n$ and $\mathbf{A}^n$ are given by (3.2) and

$$
\mathbf{B}^n=\big(b_{qp}^n\big)_{0\le p,q\le M_n},\quad b_{qp}^1=\left(\mathcal{V}_2^1\mathcal{S}_{1,p}^{(\beta)},\mathcal{S}_{1,q}^{(\beta)}\right)_{I_1},\quad b_{qp}^n=\big(\mathcal{V}_2^nL_{n,p},L_{n,q}\big)_{I_n},\quad n\ge 2.
$$

4. Existence, uniqueness and error analysis

In this section, we will investigate the existence, uniqueness and convergence of the scheme (3.1) which shows the feasibility of the Algorithm 3.1. First of all, we recall the Gronwall Lemma [29].

Lemma 4.1. *Assume that k_j and $\rho_j, j \geq 0$ are given nonnegative sequences, and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq \rho_0$ and*

$$\varepsilon_n \leq \rho_n + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \varepsilon_j, \quad n \geq 1$$

with $q_j \geq 0, j \geq 0$. Then

$$\varepsilon_n \leq \rho_n + \sum_{j=0}^{n-1} (q_j + k_j \rho_j) \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Since the solution of the VIE (1.1) with the smooth f has singularity near $t = 0$ and behaves as $\sum t^{r_i}$ while the solution is smooth at $t \neq 0$, we assume that $y|_{t \in I_1} \in L^2(I_1)$ and $\hat{\partial}^{m_1}_{\beta/2,t} y|_{t \in I_1} \in L^2_{\chi_1^{m_1}}(I_1)$ when $n = 1$, $y(t)|_{t \in I_n}$ belongs to Sobolev space $H^{m_n}_{\omega^{0,0}}(I_n)$ when $n > 1, 1 \leq m_n \leq M_n + 1$. Then let $y(t)$ be the solution of Eq. (1.1), $Y(t)$ is the global solution of Eq. (3.1), namely

$$Y(t) := Y^n(t), \quad t \in I_n, \quad 1 \leq n \leq N.$$

We first provide the existence and uniqueness of the numerical scheme (3.1) below.

Theorem 4.1. *Assume that $K(t,s) \in C(D)$, G fulfills the Lipschitz conditions*

$$|G(s, y_1) - G(s, y_2)| \leq \gamma |y_1 - y_2|, \quad \gamma \geq 0. \tag{4.1}$$

Then for any $1 \leq n \leq N$ and sufficiently small $h_{\max}^{2-2\mu}$, there exists a unique solution to (3.1).

Proof. Consider the following iteration process:

$$\begin{cases} \big(Y^{1,(m)}, \varphi\big)_{I_1} = (f, \varphi)_{I_1} + \big(\mathcal{V}_2^1 Y^{1,(m-1)}, \varphi\big)_{I_1}, & \forall \varphi \in \mathcal{P}_{M_1}^{\beta/2, \log t}(I_1), \\ \big(Y^{n,(m)}, \psi\big)_{I_n} = (f, \psi)_{I_n} + \big(\mathcal{V}_1^n Y, \psi\big)_{I_n} + \big(\mathcal{V}_2^n Y^{n,(m-1)}, \psi\big)_{I_n}, & \forall \psi \in \mathcal{P}_{M_n}(I_n), \ n \geq 2. \end{cases}$$

Thanks to the definition of the projections $\Pi^{\beta}_{I_1, M_1}$ and Π_{I_n, M_n}, it holds that

$$\begin{cases} Y^{1,(m)} = \Pi^{\beta}_{I_1, M_1}\big(f + \mathcal{V}_2^1 Y^{1,(m-1)}\big), \\ Y^{n,(m)} = \Pi_{I_n, M_n}\big(f + \mathcal{V}_1^n Y + \mathcal{V}_2^n Y^{n,(m-1)}\big). \end{cases} \tag{4.2}$$

Denote $\tilde{Y}^{n,(m)} = Y^{n,(m)} - Y^{n,(m-1)}$, $n \geq 1$, from (4.2) we have

$$\begin{cases} \tilde{Y}^{1,(m)} = \Pi^{\beta}_{I_1,M_1}\big(\mathcal{V}^1_2 Y^{1,(m-1)} - \mathcal{V}^1_2 Y^{1,(m-2)}\big), \\ \tilde{Y}^{n,(m)} = \Pi_{I_n,M_n}\big(\mathcal{V}^n_2 Y^{n,(m-1)} - \mathcal{V}^n_2 Y^{n,(m-2)}\big). \end{cases} \tag{4.3}$$

Then we can deduce that

$$\begin{aligned} \|\tilde{Y}^{1,(m)}\|^2_{I_1} &= \big\|\Pi^{\beta}_{I_1,M_1}\big(\mathcal{V}^1_2 Y^{1,(m-1)} - \mathcal{V}^1_2 Y^{1,(m-2)}\big)\big\|^2_{I_1} \\ &\leq \big\|\mathcal{V}^1_2 Y^{1,(m-1)} - \mathcal{V}^1_2 Y^{1,(m-2)}\big\|^2_{I_1}. \end{aligned}$$

Let $U_1^{(m)}(t)$ and $U_2(t)$ be the global functions defined on $[0, t_n]$ such that

$$\begin{aligned} &U_1^{(m)}(t)|_{t\in I_k} = G\big(t, Y^{k,(m)}(t)\big) - G\big(t, Y^{k,(m-1)}(t)\big), \\ &U_2(t)|_{t\in I_k} = G\big(t, y^k(t)\big) - G\big(t, Y^k(t)\big), \quad 1 \leq k \leq n. \end{aligned} \tag{4.4}$$

Combining with (2.1), (2.2) and (4.4), we have

$$\begin{aligned} &\big\|\mathcal{V}^1_2 Y^{1,(m-1)} - \mathcal{V}^1_2 Y^{1,(m-2)}\big\|^2_{I_1} \\ =& \int_{I_1}\left(\left(\frac{t}{h_1}\right)^{1-\mu}\int_{I_1}(t_1-\tau)^{-\mu}K\big(t,s(t,\tau)\big)G\big(\tilde{Y}^{1,(m-1)}(s(t,\tau))\big)\mathrm{d}\tau\right)^2\mathrm{d}t \\ =& \int_{I_1}\left(\int_0^t (t-s)^{-\mu}K(t,s)\left(G\big(Y^{1,(m-1)}(s)\big) - G\big(Y^{1,(m-2)}(s)\big)\right)\mathrm{d}s\right)^2\mathrm{d}t \\ \leq& c\int_{I_1}\left(\int_0^t (t-s)^{-\mu}U_1^{(m-1)}(s)\mathrm{d}s\right)^2\mathrm{d}t. \end{aligned}$$

With the aid of the Cauchy-Schwarz inequality and Lipschitz condition (4.1), it holds that

$$\begin{aligned} &\int_{I_1}\left(\int_0^t (t-s)^{-\mu}U_1^{(m-1)}(s)\mathrm{d}s\right)^2\mathrm{d}t \\ \leq& \int_{I_1}\left(\int_0^t (t-s)^{-\mu}\mathrm{d}s\int_0^t (t-s)^{-\mu}\big(U_1^{(m-1)}(s)\big)^2\mathrm{d}s\right)\mathrm{d}t \\ =& \int_{I_1}\left(\frac{-(t-s)^{1-\mu}}{1-\mu}\Bigg|_{s=t_0}^{s=t}\right)\left(\int_0^t (t-s)^{-\mu}\big(U_1^{(m-1)}(s)\big)^2\mathrm{d}s\right)\mathrm{d}t \\ \leq& h_1^{1-\mu}\int_{I_1}\left(\int_0^t (t-s)^{-\mu}\big(U_1^{(m-1)}(s)\big)^2\mathrm{d}s\right)\mathrm{d}t \\ \leq& h_1^{1-\mu}\int_{I_1}\big(U_1^{(m-1)}(s)\big)^2\left(\int_s^{t_1}(t-s)^{-\mu}\mathrm{d}t\right)\mathrm{d}s \\ \leq& h_1^{2-2\mu}\int_{I_1}\Big(Y^{1,(m-1)}(s) - Y^{1,(m-2)}(s)\Big)^2\mathrm{d}s \\ \leq& h_1^{2-2\mu}\|\tilde{Y}^{1,(m-1)}\|^2_{I_1}. \end{aligned}$$

Therefore, the following formula holds:

$$\|\tilde{Y}^{1,(m)}\|_{I_1}^2 \le c h_1^{2-2\mu} \|\tilde{Y}^{1,(m-1)}\|_{I_1}^2. \tag{4.5}$$

We can derive the similar results for $n \ge 2$ from relations (4.3), (2.1), (2.2), (4.1), (4.4) and Cauchy-Schwarz inequality that

$$\begin{aligned}
\|\tilde{Y}^{n,(m)}\|_{I_n}^2 &= \left\|\Pi_{I_n,M_n}\big(\mathcal{V}_2^n Y^{n,(m-1)} - \mathcal{V}_2^n Y^{n,(m-2)}\big)\right\|_{I_n} \\
&\le \left\|\mathcal{V}_2^n Y^{n,(m-1)} - \mathcal{V}_2^n Y^{n,(m-2)}\right\|_{I_n}^2 \\
&\le c \int_{I_n} \left(\int_{t_{n-1}}^t (t-s)^{-\mu} \mathrm{d}s\right) \left(\int_{t_{n-1}}^t (t-s)^{-\mu} \left(U_1^{(m-1)}(s)\right)^2 \mathrm{d}s\right) \mathrm{d}t \\
&\le c h_n^{1-\mu} \int_{I_n} \left(\int_{t_{n-1}}^t (t-s)^{-\mu} \left(U_1^{(m-1)}(s)\right)^2 \mathrm{d}s\right) \mathrm{d}t \\
&\le c h_n^{1-\mu} \int_{I_n} \left(U_1^{(m-1)}(s)\right)^2 \left(\int_s^{t_n} (t-s)^{-\mu} \mathrm{d}t\right) \mathrm{d}s \\
&\le c h_n^{2-2\mu} \|\tilde{Y}^{n,(m-1)}\|_{I_n}^2.
\end{aligned} \tag{4.6}$$

It is evident that $\|\tilde{Y}^{n,(m)}\|_{I_n} \to 0$ while $m \to \infty$ if $c h_{\max}^{2-2\mu} < 1$. □

Next, the convergence of the hybrid spectral method can be exhibited as follows:

Theorem 4.2. *Let y^n and Y^n be the solution of the problem* (2.2) *and the scheme* (3.1)*, respectively. Assume that $K(t,s) \in C(D)$, $y|_{t\in I_1} \in L^2(I_1)$ and $\hat{\partial}_{\beta/2,t}^{m_1} y|_{t\in I_1} \in L^2_{\chi_1^{m_1}}(I_1)$, $y|_{t\in I_n} \in H^{m_n}_{\omega^{0,0}}(I_n)$ with $2 \le n \le N$ and integers $1 \le m_n \le M_n + 1$, and G fulfills the Lipschitz condition* (4.1)*. Then for any $1 \le n \le N$, $0 < \mu < \frac{1}{2}$ and sufficiently small $h_{\max}^{2-2\mu}$, it holds that*

$$\begin{aligned}
&\|y^1 - Y^1\|_{I_1}^2 \le c h_1^{2m_1} M_1^{-m_1} \|\mathcal{L}^{m_1} v\|_{I_1,\chi_1^{m_1}}^2, \\
&\|y^n - Y^n\|_{I_n}^2 \le c \left(\frac{h_n}{2}\right)^{2m_n} M_n^{-2m_n} \|\partial_t^{m_n} y\|_{I_n,\omega_n^{m_n,m_n}}^2 \\
&\qquad + c h_n T^{1-2\mu} h_1^{2m_1} M_1^{-m_1} \|\mathcal{L}^{m_1} v\|_{I_1,\chi_1^{m_1}}^2 \\
&\qquad + c h_n T^{1-2\mu} \sum_{k=2}^{n-1} \left(\frac{h_k}{2}\right)^{2m_k} M_k^{-2m_k} \left\|\partial_t^{m_k} y\right\|_{I_k,\omega_k^{m_k,m_k}}^2, \quad n \ge 2.
\end{aligned} \tag{4.7}$$

Proof. It is easy to deduce from (3.1) that

$$Y^1 = \Pi_{I_1,M_1}^{\beta}\big(f + \mathcal{V}_2^1 Y^1\big), \quad Y^n = \Pi_{I_n,M_n}\big(f + \mathcal{V}_1^n Y + \mathcal{V}_2^n Y^n\big).$$

Then we have that

$$\begin{cases}
Y^1 - \Pi_{I_1,M_1}^{\beta} y^1 = \Pi_{I_1,M_1}^{\beta}\big(\mathcal{V}_2^1 Y^1 - \mathcal{V}_2^1 y^1\big), \\
Y^n - \Pi_{I_n,M_n} y^n = \Pi_{I_n,M_n}\big(\mathcal{V}_1^n Y - \mathcal{V}_1^n y\big) + \Pi_{I_n,M_n}\big(\mathcal{V}_2^n Y^n - \mathcal{V}_2^n y^n\big).
\end{cases} \tag{4.8}$$

For notational simplicity we denote

$$\begin{cases} e_1 = y^1 - Y^1 = y^1 - \Pi^{\beta}_{I_1,M_1} y^1 + \Pi^{\beta}_{I_1,M_1} y^1 - Y^1, \\ e_n = y^n - Y^n = y^n - \Pi_{I_n,M_n} y^n + \Pi_{I_n,M_n} y^n - Y^n. \end{cases} \tag{4.9}$$

Combining (4.8) and (4.9) leads to

$$\begin{cases} e_1 = y^1 - \Pi^{\beta}_{I_1,M_1} y^1 + \Pi^{\beta}_{I_1,M_1} \big(\mathcal{V}^1_2 y^1 - \mathcal{V}^1_2 Y^1\big), \\ e_n = y^n - \Pi_{I_n,M_n} y^n + \Pi_{I_n,M_n} \big(\mathcal{V}^n_1 y - \mathcal{V}^n_1 Y\big) + \Pi_{I_n,M_n} \big(\mathcal{V}^n_2 y^n - \mathcal{V}^n_2 Y^n\big). \end{cases} \tag{4.10}$$

In order to estimate errors $e_n, n \geq 1$ we define

$$\begin{aligned} &D_1 = \big\|y^1 - \Pi^{\beta}_{I_1,M_1} y^1\big\|^2_{I_1}, \quad D_2 = \big\|\Pi^{\beta}_{I_1,M_1}\big(\mathcal{V}^1_2 y^1 - \mathcal{V}^1_2 Y^1\big)\big\|^2_{I_1}, \\ &D_3 = \big\|y^n - \Pi_{I_n,M_n} y^n\big\|^2_{I_n}, \quad D_4 = \big\|\Pi_{I_n,M_n}\big(\mathcal{V}^n_1 y - \mathcal{V}^n_1 Y + \mathcal{V}^n_2 y^n - \mathcal{V}^n_2 Y^n\big)\big\|^2_{I_n}. \end{aligned}$$

According to Lemma 2.2, it is straightforward to obtain that

$$D_1 \leq c h_1^{2m_1} M_1^{-m_1} \|\mathcal{L}^{m_1} v\|^2_{I_1,\chi_1^{m_1}}. \tag{4.11}$$

Using the same method as (4.5) and (4.6) results in

$$D_2 \leq \int_{I_1} \left(\int_{t_0}^{t} (t-s)^{-\mu} K(t,s) U_2(s) \mathrm{d}s \right)^2 \mathrm{d}t \leq c h_1^{2-2\mu} \|e_1\|^2_{I_1}. \tag{4.12}$$

Thus we can obtain the following estimates from (4.11), (4.12) that

$$\|e_1\|^2_{I_1} \leq c h_1^{2m_1} M_1^{-m_1} \|\mathcal{L}^{m_1} v\|^2_{I_1,\chi_1^{m_1}}. \tag{4.13}$$

For any integer $1 \leq m_n \leq M_n + 1$ with $n \geq 2$, it is straightforward to derive the following estimate from Lemma 2.4 that

$$D_3 \leq c \left(\frac{h_n}{2}\right)^{2m_n} M_n^{-2m_n} \big\|\partial_t^{m_n} y\big\|^2_{I_n,\omega_n^{m_n,m_n}}. \tag{4.14}$$

It is easy to see

$$D_4 = \|D_5 + D_6\|^2_{I_n} \leq 2\|D_5\|^2_{I_n} + 2\|D_6\|^2_{I_n}, \tag{4.15}$$

where

$$\begin{aligned} D_5(t) &= \int_0^{t_{n-1}} (t-s)^{-\mu} K(t,s) \big(G(s,y(s)) - G(s,Y(s))\big) \mathrm{d}s, \\ D_6(t) &= \int_{t_{n-1}}^{t} (t-s)^{-\mu} K(t,s) \big(G(s,y^n(s)) - G(s,Y^n(s))\big) \mathrm{d}s. \end{aligned}$$

From (4.1), (4.4) and Cauchy-Schwarz inequality, $0 < \mu < \frac{1}{2}$, it holds that

$$\|D_5\|^2_{I_n} = \int_{I_n} \left(\int_0^{t_{n-1}} (t-s)^{-\mu} K(t,s) U_2(s) \mathrm{d}s \right)^2 \mathrm{d}t$$

$$
\begin{aligned}
&\leq c\int_{I_n}\left(\int_0^{t_{n-1}}(t-s)^{-2\mu}\mathrm{d}s\right)\left(\int_0^{t_{n-1}}\big(U_2(s)\big)^2\mathrm{d}s\right)\mathrm{d}t\\
&=c\int_{I_n}T^{1-2\mu}\bigg(\int_0^{t_{n-1}}\big(U_2(s)\big)^2\mathrm{d}s\bigg)\mathrm{d}t\\
&\leq ch_nT^{1-2\mu}\sum_{k=1}^{n-1}\|e_k\|_{I_k}^2. \qquad (4.16)
\end{aligned}
$$

Similarly, from (4.1), (4.4) and Cauchy-Schwarz inequality, it holds that

$$
\begin{aligned}
\|D_6\|_{I_n}^2 &= \int_{I_n}\left(\int_{t_{n-1}}^t (t-s)^{-\mu}K(t,s)U_2(s)\mathrm{d}s\right)^2\mathrm{d}t\\
&\leq c\int_{I_n}\left(\int_{t_{n-1}}^t(t-s)^{-\mu}\mathrm{d}s\right)\left(\int_{t_{n-1}}^t(t-s)^{-\mu}\big(U_2(s)\big)^2\mathrm{d}s\right)\mathrm{d}t\\
&= c\int_{I_n}\left(\frac{-(t-s)^{1-\mu}}{1-\mu}\bigg|_{s=t_{n-1}}^{s=t}\right)\left(\int_{t_{n-1}}^t(t-s)^{-\mu}\big(U_2(s)\big)^2\mathrm{d}s\right)\mathrm{d}t\\
&\leq ch_n^{1-\mu}\int_{I_n}\left(\int_{t_{n-1}}^t(t-s)^{-\mu}\big(U_2(s)\big)^2\mathrm{d}s\right)\mathrm{d}t\\
&\leq ch_n^{1-\mu}\int_{I_n}\big(U_2(s)\big)^2\left(\int_s^{t_n}(t-s)^{-\mu}\mathrm{d}t\right)\mathrm{d}s\\
&\leq ch_n^{2-2\mu}\int_{I_n}\big(U_2(s)\big)^2\mathrm{d}s\leq ch_n^{2-2\mu}\|e_n\|_{I_n}^2. \qquad (4.17)
\end{aligned}
$$

Thus, by virtue of (4.14)-(4.17), we can get the following formula directly

$$
\|e_n\|_{I_n}^2\leq ch_nT^{1-2\mu}\sum_{k=1}^{n-1}\|e_k\|_{I_k}^2+c\left(\frac{h_n}{2}\right)^{2m_n}M_n^{-2m_n}\big\|\partial_t^{m_n}y\big\|_{I_n,\omega_n^{m_n,m_n}}^2.
$$

Finally, thanks to Lemma 4.1 and (4.13), it is easy to show that

$$
\begin{aligned}
\|e_n\|_{I_n}^2\leq\; & c\left(\frac{h_n}{2}\right)^{2m_n}M_n^{-2m_n}\big\|\partial_t^{m_n}y\big\|_{I_n,\omega_n^{m_n,m_n}}^2\\
&+ch_nT^{1-2\mu}h_1^{2m_1}M_1^{-m_1}\|\mathcal{L}^{m_1}v\|_{I_1,\chi_1^{m_1}}^2\\
&+ch_nT^{1-2\mu}\sum_{k=2}^{n-1}\left(\frac{h_k}{2}\right)^{2m_k}M_k^{-2m_k}\big\|\partial_t^{m_k}y\big\|_{I_k,\omega_k^{m_k,m_k}}^2.
\end{aligned}
$$

The proof is complete. □

Then the global error can be derived straightforward.

Theorem 4.3. *Let y be the solution of* (1.1)*, Y be the global numerical solution of* (1.1)*. Assume that $K(t,s) \in C(D)$, $y|_{t\in I_1} \in L^2(I_1)$ and $\hat{\partial}^{m_1}_{\beta/2,t} y|_{t\in I_1} \in L^2_{\chi_1^{m_1}}(I_1)$, $y|_{t\in I_n} \in H^{m_n}_{\omega^{0,0}}(I_n)$ with $2 \le n \le N$ and integers $1 \le m_n \le M_n + 1$, G fulfills the Lipschitz condition* (4.1)*. Then for sufficiently small $h_{\max}^{2-2\mu}$, $0 < \mu < \frac{1}{2}$, we have*

$$\|y - Y\|^2_{L^2(I)} \le c\big(1 + T^{2-2\mu}\big) h_1^{2m_1} M_1^{-m_1} \|\mathcal{L}^{m_1} v\|^2_{I_1,\chi_1^m} + c\big(1 + T^{2-2\mu}\big) \sum_{n=2}^{N} \left(\frac{h_n}{2}\right)^{2m_n} M_n^{-2m_n} \big\|\partial_t^{m_n} y\big\|^2_{L^2(I)}.$$

5. Numerical results

In this section, we present some numerical results to verify the error estimates. Let $E_1(T)$ and $E_2(T)$ be the maximum error at the mesh points and the discrete L^2-error below

$$E_1(T) = \max_{\substack{1\le k\le N \\ 0\le j\le M_k}} |y(t_{k,j}) - Y(t_{k,j})|,$$

$$E_2(T) = \left(\sum_{k=1}^{N} \frac{h_k}{2} \sum_{j=0}^{M_k} \Big(y^k(t_{k,j}) - Y^k(t_{k,j})\Big)^2 \omega_{k,j}\right)^{\frac{1}{2}} \approx \left(\int_0^T \big(y(t) - Y(t)\big)^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

In actual computation, we assume $M_k = M$ and $h_n = h$.

Example 5.1. Consider the linear VIE with $T = 2, K(t,s) = 1$ and $\mu = \frac{1}{3}$ as follows:

$$y(t) = f(t) + \int_0^t (t-s)^{-\mu} K(t,s) y(s) \mathrm{d}s, \quad t \in [0,T], \tag{5.1}$$

where the exact solution is $y(t) = t^2 \cos(t)$. We plot the maximum error for our method in Fig. 1. The maximum error curves demonstrate that our method is exponentially convergent for VIEs with smooth solutions. Comparing with the convergence rate (see Fig. 2) of the hybrid spectral element method proposed by Sheng and Shen [28], our method performs better for this same case. At the same time the numerical results show that our method performs as well as the classical polynomial h-p method.

Example 5.2. Consider the nonlinear VIEs with $T = 3, K(t,s) = 1$

$$y(t) = t^{\frac{1}{2}} \exp(t) + \frac{4}{3} t^{\frac{3}{2}} - \int_0^t (t-s)^{-\frac{1}{2}} \exp(-2s) y^2(s) \mathrm{d}s, \quad t \in [0,T] \tag{5.2}$$

with the exact solution is $y(t) = \sqrt{t}\exp(t)$ (cf. [28]). Sheng and Shen [28] employed the shifted generalized Jacobi function $t^{1-\mu} P^{(-\mu,1-\mu)}(2t-1)$ as the basis in the first interval, so their method is highly efficient for solving problems with solutions behave as $t^{1-\mu} u(t)$ where $u(t)$ is a smooth function. Similarly, the theoretical result provided

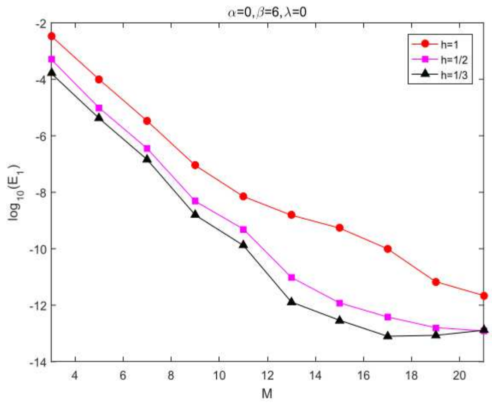

Figure 1: Our method.

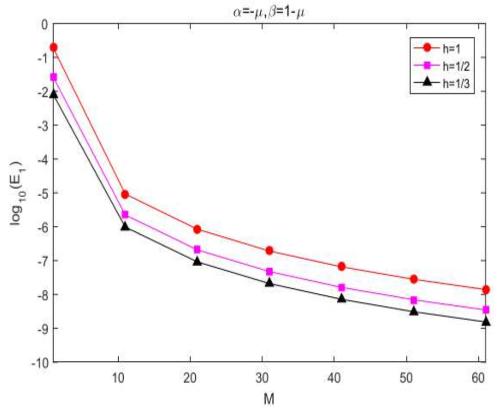

Figure 2: Sheng and Shen's method [28].

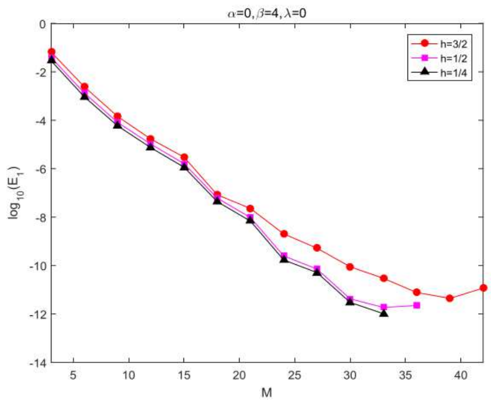

Figure 3: Maximum error.

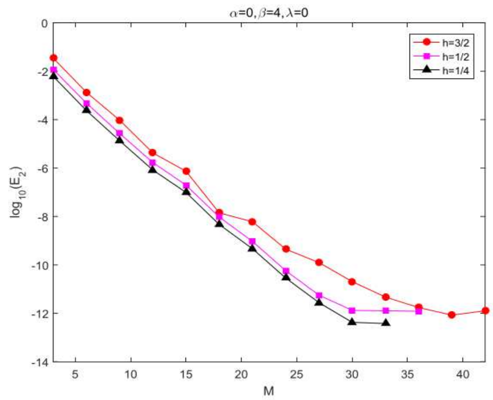

Figure 4: Discrete L^2-error.

in the previous section shows that our new method is also efficient for this one-point singular solution since we use the GLOFs $\mathcal{S}_n^{(\alpha,\beta,\lambda)}(t/t_1)$ for the first interval $I_1=[0,t_1]$. In order to verify this merit we plot the convergence curves for Example 5.2 with $\alpha=0,\beta=4,\lambda=0$ in Fig. 3 (Maximum error) and Fig. 4 (L^2-error). All the error curves show that our new hybrid spectral method is robust for the singular solution with one-point singularity.

Now we use our method to solve the following more challenged one.

Example 5.3. Consider the λ-polynomial solution with $T=2, K(t,s)=1$.

$$y(t)=f(t)+\int_0^t (t-s)^{-\mu}K(t,s)y(s)\mathrm{d}s, \quad t\in[0,T], \tag{5.3}$$

where the exact solution $y(t)=t^{\gamma_1}+t^{\gamma_2}$.

We plot the maximum errors and the discrete L^2-errors for Example 5.3 with (a): $\mu=0.1,\gamma_1=\frac{1}{3},\gamma_2=\frac{1}{2}$ and (b): $\mu=0.9,\gamma_1=1.1,\gamma_2=2.3$ in Figs. 5-8.

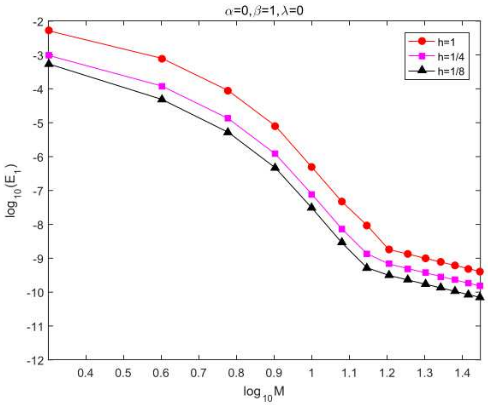

Figure 5: Maximum error for Example 5.3(a).

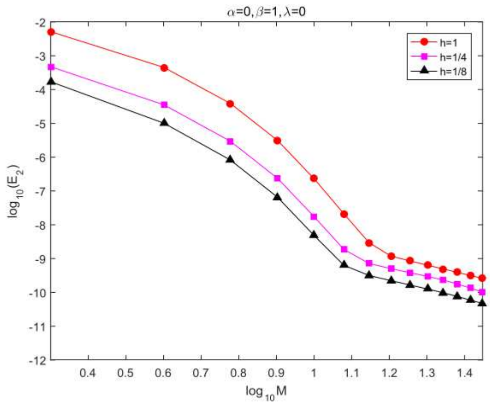

Figure 6: Discrete L^2-error for Example 5.3(a).

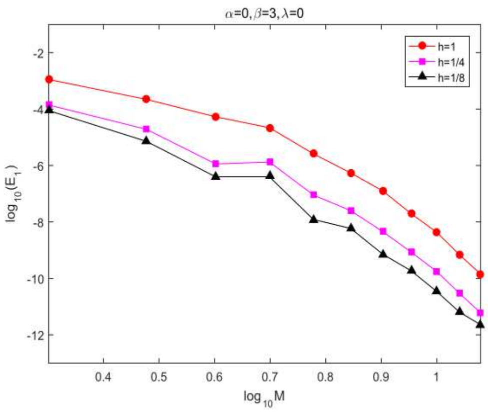

Figure 7: Maximum error for Example 5.3(b).

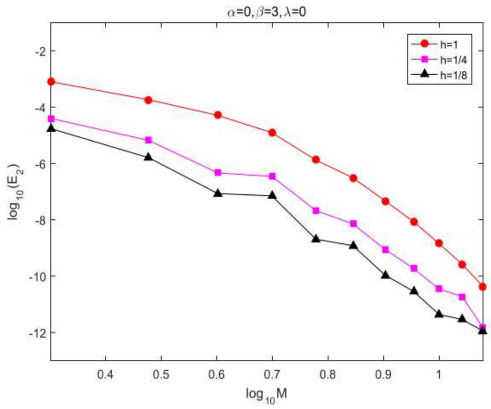

Figure 8: Discrete L^2-error for Example 5.3(b).

The above three examples show that our hybrid spectral method can handle problems with both smooth and singular solution as the classical polynomial h-p approximation and Sheng and Shen's method [28]. Further, as theoretical results stated in the previous section and the validation from Fig. 5 to Fig. 8 shown that our method is robust for the VIEs with more complicated one-point singularity.

6. Concluding remarks

In this paper we provided an efficient hybrid spectral method for solving nonlinear VIEs with weakly singular kernel. In order to deal with weak singularity we divide the interval and use the idea of the multi-step method and use GLOFs as the basis in the first interval. Comparing with the classical h-p polynomial approximation, the proposed hybrid spectral method can exponentially approximate both singular solutions and smooth solutions on the general quasi-uniform mesh. In addition, we provide the existence, uniqueness and convergence of this hybrid spectral method. Numerical results show that the proposed method is highly efficient for solving the VIEs with weakly singular kernels. Undoubtedly, the new method can be straightforwardly applied to solve more general nonlocal problems with one-point singularity.

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