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Galerkin Finite Element Approximation for Semilinear Stochastic Time-Tempered Fractional Wave Equations with Multiplicative Gaussian Noise and Additive Fractional Gaussian Noise

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Abstract. To model wave propagation in inhomogeneous media with frequency dependent power-law attenuation, it is needed to use the fractional powers of symmetric coercive elliptic operators in space and the Caputo tempered fractional derivative in time. The model studied in this paper is semilinear stochastic space-time fractional wave equations driven by infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise, because of the potential fluctuations of the external sources. The purpose of this work is to discuss the Galerkin finite element approximation for the semilinear stochastic fractional wave equation. First, the space-time multiplicative Gaussian noise and additive fractional Gaussian noise are discretized, which results in a regularized stochastic fractional wave equation while introducing a modeling error in the mean-square sense. We further present a complete regularity theory for the regularized equation. A standard finite element approximation is used for the spatial operator, and a mean-square priori estimates for the modeling error and the approximation error to the solution of the regularized problem are established. Finally, numerical experiments are performed to confirm the theoretical analysis.

AMS subject classifications: 35R11, 60H15, 65M12, 65M60, 60G22

Key words: Galerkin finite element method, semilinear stochastic time-tempered fractional wave equation, fractional Laplacian, multiplicative Gaussian noise, additive fractional Gaussian noise.

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1. Introduction

The classical wave equation well models the wave propagation in an ideal medium. However, the wave propagation in complex inhomogeneous media generally has frequency-dependent attenuation, being observed in a wide range of areas including acoustics, viscous damping in the seismic isolation of buildings, structural vibration, and seismic wave propagation [7, 21, 24, 31]. The striking power-law feature of the attenuated wave propagation implies that the Laplacian in the classical equation should be replaced by fractional powers of symmetric coercive elliptic operators in space, while the time-tempered derivative should be substituted for second time derivative. Because of the finite time/space scale, the tempered power-law distribution in some sense becomes more reasonable choice compared with the pure power-law one [23]. There are already some discussions on the numerical methods or correct ways of specifying the boundary conditions for tempered fractional differential equations; see, e.g., [11, 12, 15, 34] and the references therein. As for the fractional wave equations, there are also some progresses not only on their numerical methods [9, 14, 30, 33] but also on their fundamental solutions and properties [4, 17].

Random effects arise naturally in practically physical systems; the ones considered in this paper are on the fluctuations of the external sources, and the fluctuations include both infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise, which drive the semilinear space-time fractional wave equations. The multiplicative noise can capture the effects of geometrical confinements [20]. The fractional Gaussian noise is the formal derivative of the fractional Brownian motion (FBM) B^H, being a centered Gaussian process with a special covariance function determined by Hurst parameter $H \in (0,1)$. For $H = 1/2$, $B^{1/2}$ is the standard Brownian motion, the formal time derivative of which is white noise. For $H \neq 1/2$, B^H behaves in a way completely different from the standard Brownian motion; especially, neither is a semi-martingale nor a Markov process. In addition, the FBM with Hurst parameter $H \in (1/2, 1)$ enjoys the property of a long range memory, which roughly implies that the decay of stochastic dependence with respect to the past is only sub-exponentially slow. This long-range dependence property of the FBM makes it a realistic choice of noise for problems with long memory in the applied sciences.

With the above introduction of the fractional wave equation and the external noises, now we propose the model, which is a space-time fractional wave equation driven by three nonlinear external source terms: a deterministic term and two stochastic terms, being respectively Gaussian noise and fractional Gaussian noise. Specifically, the model is a semilinear stochastic time tempered fractional wave equation with $3/2 < \alpha < 2$, $1/2 < \beta < 1$, $1/2 < H < 1$, and $\nu > 0$

$$
\begin{aligned}
&{}_0^c\partial_t^{\alpha,\nu} u(t,x) + (-\Delta)^\beta u(t,x) \\
&= f\big(t, u(t,x)\big) + g\big(t, u(t,x)\big)\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x} + h(t)\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x} \quad \text{in } (0,T] \times \mathcal{D}, \qquad (1.1\text{a})
\end{aligned}
$$

$$u(t,x) = 0 \qquad \text{on } (0,T] \times \partial\mathcal{D}, \tag{1.1b}$$
$$u(0,x) = a(x), \quad \partial_t u(t,x)|_{t=0} = b(x) \quad \text{in } \mathcal{D}, \tag{1.1c}$$

where $\mathcal{D} \subset \mathbb{R}^d, d = 1,2,3,$ is a bounded convex polygonal domain with the boundary $\partial\mathcal{D}$, ${}_0^c\partial_t^{\alpha,\nu}$ denotes the left-sided Caputo tempered fractional derivative of order α with respect to t, $(-\Delta)^\beta$ is the fractional Laplacian, the definition of which is based on the spectral decomposition of the Dirichlet Laplacian, as adopted in [26], $\frac{\partial^2\mathbb{W}(t,x)}{\partial t\partial x}$ and $\frac{\partial^2\mathbb{W}^H(t,x)}{\partial t\partial x}$, respectively, represent the infinite dimensional Gaussian noise and fractional Gaussian noise defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and the initial data a and b are $\mathcal{F}_0$-measurable random variables. Assumptions on the smoothness of the nonlinearities f, g, and h will be given below.

Numerical approximations of stochastic wave equations with classical derivatives have been considered in recent literatures; see, e.g., [1,8,13,32]. Stochastic solutions to wave equations with additive or multiplicative fractional Gaussian noise have been studied in, for example, [3,5,6,28,29] and the references therein. There has, however, been little mention of numerical approximations for semilinear stochastic fractional wave equations with fractional Gaussian noise even for the linear case. Very recently, we investigated the Galerkin finite element approximations for linear stochastic space-time fractional wave equations with an infinite dimensional additive noise [21]. The purpose of this paper is to consider the Galerkin finite element approximations for semilinear stochastic fractional wave equations with fractional Laplacian in space and Caputo tempered fractional time derivative driven by infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise. The novelty and the difficulties of this work are in three aspects:

(i) The nonlinear terms and nonlinear multiplicative noises (in comparison with our results recently published in [21], the analysis of the nonlinear parts requires different mathematical machineries in order to derive error estimates).

(ii) The fractional Gaussian noise term (since the fractional Brownian motion, neither is a semi-martingale nor a Markov process, and does not have the property of independent increments, some new ideas for dealing with fractional Gaussian noise are developed here).

(iii) The error estimates (in order to conduct new and very complicated error estimates, here we need to use a complete solution theory, e.g., existence, uniqueness, and regularity, for semilinear stochastic fractional wave equations with multiplicative Gaussian noise and additive fractional Gaussian noise).

This paper is organized as follows. In Section 2, we first introduce some basic definitions, notations, and necessary preliminaries, and then recall the existence, uniqueness, and regularity for the semilinear stochastic fractional wave equation. In Section 3, we discretize the space-time multiplicative Gaussian noise and additive fractional Gaussian noise, which result in a regularized semilinear stochastic fractional wave equation while introducing a modeling error in the mean-square sense. The convergence order of the modeling error and the regularity of the regularized equation are well established. Section 4 is devoted to providing the finite element scheme for the regularized semilinear stochastic fractional wave equation, and the corresponding very detailed mean-square error estimates are presented. In Section 5, the numerical experiments are performed to confirm the convergence orders of the modeling error and the finite element approximations to the regularized equation. We conclude the paper with some discussions in the last section.

2. Preliminaries

In this section, we recall some basic definitions, notations, and necessary preliminaries, collect useful facts on the Mittag-Leffler function, the Brownian motion, and the fractional Brownian motion, and recall the existence, uniqueness, and regularity results of mild solutions to (1.1).

2.1. Fractional Laplacian and Caputo tempered fractional derivative

The operator $-\Delta : L^2(\mathcal{D}) \to L^2(\mathcal{D})$, with the domain

$$\mathrm{Dom}(-\Delta) = \{u \in H_0^1(\mathcal{D}), \Delta u \in L^2(\mathcal{D})\}$$

is positive, unbounded, and closed, and its inverse is compact. Hence, the spectrum of the operator $-\Delta$ is discrete, real, positive, and accumulates at infinity. Moreover, the eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^+} \subset H_0^1(\mathcal{D})$ satisfying

$$\begin{cases} -\Delta\varphi_k(x) = \lambda_k\varphi_k(x) & \text{in } \mathcal{D}, \\ \varphi_k(x) = 0 & \text{on } \partial\mathcal{D}, \quad k \in \mathbb{N}^+ \end{cases} \tag{2.1}$$

form an orthonormal basis of $L^2(\mathcal{D})$. Consequently, $\{\varphi_k\}_{k\in\mathbb{N}^+}$ is an orthogonal basis of $H_0^1(\mathcal{D})$ and $\|\nabla_x\varphi_k\|_{L^2(\mathcal{D})} = \sqrt{\lambda_k}$.

For any $s \in \mathbb{R}$, we denote by $\mathbb{H}^s(\mathcal{D}) \subset L^2(\mathcal{D})$ the Hilbert space induced by the norm

$$\|u\|_{\mathbb{H}^s}^2 = \sum_{k=1}^{\infty} \lambda_k^s (u, \varphi_k)^2.$$

In particular, $\mathbb{H}^0(\mathcal{D}) = L^2(\mathcal{D})$ with $\|\cdot\|$ denoting the norm in $\mathbb{H}^0(\mathcal{D})$ and $(\cdot,\cdot)$ denoting the inner product of $\mathbb{H}^0(\mathcal{D})$, $\mathbb{H}^1(\mathcal{D}) = H_0^1(\mathcal{D})$, and $\mathbb{H}^2(\mathcal{D}) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Then for any $u \in \mathbb{H}^{2s}$, we have

$$(-\Delta)^s u = \sum_{k=1}^{\infty} \lambda_k^s (u, \varphi_k)\varphi_k,$$

which is the fractional Laplacian, also adopted in [26].

Then we give some concepts of fractional calculus. For more details, the reader can consult [10, 12], [18, p. 91], and [27, p. 78].

Definition 2.1. *The left fractional integral of order $\alpha > 0$ for a function u is defined as*

$$ {}_0I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad t>0, $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The left Caputo fractional derivative of order $\alpha > 0$ for a function u is defined as*

$$ {}_0^c\partial_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{\partial^n u(s)}{\partial s^n}ds, \quad t>0, \quad 0\le n-1<\alpha<n, $$

where the function $u(t)$ has absolutely continuous derivatives up to order $n-1$.

Definition 2.3. *For $\alpha > 0, \nu > 0$ the left tempered fractional integral of order α for a function u is defined as*

$$ {}_0I_t^{\alpha,\nu}u(t) := e^{-\nu t}{}_0I_t^{\alpha}\left[e^{\nu t}u(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}u(s)ds, \quad t>0. $$

Definition 2.4. *For $\alpha > 0, \nu > 0$, the left Caputo tempered fractional derivative of order α for a function u is defined as*

$$ \begin{aligned} &{}_0^c\partial_t^{\alpha,\nu}u(t) := e^{-\nu t}{}_0^c\partial_t^{\alpha}\left[e^{\nu t}u(t)\right] \\ &= e^{-\nu t}\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{e^{\nu s}}{(t-s)^{\alpha-n+1}}\left(\frac{\partial}{\partial s}+\nu\right)^n u(s)ds, \quad t>0, \quad 0\le n-1<\alpha<n, \end{aligned} $$

where the function $u(t)$ has absolutely continuous derivatives up to order $n-1$, and

$$ \left(\frac{\partial}{\partial s}+\nu\right)^n = \left(\frac{\partial}{\partial s}+\nu\right)\left(\frac{\partial}{\partial s}+\nu\right)\cdots\left(\frac{\partial}{\partial s}+\nu\right). $$

If u is an abstract function belonging to $\mathbb{H}^s$ $(s\ge 0)$, then the integrals which appear in the above definitions are taken in Bochner's sense. A measurable function $u:[0,\infty)\to\mathbb{H}^s$ is Bochner-integrable if $\|u\|_{\mathbb{H}^s}$ is Lebesgue-integrable.

2.2. Mittag-Leffler function

Throughout this paper, we shall frequently use the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined as follows:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z\in\mathbb{C}, \tag{2.2} $$

which is a two-parameter family of entire functions in z of order α^{-1} and type 1 [18, p. 42], and generalizes the exponential function in the sense that $E_{1,1}(z)=e^z$. For later use, we collect some results in the next lemma; see [18, 27].

Lemma 2.1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is an arbitrary real number such that $\pi\alpha/2 < \mu < \min(\pi, \pi\alpha)$. Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi. \tag{2.3}$$

Moreover, for $\lambda > 0, \alpha > 0$, and positive integer $m \in \mathbb{N}$, we have

$$\frac{d^m}{dt^m} E_{\alpha,1}\big(-\lambda^\beta t^\alpha\big) = -\lambda^\beta t^{\alpha-m} E_{\alpha,\alpha-m+1}\big(-\lambda^\beta t^\alpha\big), \quad t > 0, \tag{2.4}$$

$$\frac{d}{dt}\Big(t E_{\alpha,2}\big(-\lambda^\beta t^\alpha\big)\Big) = E_{\alpha,1}\big(-\lambda^\beta t^\alpha\big), \quad t \geq 0. \tag{2.5}$$

2.3. Infinite dimensional Gaussian noise and fractional Gaussian noise

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. We define $L^2(\Omega; \mathbb{H}^s(\mathcal{D}))$ as the separable Hilbert space of all strongly measurable, square-integrable random variables ω, with values in $\mathbb{H}^s(\mathcal{D})$ such that

$$\|\omega\|^2_{L^2(\Omega;\mathbb{H}^s(\mathcal{D}))} = \mathbb{E}\|\omega\|^2_{\mathbb{H}^s},$$

where $\mathbb{E}$ denotes the expectation. In the sequel, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

Definition 2.5. *The two-sided one-dimensional FBM with Hurst index $H \in (0,1)$ is a Gaussian process $\xi^H = \{\xi^H(t), t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, having the following properties:*

(i) $\xi^H(0) = 0$,

(ii) $\mathbb{E}\xi^H(t) = 0,\ t \in \mathbb{R}$,

(iii) $\mathbb{E}\left[\xi^H(t)\xi^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\ t, s \in \mathbb{R}$.

Remark 2.1. For $H = 1/2$, we set $\xi^{1/2}(t) = \xi(t)$, where ξ is a standard Brownian motion; in this case the increments of the process are independent. On the contrary, for $H \neq 1/2$ the increments are not independent.

Let $\mathbb{U}$ be a separable Hilbert space endowed with a Hilbert basis $\{e_k\}_{k\geq 1}$. We then consider $\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x}$ and $\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x}$, respectively, the $\mathbb{U}$-valued Gaussian noise and fractional Gaussian noise defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ such that

$$\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varsigma_k(t) \dot{\xi}_k(t) e_k(x), \tag{2.6}$$

$$\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varrho_k(t) \dot{\xi}_k^H(t) e_k(x), \tag{2.7}$$

where $\varsigma_k(t)$ and $\varrho_k(t)$ are continuous functions, rapidly decaying with the increase of k to ensure the convergence of the series, $\{\xi_k\}_{k=1}^{\infty}$ and $\{\xi_k^H\}_{k=1}^{\infty}$, respectively, are the sequences of mutually independent one-dimensional standard Brownian motions and fractional Brownian motions with Hurst index $H \in (1/2, 1)$; $\dot{\xi}_k(t) = \frac{d\xi_k(t)}{dt}$, $k = 1, 2, \ldots$ is the white noise, the formal derivative of the Brownian motion $\xi_k(t)$, $k = 1, 2, \ldots,$ and $\dot{\xi}_k^H(t) = \frac{d\xi_k^H(t)}{dt}$, $k = 1, 2, \ldots$ is the fractional Gaussian noise, the formal derivative of the fractional Brownian motion $\xi_k^H(t)$, $k = 1, 2, \ldots$.

We define $\mathbb{H} := L^2(\mathcal{D})$ and denote the space of bounded linear operators from $\mathbb{U}$ to $\mathbb{H}$ by $\mathcal{L}(\mathbb{U}, \mathbb{H})$. Let

$$\mathcal{L}_2^0(\mathbb{U}, \mathbb{H}) = \left\{ R \in \mathcal{L}(\mathbb{U}, \mathbb{H}) : \sum_{k=1}^{\infty} \|Re_k\|^2 < \infty \right\}$$

be the set of Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$, and endow this set with the inner product $(R, S)_{\mathcal{L}_2^0} = \sum_k (Re_k, Se_k)$, so that $\mathcal{L}_2^0(\mathbb{U}, \mathbb{H})$ can be considered as a Hilbert space with the norm

$$\|R\|_{\mathcal{L}_2^0} = \left(\sum_{k=1}^{\infty} \|Re_k\|^2 \right)^{\frac{1}{2}}.$$

The following notations will be used throughout the paper.

Remark 2.2. For $g, h \in L^2(0, T; \mathcal{L}_2^0(\mathbb{U}, \mathbb{H}))$,

$$\frac{\partial^2 \mathbb{W}(t, x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varsigma_k(t) \dot{\xi}_k(t) e_k(x),$$
$$\frac{\partial^2 \mathbb{W}^H(t, x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varrho_k(t) \dot{\xi}_k^H(t) e_k(x),$$

we introduce the notations $g(t)\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x}$ and $h(t)\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x}$ to formally represent

$$\begin{aligned} g(t)\frac{\partial^2 \mathbb{W}(t, x)}{\partial t \partial x} &= \sum_{k=1}^{\infty} g(t) \cdot e_k \varsigma_k(t) \dot{\xi}_k(t) \\ &= \sum_{j,k=1}^{\infty} \big(g(t) \cdot e_k, \varphi_j\big) \varphi_j \varsigma_k(t) \dot{\xi}_k(t) = \sum_{j,k=1}^{\infty} g^{j,k}(t) \varphi_j \varsigma_k(t) \dot{\xi}_k(t), \end{aligned} \tag{2.8}$$

and in a similar way,

$$h(t)\frac{\partial^2 \mathbb{W}^H(t, x)}{\partial t \partial x} = \sum_{j,k=1}^{\infty} h^{j,k}(t) \varphi_j \varrho_k(t) \dot{\xi}_k^H(t), \tag{2.9}$$

where we have used the decomposition

$$g(t) \cdot e_k = \sum_{j=1}^{\infty} g^{j,k}(t) \varphi_j, \quad g^{j,k}(t) := (g(t) \cdot e_k, \varphi_j),$$

$$h(t)\cdot e_k=\sum_{j=1}^{\infty}h^{j,k}(t)\varphi_j,\quad h^{j,k}(t):=(h(t)\cdot e_k,\varphi_j),$$

which make sense since, from assumptions, $g(t)\cdot e_k$ and $h(t)\cdot e_k$ belong to $\mathbb{H}$ and $\{\varphi_j\}_{j\in\mathbb{N}^+}$ is a Hilbert basis of $\mathbb{H}$.

The following proposition plays an important role in the proof of this paper (see, for instance, [19, 25]).

Proposition 2.1. *For $H>1/2$ and $f,g\in L^2([0,T];\mathbb{R})$, we have*

$$\begin{aligned}
&\mathbb{E}\int_0^T f(s)d\xi^H(s)=0,\\
&\mathbb{E}\left[\int_0^T f(s)d\xi(s)\int_0^T g(s)d\xi(s)\right]=\int_0^T\mathbb{E}[f(s)g(s)]ds,\\
&\mathbb{E}\left[\int_0^T f(s)d\xi^H(s)\int_0^T g(s)d\xi^H(s)\right]\\
&=H(2H-1)\int_0^T\int_0^T f(s)g(r)|s-r|^{2H-2}drds.
\end{aligned}\tag{2.10}$$

Proof. Here, we just need to prove (2.10). Thus

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T f(s)d\xi^H(s)\int_0^T g(s)d\xi^H(s)\right]\\
&=\mathbb{E}\left[\int_0^T\int_s^T f(r)(r-s)^{H-\frac{3}{2}}\left(\frac{s}{r}\right)^{\frac{1}{2}-H}drd\xi(s)\int_0^T\int_s^T g(r)(r-s)^{H-\frac{3}{2}}\left(\frac{s}{r}\right)^{\frac{1}{2}-H}drd\xi(s)\right]\\
&=\int_0^T\mathbb{E}\left[\int_s^T\int_s^T f(r_1)g(r_2)(r_1-s)^{H-\frac{3}{2}}\left(\frac{s}{r_1}\right)^{\frac{1}{2}-H}(r_2-s)^{H-\frac{3}{2}}\left(\frac{s}{r_2}\right)^{\frac{1}{2}-H}dr_1dr_2\right]ds\\
&=\int_0^T\int_0^{r_2}\int_s^T f(r_1)g(r_2)(r_1-s)^{H-\frac{3}{2}}\left(\frac{s}{r_1}\right)^{\frac{1}{2}-H}(r_2-s)^{H-\frac{3}{2}}\left(\frac{s}{r_2}\right)^{\frac{1}{2}-H}dr_1dsdr_2\\
&=\int_0^T\int_0^T\int_0^{r_1\wedge r_2} f(r_1)g(r_2)s^{1-2H}(r_1-s)^{H-\frac{3}{2}}(r_2-s)^{H-\frac{3}{2}}dsr_1{}^{H-\frac{1}{2}}r_2{}^{H-\frac{1}{2}}dr_1dr_2\\
&=H(2H-1)\int_0^T\int_0^T f(r_1)g(r_2)|r_1-r_2|^{2H-2}dr_1dr_2,
\end{aligned}$$

where $r_1\wedge r_2=\min\{r_1,r_2\}$, and we have used the equation

$$|r_1-r_2|^{2H-2}=\frac{(r_1r_2)^{H-\frac{1}{2}}}{B\big(2-2H,H-\frac{1}{2}\big)}\int_0^{r_1\wedge r_2}s^{1-2H}(r_1-s)^{H-\frac{3}{2}}(r_2-s)^{H-\frac{3}{2}}ds.$$

The proof is complete. □

As a simple consequence of Proposition 2.1, we have the following result.

Lemma 2.2. *Let $H > 1/2$ and $\phi \in L^2([0,T];\mathbb{R})$. Then for any $t_1, t_2 \in [0,T]$ with $t_2 > t_1$,*

$$\mathbb{E}\left|\int_{t_1}^{t_2} \phi(s)d\xi^H(s)\right|^2 \leq 2H(t_2-t_1)^{2H-1}\int_{t_1}^{t_2}|\phi(s)|^2 ds.$$

Proof. By Proposition 2.1, we find that

$$\begin{aligned}\mathbb{E}\left|\int_{t_1}^{t_2} \phi(s)d\xi^H(s)\right|^2 &\leq H(2H-1)\int_{t_1}^{t_2}\int_{t_1}^{t_2}\big(|\phi(s)||\phi(r)|\big)|s-r|^{2H-2}drds \\ &\leq H(2H-1)\int_{t_1}^{t_2}\int_{t_1}^{t_2}\big(|\phi(s)|^2\big)|s-r|^{2H-2}drds \\ &\leq 2H(t_2-t_1)^{2H-1}\int_{t_1}^{t_2}|\phi(s)|^2 ds,\end{aligned}$$

which completes the proof. □

2.4. Regularity of the solution

Now we recall the existence, uniqueness, and regularity results of mild solutions to (1.1), see [22] for more details.

Let $v(t,x) = e^{\nu t}u(t,x)$. Then by Definitions 2.3-2.4, we rewrite (1.1) as

$$\begin{cases} {}_0^c\partial_t^\alpha v + (-\Delta)^\beta v \\ = e^{\nu t}\left[f(t,u) + g(t,u)\dfrac{\partial^2 \mathbb{W}(t,x)}{\partial t\partial x} + h(t)\dfrac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x}\right] & \text{in } (0,T]\times\mathcal{D}, \\ v(t,x) = 0 & \text{on } (0,T]\times\partial\mathcal{D}, \\ v(0,x) = a(x), \ \ \partial_t v(0,x) = \nu a(x) + b(x) & \text{in } \mathcal{D}, \end{cases} \tag{2.11}$$

where ${}_0^c\partial_t^\alpha$ denotes the left-sided Caputo fractional derivative of order α with respect to t. We assume that $a \in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}})$ and $b \in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha})$ with $\widetilde{\gamma} = \max\{\gamma, \beta/\alpha\}$ for some regularity parameter $\gamma > 0$.

In order to ensure the existence and uniqueness of problem (1.1), we list the following conditions:

(A$_1$) There exists a positive constant l such that the functions $f : \mathbb{R}\times\mathbb{H} \to \mathbb{H}$, $g : \mathbb{R}\times\mathbb{H} \to \mathcal{L}_2^0(\mathbb{U},\mathbb{H})$, and $h : \mathbb{R} \to \mathcal{L}_2^0(\mathbb{U},\mathbb{H})$ satisfy

$$\begin{aligned}&\|f(t_1,u_1) - f(t_2,u_2)\| + \|g(t_1,u_1) - g(t_2,u_2)\|_{\mathcal{L}_2^0} + \|h(t_1)-h(t_2)\|_{\mathcal{L}_2^0} \\ &\leq l\big(|t_1-t_2| + \|u_1-u_2\|\big), \quad \text{if} \quad 0 \leq \gamma \leq \frac{\beta}{\alpha}\end{aligned}$$

for all t_1, $t_2 \in \mathbb{R}$ and u_1, $u_2 \in \mathbb{H}$

$$\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}\big(f(t_1,u_1)-f(t_2,u_2)\big)\right\| + \left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}\big(g(t_1,u_1)-g(t_2,u_2)\big)\right\|_{\mathcal{L}_2^0}$$

$$
\begin{aligned}
&+\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}\big(h(t_1)-h(t_2)\big)\right\|_{\mathcal{L}_2^0}\\
&\le l\bigg(|t_1-t_2|+\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}(u_1-u_2)\right\|\bigg),\quad \text{if}\quad \gamma>\frac{\beta}{\alpha}
\end{aligned}
$$

for all $t_1, t_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{H}^{2\gamma-2\beta/\alpha}$

$$
\|f(t,u)\|+\|g(t,u)\|_{\mathcal{L}_2^0}+\|h(t)\|_{\mathcal{L}_2^0}\le l\big(1+\|u\|\big),\quad \text{if}\quad 0\le\gamma\le\frac{\beta}{\alpha}
$$

for all $t\in\mathbb{R}$ and $u\in\mathbb{H}$

$$
\begin{aligned}
&\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}f(t,u)\right\|+\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}g(t,u)\right\|_{\mathcal{L}_2^0}+\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}h(t)\right\|_{\mathcal{L}_2^0}\\
&\le l\bigg(1+\left\|(-\Delta)^{\gamma-\frac{\beta}{\alpha}}u\right\|\bigg),\quad \text{if}\quad \gamma>\frac{\beta}{\alpha}
\end{aligned}
$$

for all $t\in\mathbb{R}$ and $u\in\mathbb{H}^{2\gamma-2\beta/\alpha}$.

($\mathbf{A}_2$) $\{\varsigma_k(t)\}$, $\{\varrho_k(t)\}$, and their derivatives are uniformly bounded by

$$
|\varsigma_k(t)|\le\mu_k,\quad |\varrho_k(t)|\le\widetilde{\mu}_k,\quad |\varsigma_k'(t)|\le\gamma_k,\quad |\varrho_k'(t)|\le\widetilde{\gamma}_k,\quad \forall t\in[0,T],
$$

and the series $(\{\mu_k\},\{\widetilde{\mu}_k\},\{\gamma_k\},\{\widetilde{\gamma}_k\})$ is rapidly decaying with the increase of k.

Remark 2.3. It is worth mentioning that if $\alpha=2$ and $\beta=1$, then the condition ($\mathbf{A_1}$) will reduce to the corresponding condition (9) in [1].

First, we give a representation of the mild solution to problem (1.1) using the Dirichlet eigenpairs $\{(\lambda_k,\varphi_k)\}_{k=1}^{\infty}$.

Lemma 2.3. *The solution u to problem* (1.1) *with $3/2<\alpha<2, 1/2<\beta\le 1$, $\nu>0$, and $1/2<H<1$ is given by*

$$
\begin{aligned}
u(t,x)=&\int_{\mathcal{D}}\mathcal{T}_{\alpha,\beta}^{\nu}(t,x,y)a(y)dy+\int_{\mathcal{D}}\mathcal{R}_{\alpha,\beta}^{\nu}(t,x,y)b(y)dy\\
&+\int_0^t\int_{\mathcal{D}}\mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)f\big(s,u(s,y)\big)dyds\\
&+\int_0^t\int_{\mathcal{D}}\mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)g\big(s,u(s,y)\big)d\mathbb{W}(s,y)\\
&+\int_0^t\int_{\mathcal{D}}\mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)h(s)d\mathbb{W}^H(s,y),
\end{aligned}\tag{2.12}
$$

where the stochastic integral with respect to $\mathbb{W}$ is understood in the Itô sense, and the stochastic integral with respect to $\mathbb{W}^H$ is the Riemann integral [5, 19, 25]. Here

$$
\mathcal{T}_{\alpha,\beta}^{\nu}(t,x,y)=e^{-\nu t}\sum_{k=1}^{\infty}\Big(E_{\alpha,1}\big(-\lambda_k^{\beta}t^{\alpha}\big)+\nu tE_{\alpha,2}\big(-\lambda_k^{\beta}t^{\alpha}\big)\Big)\varphi_k(x)\varphi_k(y)\tag{2.13}
$$

is the fundamental solution of

$$\begin{cases} {}_0^c\partial_t^{\alpha,\nu} v(t,x) + (-\Delta)^\beta v(t,x) = 0 & \text{in } (0,T]\times\mathcal{D}, \\ v(t,x)=0 & \text{on } (0,T]\times\partial\mathcal{D}, \\ v(0,x)=\phi(x),\ \ \partial_t v(t,x)|_{t=0}=0 & \text{in } \mathcal{D}, \end{cases}$$

so that

$$v(t,x) = \int_{\mathcal{D}} \mathcal{T}_{\alpha,\beta}^{\nu}(t,x,y)\phi(y)dy,$$

and

$$\mathcal{R}_{\alpha,\beta}^{\nu}(t,x,y) = te^{-\nu t}\sum_{k=1}^{\infty} E_{\alpha,2}\big(-\lambda_k^\beta t^\alpha\big)\varphi_k(x)\varphi_k(y) \tag{2.14}$$

is the fundamental solution of

$$\begin{cases} {}_0^c\partial_t^{\alpha,\nu} v(t,x) + (-\Delta)^\beta v(t,x) = 0 & \text{in } (0,T]\times\mathcal{D}, \\ v(t,x)=0 & \text{on } (0,T]\times\partial\mathcal{D}, \\ v(0,x)=0,\ \ \partial_t v(0,x)=\psi(x) & \text{in } \mathcal{D}, \end{cases}$$

so that

$$v(t,x) = \int_{\mathcal{D}} \mathcal{R}_{\alpha,\beta}^{\nu}(t,x,y)\psi(y)dy.$$

For (1.1) with the initial data $v(0,x)=\partial_t v(0,x)\equiv 0$, we shall use the operator defined by

$$\mathcal{S}_{\alpha,\beta}^{\nu}(t,x,y) = t^{\alpha-1}e^{-\nu t}\sum_{k=1}^{\infty} E_{\alpha,\alpha}\big(-\lambda_k^\beta t^\alpha\big)\varphi_k(x)\varphi_k(y) \tag{2.15}$$

and

$$\begin{aligned} v(t,x) = &\int_0^t\int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)f\big(s,u(s,y)\big)dyds \\ &+ \int_0^t\int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)g\big(s,u(s,y)\big)d\mathbb{W}(s,y) \\ &+ \int_0^t\int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^{\nu}(t-s,x,y)h(s)d\mathbb{W}^H(s,y). \end{aligned}$$

The following result is on stability estimates for the homogeneous problem of (1.1).

Lemma 2.4. *Let u be the solution of* (1.1) *with $f=g=h=0$ and the initial data $u(0,x)=a(x),\partial_t u(0,x)=b(x)$. Then for all $t>0$,*

$$\|u(t)\|_{\mathbb{H}^p} \le \begin{cases} C(1+\nu t)e^{-\nu t}t^{-\frac{\alpha(p-q)}{2\beta}}\|a\|_{\mathbb{H}^q} + Ce^{-\nu t}t^{1-\frac{\alpha(p-r)}{2\beta}}\|b\|_{\mathbb{H}^r}, & 0\le q,r\le p\le 2\beta, \\ C(1+\nu t)e^{-\nu t}t^{-\alpha}\|a\|_{\mathbb{H}^q} + Ce^{-\nu t}t^{1-\alpha}\|b\|_{\mathbb{H}^r}, & q,r>p, \end{cases}$$

and

$$\big\|{}_0^c\partial_t^{\alpha,\nu}u(t)\big\|_{\mathbb{H}^p} \le C(1+\nu t)e^{-\nu t}t^{-\alpha-\frac{\alpha(p-q)}{2\beta}}\|a\|_{\mathbb{H}^q} + Ce^{-\nu t}t^{1-\alpha-\frac{\alpha(p-r)}{2\beta}}\|b\|_{\mathbb{H}^r},$$

where $0\le p\le q$ and $r\le p+2\beta$.

Remark 2.4. It follows from the proof of Lemma 2.4 that

$$\|u(t)\|_{\mathbb{H}^p} \le C(1+\nu t)e^{-\nu t}t^{-\frac{\alpha(p-q)}{2\beta}}\|a\|_{\mathbb{H}^q} + Ce^{-\nu t}t^{1-\frac{\alpha(p-r)}{2\beta}}\|b\|_{\mathbb{H}^r}$$

also holds true for all $t > 0, 0 \le q, r \le p$ and $p-q, p-r \le 2\beta$.

The following result shows the existence, uniqueness, decay, and regularity properties of mild solutions.

Proposition 2.2. *Let (a, b) be $\mathcal{F}_0$-adapted random variable and*

$$\|a\|_{L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))} + \|b\|_{L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}(\mathcal{D}))} < \infty$$

with $\widetilde{\gamma} = \max\{\gamma, \beta/\alpha\}$, and that the functions f, g, and h satisfy $(\mathbf{A_1})$ for some $\gamma \ge 0$. Let $(\mathbf{A_2})$ holds, $3/2 < \alpha < 2, 1/2 < \beta \le 1, \nu > 0$, and $1/2 < H < 1$. Then problem (1.1) *has a unique mild solution $u \in C([0,T]; L^2(\Omega; \mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))$ given by* (2.12) *for each $t \in [0,T]$, and $\sup_{0\le t\le T}\|\partial_t u(t)\|_{\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}} \le C$. Moreover, if $\gamma > \beta/\alpha$, then we obtain that for any $\delta \in (0, 2\beta - 3\beta/\alpha)$,*

$$t^{\frac{\alpha\delta}{2\beta}}u \in C\left([0,T]; L^2\left(\Omega; \mathbb{H}^{2\widetilde{\gamma}+\delta}(\mathcal{D})\right)\right) \text{ with value zero at } t = 0,$$

and for $0 \le \theta_1 \le \theta_2 \le T$,

$$\mathbb{E}\big\|u(\theta_2) - u(\theta_1)\big\|^2 \le C|\theta_2 - \theta_1|^{2\alpha-2} \times \left(\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + \mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} + \sup_{s\in[0,T]}\mathbb{E}\left(1 + \|u(s)\|^2_{\mathbb{H}^{2\gamma}}\right)\right).$$

3. Regularity and approximation of Gaussian noise and fractional Gaussian noise

Now we define a partition of $[0,T]$ by intervals $[t_i, t_{i+1}]$ for $i = 1, 2, \ldots, N$, where $t_i = (i-1)\tau, \tau = T/N$. A sequence of noise which approximates the space-time Gaussian noise is defined as

$$\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varsigma_k^n(t) e_k(x) \left(\sum_{i=1}^{N} \frac{1}{\sqrt{\tau}} \xi_{ki}\chi_i(t)\right),$$

and another sequence of noise which approximates the space-time fractional Gaussian noise with Hurst parameter $H \in (1/2, 1)$ is defined as

$$\frac{\partial^2 \mathbb{W}_n^H(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \varrho_k^n(t) e_k(x) \left(\sum_{i=1}^{N} \frac{1}{\tau^{1-H}} \xi_{ki}^H \chi_i(t)\right),$$

where $\chi_i(t)$ is the characteristic function for the ith time subinterval,

$$\xi_{ki} = \frac{1}{\sqrt{\tau}} \int_{t_i}^{t_{i+1}} d\xi_k(t) = \frac{1}{\sqrt{\tau}} \big(\xi_k(t_{i+1}) - \xi_k(t_i)\big) \sim \mathcal{N}(0,1),$$
$$\xi_{ki}^H = \frac{1}{\tau^H} \int_{t_i}^{t_{i+1}} d\xi_k^H(t) = \frac{1}{\tau^H} \big(\xi_k^H(t_{i+1}) - \xi_k^H(t_i)\big) \sim \mathcal{N}(0,1),$$

$\varsigma_k^n(t)$ and $\varrho_k^n(t)$, respectively, are the approximations of $\varsigma_k(t)$ and $\varrho_k(t)$ in the space direction. Then $\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t \partial x}$ and $\frac{\partial^2 \mathbb{W}_n^H(t,x)}{\partial t \partial x}$ are, respectively, substituted for $\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x}$ and $\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x}$ in (1.1) to obtain the equation

$$\begin{cases} {}_0^c\partial_t^{\alpha,\nu} u_n(t,x) + (-\Delta)^\beta u_n(t,x) & \\ = f\big(t,u_n(t,x)\big) + g\big(t,u_n(t,x)\big)\dfrac{\partial^2 \mathbb{W}_n(t,x)}{\partial t \partial x} + h(t)\dfrac{\partial^2 \mathbb{W}_n^H(t,x)}{\partial t \partial x} & \text{in } (0,T]\times\mathcal{D}, \\ u_n(t,x) = 0 & \text{on } (0,T]\times\partial\mathcal{D}, \\ u_n(0,x) = a(x), \ \ \partial_t u_n(0,x) = b(x) & \text{in } \mathcal{D}. \end{cases} \tag{3.1}$$

As a simple consequence of Lemma 2.3, we get an integral formulation of (3.1).

Lemma 3.1. *The solution u_n to problem* (3.1) *with $3/2 < \alpha < 2, 1/2 < \beta \le 1, \nu > 0$, and $1/2 < H < 1$ is given by*

$$\begin{aligned} u_n(t,x) = & \int_{\mathcal{D}} \mathcal{T}_{\alpha,\beta}^\nu(t,x,y)a(y)dy + \int_{\mathcal{D}} \mathcal{R}_{\alpha,\beta}^\nu(t,x,y)b(y)dy \\ & + \int_0^t \int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^\nu(t-s,x,y) f\big(s,u_n(s,y)\big)dyds \\ & + \int_0^t \int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^\nu(t-s,x,y) g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y) \\ & + \int_0^t \int_{\mathcal{D}} \mathcal{S}_{\alpha,\beta}^\nu(t-s,x,y) h(s) d\mathbb{W}_n^H(s,y), \end{aligned} \tag{3.2}$$

where $\mathcal{T}_{\alpha,\beta}^\nu(t,x,y), \mathcal{R}_{\alpha,\beta}^\nu(t,x,y)$, and $\mathcal{S}_{\alpha,\beta}^\nu(t-s,x,y)$ are given in Lemma 2.3*.*

The following theorem shows the regularity of the solution of (3.2), which will be used in the error analysis.

Theorem 3.1. *Assume $\{\varsigma_k^n(t)\}$ and $\{\varrho_k^n(t)\}$ are uniformly bounded by $|\varsigma_k^n| \le \mu_k^n$ and $|\varrho_k^n| \le \widetilde{\mu}_k^n$ for all $t \in [0,T]$, and the series $(\{\mu_k^n\},\{\widetilde{\mu}_k^n\})$ is rapidly decaying with the increase of k. Further assume that the functions f, g, and h satisfy* ($\boldsymbol{A}_1$) *for some $\gamma \ge 0$. Let* ($\boldsymbol{A}_2$) *holds, $3/2 < \alpha < 2, 1/2 < \beta \le 1, \nu > 0$, and $1/2 < H < 1$, and let u_n be the solution to* (3.2)*, where the $\mathcal{F}_0$-adapted random initial values satisfy $a \in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))$, $b \in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}(\mathcal{D}))$ with $\widetilde{\gamma} = \max(\gamma,\beta/\alpha)$. Then it holds that*

$$\begin{aligned} \sup_{0\le t\le T} \mathbb{E}\|u_n(t)\|_{\mathbb{H}^{2\widetilde{\gamma}}}^2 \le & \big(C + C(\mu_1^n)^2\big)\,\mathbb{E}\|a\|_{\mathbb{H}^{2\widetilde{\gamma}}}^2 + \big(C + C(\mu_1^n)^2\big)\,\mathbb{E}\|b\|_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}^2 \\ & + \big(C + C(\mu_1^n)^2 + C(\widetilde{\mu}_1^n)^2\big)\left(1 + (\mu_1^n)^2\right), \end{aligned}$$

$$\sup_{0\le t\le T}\mathbb{E}\|\partial_t u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\le C\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}+C\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}+C\left(\widetilde{\mu}_1^n\right)^2 + \left(C+C\left(\mu_1^n\right)^2\right)\left(1+\sup_{0\le t\le T}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right),$$

$$\sup_{0\le t\le T}\mathbb{E}\left\|{}_0^c\partial_t^{\alpha,\nu}u_n(t)\right\|^2_{\mathbb{H}^{2\widetilde{\gamma}-2\beta}}\le C\left(\widetilde{\mu}_1^n\right)^2\tau^{2H-2} + \left(C+C\left(\mu_1^n\right)^2\tau^{-1}\right)\left(1+\sup_{0\le t\le T}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right),$$

and for $0\le\theta_1\le\theta_2\le T$*,*

$$\begin{aligned}&\mathbb{E}\left\|u_n(\theta_2)-u_n(\theta_1)\right\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\\ &\le C|\theta_2-\theta_1|^2\Bigg(\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}+\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}+1+\left(\mu_1^n\right)^2+\left(\widetilde{\mu}_1^n\right)^2\\ &\quad+\sup_{0\le s\le T}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}+\left(\mu_1^n\right)^2\sup_{0\le s\le T}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\Bigg).\end{aligned}$$

The proof of Theorem 3.1 is given in Appendix A.

In order to prove that the solution u_n of (3.1) indeed approximates u, the solution of (1.1), first we need the assumptions on $\{\varsigma_k(t)\}, \{\varsigma_k^n(t)\}, \{\varrho_k(t)\}$, and $\{\varrho_k^n(t)\}$.

($\mathbf{A_3}$) Assume that $\{\varsigma_k(t)\}, \{\varrho_k(t)\}$ and their derivatives are uniformly bounded by

$$|\varsigma_k(t)|\le\mu_k,\quad |\varrho_k(t)|\le\widetilde{\mu}_k,\quad |\varsigma_k'(t)|\le\gamma_k,\quad |\varrho_k'(t)|\le\widetilde{\gamma}_k,\quad \forall t\in[0,T],$$

and that the coefficients $\{\varsigma_k^n(t)\}$ and $\{\varrho_k^n(t)\}$ are constructed such that

$$\begin{aligned}&\left|\varsigma_k(t)-\varsigma_k^n(t)\right|\le\eta_k^n, &&\left|\varrho_k(t)-\varrho_k^n(t)\right|\le\widetilde{\eta}_k^n, &&\left|\varsigma_k^n(t)\right|\le\mu_k^n,\\ &\left|\varrho_k^n(t)\right|\le\widetilde{\mu}_k^n, &&\left|(\varsigma_k^n)'(t)\right|\le\gamma_k^n, &&\left|(\varrho_k^n)'(t)\right|\le\widetilde{\gamma}_k^n,\quad\forall t\in[0,T]\end{aligned}$$

with positive sequences $\{\eta_k^n\}$ and $\{\widetilde{\eta}_k^n\}$ being arbitrarily chosen, and $\{\mu_k^n\},\{\widetilde{\mu}_k^n\}$, $\{\gamma_k^n\}$, and $\{\widetilde{\gamma}_k^n\}$ being related to $\{\eta_k^n\mu_k\},\{\widetilde{\eta}_k^n\widetilde{\mu}_k\},\{\gamma_k\}$, and $\{\widetilde{\gamma}_k\}$. The series $(\{\mu_k\},\{\widetilde{\mu}_k\},\{\gamma_k\},\{\widetilde{\gamma}_k\},\{\gamma_k^n\},\{\widetilde{\gamma}_k^n\},\{\mu_k^n\},\{\widetilde{\mu}_k^n\})$ are rapidly decaying with the increase of k, and the series $(\{\eta_k^n\},\{\widetilde{\eta}_k^n\})$ is required to rapidly decay to ensure the convergence of the series in Theorem 3.2.

Theorem 3.2. *Assume that the functions f, g and h satisfy ($\boldsymbol{A}_1$) for some $\gamma\ge 0$, and that ($\boldsymbol{A}_3$) holds, $3/2<\alpha<2, 1/2<\beta\le 1, \nu>0$ and $1/2<H<1$. Let u_n and u be the solutions of* (3.2) *and* (1.1)*, respectively, where the $\mathcal{F}_0$-adapted random initial values satisfy $a\in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))$, $b\in L^2(\Omega;\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}(\mathcal{D}))$ with $\widetilde{\gamma}=\max(\gamma,\beta/\alpha)$. Then for some constant $C>0$ independent of τ,*

$$\mathbb{E}\left\|u(t)-u_n(t)\right\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\le C\tau^2+C\sum_{l=1}^{\infty}\left(\eta_l^n\right)^2+C\sum_{l=1}^{\infty}\left(\widetilde{\eta}_l^n\right)^2,\quad t>0,$$

provided that the infinite series are all convergent.

Proof. Without loss of generality, we assume that there exists a positive integer N_t such that $t = t_{N_t+1}$. Subtracting (3.2) from (2.12), we have

$$u(t,x) - u_n(t,x) = \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3 + \mathscr{A}_4 + \mathscr{A}_5 + \mathscr{A}_6 + \mathscr{A}_7 + \mathscr{A}_8, \tag{3.3}$$

where

$$\begin{aligned}
\mathscr{A}_1 = & \int_0^t \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \big(f\big(s,u(s,y)\big) - f\big(s,u_n(s,y)\big)\big)\, dyds, \\
\mathscr{A}_2 = & \int_0^t \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \sum_{l=1}^{\infty} \big(g\big(s,u(s,y)\big) - g\big(s,u_n(s,y)\big)\big) \cdot e_l(y)\varsigma_l(s) dy d\xi_l(s), \\
\mathscr{A}_3 = & \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \sum_{l=1}^{\infty} \big(g\big(s,u_n(s,y)\big) \cdot e_l(y)\big)\, \varsigma_l(s) dy d\xi_l(s) \\
& - \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} \\
& \times \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \sum_{l=1}^{\infty} \big(g\big(t_i,u_n(t_i,y)\big) \cdot e_l(y)\big)\varsigma_l(t_i) dy d\xi_l(s), \\
\mathscr{A}_4 = & \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \sum_{l=1}^{\infty} \big(g\big(t_i,u_n(t_i,y)\big) \cdot e_l(y)\big) \big(\varsigma_l(t_i) - \varsigma_l^n(t_i)\big) dy d\xi_l(s), \\
\mathscr{A}_5 = & \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \\
& \times \sum_{l=1}^{\infty} \big(g\big(t_i,u_n(t_i,y)\big) \cdot e_l(y)\big)\varsigma_l^n(t_i) dy d\xi_l(s) \\
& - \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(x)\varphi_k(y)
\end{aligned}$$

$$\times \sum_{l=1}^{\infty} \big(g\big(s, u_n(s,y)\big) \cdot e_l(y)\big) \varsigma_l^n(s) \frac{\xi_l(t_{i+1}) - \xi_l(t_i)}{\tau} dyds,$$

$$\mathscr{A}_6 = \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(x)\varphi_k(y)$$
$$\times \sum_{l=1}^{\infty} \big(h(s) \cdot e_l(y)\big) \varrho_l(s) dy d\xi_l^H(s) - \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)}$$
$$\times \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y) \sum_{l=1}^{\infty} \big(h(t_i) \cdot e_l(y)\big) \varrho_l(t_i) dy d\xi_l^H(s),$$

$$\mathscr{A}_7 = \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y)$$
$$\times \sum_{l=1}^{\infty} \big(h(t_i) \cdot e_l(y)\big) \big(\varrho_l(t_i) - \varrho_l^n(t_i)\big) dy d\xi_l^H(s),$$

$$\mathscr{A}_8 = \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varphi_k(x)\varphi_k(y)$$
$$\times \sum_{l=1}^{\infty} \big(h(t_i) \cdot e_l(y)\big) \varrho_l^n(t_i) dy d\xi_l^H(s) - \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)}$$
$$\times \sum_{k=1}^{\infty} E_{\alpha,\alpha}\Big(-\lambda_k^{\beta}(t-s)^{\alpha}\Big) \varphi_k(x)\varphi_k(y) \sum_{l=1}^{\infty} \big(h(s) \cdot e_l(y)\big) \varrho_l^n(s) \frac{\xi_l^H(t_{i+1}) - \xi_l^H(t_i)}{\tau} dyds,$$

and

$$d\mathbb{W}(s,y) = \frac{\partial^2 \mathbb{W}}{\partial s \partial y} dyds = \sum_{k=1}^{\infty} \varsigma_k(s) e_k(y) dy d\xi_k(s),$$
$$d\mathbb{W}_n(s,y) = \frac{\partial^2 \mathbb{W}_n}{\partial s \partial y} dyds = \sum_{k=1}^{\infty} \varsigma_k^n(s) e_k(y) \left(\sum_{i=1}^{N_t} \frac{1}{\sqrt{\tau}} \xi_{ki} \chi_i(s)\right) dyds,$$
$$d\mathbb{W}^H(s,y) = \frac{\partial^2 \mathbb{W}^H}{\partial s \partial y} dyds = \sum_{k=1}^{\infty} \varrho_k(s) e_k(y) dy d\xi_k^H(s),$$
$$d\mathbb{W}_n^H(s,y) = \frac{\partial^2 \mathbb{W}_n^H}{\partial s \partial y} dyds = \sum_{k=1}^{\infty} \varrho_k^n(s) e_k(y) \left(\sum_{i=1}^{N_t} \frac{1}{\tau^{1-H}} \xi_{ki}^H \chi_i(s)\right) dyds.$$

For $\mathscr{A}_1$, since $\{\varphi_k\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$, by the assumption on f given in $(\mathbf{A_1})$, Lemma 2.1, and Hölder's inequality, we have

$$\mathbb{E}\|\mathscr{A}_1\|_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}^2 = \mathbb{E} \sum_{k=1}^{\infty} \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}} \bigg| \int_0^t \int_{\mathcal{D}} (t-s)^{\alpha-1} e^{-\nu(t-s)} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varphi_k(y)$$

$$
\begin{aligned}
&\qquad\qquad \times \big(f\big(s,u(s,y)\big)-f\big(s,u_n(s,y)\big)\big)\,dyds\Big|^2 \\
&\le CT\mathbb{E}\sum_{k=1}^{\infty}\int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)}\frac{1}{(1+\lambda_k^{\beta}(t-s)^{\alpha})^2} \\
&\quad \times \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(f\big(s,u(s,y)\big)-f\big(s,u_n(s,y)\big),\varphi_k\big)^2\,ds \\
&\le C\int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)}\mathbb{E}\|u(s)-u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}ds. \qquad (3.4)
\end{aligned}
$$

Since $\{\varphi_k\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$ and $\{\xi_l\}_{l=1}^{\infty}$ is a family of mutually independent one-dimensional standard Brownian motions, then by (2.8),($\mathbf{A_1}$),($\mathbf{A_3}$), and the Itô isometry, we have

$$
\begin{aligned}
\mathbb{E}\|\mathscr{A}_2\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} &= \mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\bigg|\int_0^t\int_{\mathcal{D}}(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\varphi_k(y) \\
&\qquad \times\sum_{j,l=1}^{\infty}\big(\big(g(s,u(s))-g(s,u_n(s))\big)\cdot e_l,\varphi_j\big)\varphi_j(y)\varsigma_l(s)dyd\xi_l(s)\bigg|^2 \\
&\le C\mathbb{E}\sum_{k,l=1}^{\infty}\int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)}\frac{1}{\big(1+\lambda_k^{\beta}(t-s)^{\alpha}\big)^2} \\
&\quad\times\left|\lambda_k^{\widetilde{\gamma}-\frac{\beta}{\alpha}}\big(\big(g(s,u(s))-g(s,u_n(s))\big)\cdot e_l,\varphi_k\big)\varsigma_l(s)\right|^2 ds \\
&\le C\int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)}\mathbb{E}\|u(s)-u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}ds. \qquad (3.5)
\end{aligned}
$$

We observe that for $s\in[t_i,t_{i+1}]$, by (A.5), Lemma 2.1, Lagrange's mean value theorem, and the equality (see [27, Eq. (1.82)])

$$
\frac{d}{d\tau}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-\tau)^{\alpha}\right)=-(t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-\tau)^{\alpha}\right),
$$

we have

$$
\begin{aligned}
&\Big|(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\varsigma_l(s) \\
&\qquad -(t-t_i)^{\alpha-1}e^{-\nu(t-t_i)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\varsigma_l(t_i)\Big| \\
&\le e^{-\nu(t-t_i)}|\varsigma_l(t_i)|\left|(t-t_i)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)-(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right| \\
&\qquad +(t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right||\varsigma_l(t_i)||e^{-\nu(t-t_i)}-e^{-\nu(t-s)}| \\
&\qquad +(t-s)^{\alpha-1}e^{-\nu(t-s)}\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right||\varsigma_l(t_i)-\varsigma_l(s)| \\
&\le \mu_1\int_{t_i}^{s}(t-r)^{\alpha-2}\left|E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-r)^{\alpha}\right)\right|dr+C\mu_1\tau\frac{1}{1+\lambda_k^{\beta}(t-s)^{\alpha}}(t-s)^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\gamma_1\tau\frac{1}{1+\lambda_k^{\beta}(t-s)^{\alpha}}(t-s)^{\alpha-1}\\
&\le C\tau(t-s)^{\alpha-2}+C\tau(t-s)^{\alpha-1}. \qquad (3.6)
\end{aligned}
$$

In view of $(\mathbf{A_1})$,(A.5), and (A.14), we obtain

$$
\begin{aligned}
\mathbb{E}\|\mathscr{A}_3\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} &\le C\mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\Big((t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\varsigma_l(s)\\
&\qquad -(t-t_i)^{\alpha-1}e^{-\nu(t-t_i)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\varsigma_l(t_i)\Big)^2\\
&\quad\times\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(g(t_i,u_n(t_i))\cdot e_l,\varphi_k\big)^2ds\\
&\quad + C\mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right|^2\left|\varsigma_l(s)\right|^2\\
&\quad\times\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(\big(g(s,u_n(s))-g(t_i,u_n(t_i))\big)\cdot e_l,\varphi_k\big)^2ds\\
&\le C\mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\tau^2\left((t-s)^{2\alpha-4}+(t-s)^{2\alpha-2}\right)\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\\
&\quad\times\big(g(t_i,u_n(t_i))\cdot e_l,\varphi_k\big)^2ds\\
&\quad + C\mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}\frac{1}{(1+\lambda_k^{\beta}(t-s)^{\alpha})^2}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\\
&\quad\times\big(\big(g(s,u_n(s))-g(t_i,u_n(t_i))\big)\cdot e_l,\varphi_k\big)^2ds\\
&\le C\tau^2\mathbb{E}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\big((t-s)^{2\alpha-4}+(t-s)^{2\alpha-2}\big)\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}g\big(t_i,u_n(t_i)\big)\right\|^2_{\mathcal{L}_2^0}ds\\
&\quad + C\mathbb{E}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}\\
&\quad\times\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}\big(g(s,u_n(s))-g(t_i,u_n(t_i))\big)\right\|^2_{\mathcal{L}_2^0}ds\le C\tau^2. \qquad (3.7)
\end{aligned}
$$

For $\mathscr{A}_4$, in a similar way as above, we get from (A.5) and (A.14) that

$$
\begin{aligned}
\mathbb{E}\|\mathscr{A}_4\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} &\le \mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-t_i)^{2\alpha-2}e^{-2\nu(t-t_i)}\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\right|^2\\
&\quad\times\left|\varsigma_l(t_i)-\varsigma_l^n(t_i)\right|^2\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(g\big(t_i,u_n(t_i)\big)\cdot e_l,\varphi_k\big)^2ds\\
&\le C\mathbb{E}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-t_i)^{2\alpha-2}e^{-2\nu(t-t_i)}\sum_{l=1}^{\infty}(\eta_l^n)^2
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\| (-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}} g\big(t_i, u_n(t_i)\big) \right\|_{\mathcal{L}_2^0}^2 ds \\
&\le C \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} \sum_{l=1}^{\infty} (\eta_l^n)^2 \left(1 + \mathbb{E}\|u_n(t_i)\|_{\mathbb{H}^{2\widetilde{\gamma}}}^2\right) ds \le C \sum_{l=1}^{\infty} (\eta_l^n)^2 . \qquad (3.8)
\end{aligned}
$$

For $\mathscr{A}_5$, similar to the arguments in (3.6) and (3.7), we find that

$$
\begin{aligned}
\mathbb{E}\|\mathscr{A}_5\|_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}^2 = \mathbb{E} \sum_{k=1}^{\infty} \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}} \Bigg| & \sum_{i=1}^{N_t} \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \\
& \times \sum_{l=1}^{\infty} \big(g(t_i, u_n(t_i)) \cdot e_l, \varphi_k\big) \varsigma_l^n(t_i) ds d\xi_l(r) \\
& - \sum_{i=1}^{N_t} \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (t-s)^{\alpha-1} e^{-\nu(t-s)} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \\
& \times \sum_{l=1}^{\infty} \big(g(s, u_n(s)) \cdot e_l, \varphi_k\big) \varsigma_l^n(s) ds d\xi_l(r) \Bigg|^2 \\
\le C\mathbb{E} \sum_{k,l=1}^{\infty} \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} & \Bigg((t-t_i)^{\alpha-1} e^{-\nu(t-t_i)} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right) \varsigma_l^n(t_i) \\
& - (t-s)^{\alpha-1} e^{-\nu(t-s)} E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \varsigma_l^n(s) \Bigg)^2 \\
\times \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}} & \big(g(t_i, u_n(t_i)) \cdot e_l, \varphi_k\big)^2 ds \\
+ C\mathbb{E} \sum_{k,l=1}^{\infty} \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} & (t-s)^{2\alpha-2} e^{-2\nu(t-s)} \left| E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right|^2 \left|\varsigma_l^n(s)\right|^2 \\
\times \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}} & \Big(\big(g(t_i, u_n(t_i)) - g(s, u_n(s))\big) \cdot e_l, \varphi_k\Big)^2 ds \\
\le C\tau^2 \mathbb{E} \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} & \left((t-s)^{2\alpha-4} + (t-s)^{2\alpha-2}\right) \\
\times \left\| (-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}} g\big(t_i, u_n(t_i)\big) \right\|_{\mathcal{L}_2^0}^2 & ds + C\mathbb{E} \sum_{i=1}^{N_t} \int_{t_i}^{t_{i+1}} (t-s)^{2\alpha-2} \\
\times \left\| (-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}} \big(g(t_i, u_n(t_i)) - g(s, u_n(s))\big) \right\|_{\mathcal{L}_2^0}^2 & ds \le C\tau^2. \qquad (3.9)
\end{aligned}
$$

By $(\mathbf{A_1})$, (A.5), (A.14), and the similar argument as in (3.6), we obtain

$$
\mathbb{E}\|\mathscr{A}_6\|_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}^2 = \mathbb{E} \sum_{k=1}^{\infty} \lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}} \Bigg| \int_0^t \sum_{i=1}^{N_t} \chi_i(s) \Big((t-s)^{\alpha-1} e^{-\nu(t-s)}
$$

$$\times E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\sum_{l=1}^{\infty}\big(h(s)\cdot e_l,\varphi_k\big)\varrho_l(s)-(t-t_i)^{\alpha-1}e^{-\nu(t-t_i)}$$
$$\times E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\sum_{l=1}^{\infty}\big(h(t_i)\cdot e_l,\varphi_k\big)\varrho_l(t_i)\Big)d\xi_l^H(s)\Big|^2$$
$$\le CHT^{2H-1}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\int_0^t\Big|\sum_{i=1}^{N_t}\chi_i(s)\Big((t-s)^{\alpha-1}e^{-\nu(t-s)}$$
$$\times E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\big(h(s)\cdot e_l,\varphi_k\big)\varrho_l(s)-(t-t_i)^{\alpha-1}e^{-\nu(t-t_i)}$$
$$\times E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\big(h(t_i)\cdot e_l,\varphi_k\big)\varrho_l(t_i)\Big)\Big|^2ds$$
$$\le CT^{2H-1}\tau^2\int_0^t(t-s)^{2\alpha-4}\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}h(t_i)\right\|^2_{\mathcal{L}_2^0}ds$$
$$+CT^{2H-1}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}$$
$$\times\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}\big(h(t_i)-h(s)\big)\right\|^2_{\mathcal{L}_2^0}ds\le C\tau^2. \tag{3.10}$$

For $\mathscr{A}_7$ and $\mathscr{A}_8$, arguing as in (3.10), we have

$$\mathbb{E}\|\mathscr{A}_7\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\le CHT^{2H-1}\mathbb{E}\sum_{k,l=1}^{\infty}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-t_i)^{2\alpha-2}e^{-2\nu(t-t_i)}$$
$$\times\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-t_i)^{\alpha}\right)\right|^2\left|\varrho_l(t_i)-\varrho_l^n(t_i)\right|^2\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(h(t_i)\cdot e_l,\varphi_k\big)^2ds$$
$$\le CT^{2H-1}\mathbb{E}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-t_i)^{2\alpha-2}e^{-2\nu(t-t_i)}$$
$$\times\sum_{l=1}^{\infty}(\widetilde{\eta}_l^n)^2\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}h(t_i)\right\|^2_{\mathcal{L}_2^0}ds\le C\sum_{l=1}^{\infty}(\widetilde{\eta}_l^n)^2, \tag{3.11}$$

$$\mathbb{E}\|\mathscr{A}_8\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\le C\tau^2\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\left((t-s)^{2\alpha-4}+(t-s)^{2\alpha-2}\right)\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}h(t_i)\right\|^2_{\mathcal{L}_2^0}ds$$
$$+CT^{2H-1}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}$$
$$\times\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}(h(t_i)-h(s))\right\|^2_{\mathcal{L}_2^0}ds\le C\tau^2. \tag{3.12}$$

Collecting the above estimates, we find that

$$e^{2\nu t}\mathbb{E}\|u(t)-u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\le Ce^{2\nu t}\tau^2+Ce^{2\nu t}\sum_{l=1}^{\infty}(\eta_l^n)^2+Ce^{2\nu t}\sum_{l=1}^{\infty}(\widetilde{\eta}_l^n)^2$$

$$+ C\int_0^t (t-s)^{2\alpha-2}e^{2\nu s}\mathbb{E}\|u(s)-u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}ds.$$

Thus the Grönwall's lemma for singular kernels [16] leads to

$$\mathbb{E}\|u(t)-u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \le C\tau^2 + C\sum_{l=1}^{\infty}(\eta_l^n)^2 + C\sum_{l=1}^{\infty}(\widetilde{\eta}_l^n)^2, \quad t>0. \tag{3.13}$$

The proof is complete. □

4. Galerkin finite element approximation

In this section, we provide a Galerkin FEM scheme and derive the corresponding error estimates.

4.1. Spatially Galerkin FEM and its properties

Let $\mathcal{T}_{\bar{h}}$ be a shape regular and quasi-uniform triangulation of the convex polygonal domain $\mathcal{D}$. Let $\mathcal{S}_{\bar{h}} \subset \mathbb{H}^\beta(\mathcal{D})$ be the space of piecewise polynomial functions with respect to $\tau_{\bar{h}}$, which are zero on the boundary of $\mathcal{D}$.

On the space $\mathcal{S}_{\bar{h}}$ we define the orthogonal L^2-projection $P_{\bar{h}}:\mathbb{H}^0(\mathcal{D})\to\mathcal{S}_{\bar{h}}$ and the generalized Ritz projection $R_{\bar{h}}:\mathbb{H}^\beta(\mathcal{D})\to\mathcal{S}_{\bar{h}}$, respectively, by

$$\begin{aligned} &(P_{\bar{h}}\psi,\chi) = (\psi,\chi), && \forall\chi\in\mathcal{S}_{\bar{h}},\\ &\left((-\Delta)^{\frac{\beta}{2}}R_{\bar{h}}\psi,(-\Delta)^{\frac{\beta}{2}}\chi\right) = \left((-\Delta)^{\frac{\beta}{2}}\psi,(-\Delta)^{\frac{\beta}{2}}\chi\right), && \forall\chi\in\mathcal{S}_{\bar{h}}. \end{aligned}$$

The projection $R_{\bar{h}}$ of ψ is unique, since $\psi\in\mathbb{H}^\beta(\mathcal{D})$ and it equals to zero on the boundary.

In the next lemma, we establish the error estimates for $P_{\bar{h}}\psi$ and $R_{\bar{h}}\psi$, see [21] for details.

Lemma 4.1. *The operators $P_{\bar{h}}$ and $R_{\bar{h}}$ satisfy*

$$\|P_{\bar{h}}\psi-\psi\| + \bar{h}^\beta\left\|(-\Delta)^{\frac{\beta}{2}}(P_{\bar{h}}\psi-\psi)\right\| \le C\bar{h}^q\|\psi\|_{\mathbb{H}^q} \quad \textit{for} \quad \psi\in\mathbb{H}^q, \quad q\in[\beta,r], \tag{4.1}$$

$$\|R_{\bar{h}}\psi-\psi\| + \bar{h}^\beta\left\|(-\Delta)^{\frac{\beta}{2}}(R_{\bar{h}}\psi-\psi)\right\| \le C\bar{h}^q\|\psi\|_{\mathbb{H}^q} \quad \textit{for} \quad \psi\in\mathbb{H}^q, \quad q\in[\beta,r]. \tag{4.2}$$

Remark 4.1. The number r refers to the order of accuracy of the family $\{S_{\bar{h}}\}$. In the case $r=2$, $S_{\bar{h}}$ is a piecewise linear finite element subspace. For the case $r>2$, $S_{\bar{h}}$ often consists of piecewise polynomials of degree at most $r-1$ on a triangulation $\tau_{\bar{h}}$. For instance, $r=4$ in the case of piecewise cubic polynomial subspaces.

The discrete fractional Laplacian $(-\Delta_{\bar{h}})^\beta : \mathcal{S}_{\bar{h}} \to \mathcal{S}_{\bar{h}}$ is then defined by

$$\Big((-\Delta_{\bar{h}})^\beta \psi, \chi\Big) = \Big((-\Delta)^{\frac{\beta}{2}}\psi, (-\Delta)^{\frac{\beta}{2}}\chi\Big), \quad \forall \psi, \chi \in \mathcal{S}_{\bar{h}}, \tag{4.3}$$

and thus we can write the spatial FEM approximation of (3.1) as

$$\begin{aligned} &{}_0^c\partial_t^{\alpha,\nu} u_n^{\bar{h}}(t,x) + (-\Delta_{\bar{h}})^\beta u_n^{\bar{h}}(t,x) \\ &= P_{\bar{h}}\left(f\left(t, u_n^{\bar{h}}(t,x)\right) + g\left(t, u_n^{\bar{h}}(t,x)\right)\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t \partial x} + h(t)\frac{\partial^2 \mathbb{W}_n^H(t,x)}{\partial t\partial x}\right), \quad 0 < t \le T \end{aligned} \tag{4.4}$$

with $u_n^{\bar{h}}(0) = a_{\bar{h}}$ and $\partial_t u_n^{\bar{h}}(0) = b_{\bar{h}}$, where $a_{\bar{h}} = P_{\bar{h}}a$, $b_{\bar{h}} = P_{\bar{h}}b$.

Now we give a representation of the solution of (4.4) using the eigenvalues and eigenfunctions $\{\lambda_k^{\bar{h},\beta}\}_{k=1}^M$ and $\{\varphi_k^{\bar{h}}\}_{k=1}^M$ of the discrete fractional Laplacian $(-\Delta_{\bar{h}})^\beta$. Since we know that the operator $(-\Delta_{\bar{h}})^\beta$ is symmetrical, $\{\varphi_k^{\bar{h}}\}_{k=1}^M$ is orthogonal. Take $\{\varphi_k^{\bar{h}}\}_{k=1}^M$ as the orthonormal bases in $\mathcal{S}_{\bar{h}}$ and define the discrete analogues of (2.13)-(2.15) by

$$\widehat{\mathcal{T}}_{\alpha,\beta}^\nu(t,x,y) = e^{-\nu t}\sum_{k=1}^M \left(E_{\alpha,1}\left(-\lambda_k^{\bar{h},\beta}t^\alpha\right) + \nu t E_{\alpha,2}\left(-\lambda_k^{\bar{h},\beta}t^\alpha\right)\right)\varphi_k^{\bar{h}}(x)\varphi_k^{\bar{h}}(y), \tag{4.5}$$

$$\widehat{\mathcal{R}}_{\alpha,\beta}^\nu(t,x,y) = te^{-\nu t}\sum_{k=1}^M E_{\alpha,2}\left(-\lambda_k^{\bar{h},\beta}t^\alpha\right)\varphi_k^{\bar{h}}(x)\varphi_k^{\bar{h}}(y), \tag{4.6}$$

$$\widehat{\mathcal{S}}_{\alpha,\beta}^\nu(t,x,y) = t^{\alpha-1}e^{-\nu t}\sum_{k=1}^M E_{\alpha,\alpha}\left(-\lambda_k^{\bar{h},\beta}t^\alpha\right)\varphi_k^{\bar{h}}(x)\varphi_k^{\bar{h}}(y). \tag{4.7}$$

Then the solution $u_n^{\bar{h}}$ of the discrete problem (4.4) can be expressed by

$$\begin{aligned} u_n^{\bar{h}}(t,x) = &\int_{\mathcal{D}} \widehat{\mathcal{T}}_{\alpha,\beta}^\nu(t,x,y)a_{\bar{h}}(y)dy + \int_{\mathcal{D}} \widehat{\mathcal{R}}_{\alpha,\beta}^\nu(t,x,y)b_{\bar{h}}(y)dy \\ &+ \int_0^t\int_{\mathcal{D}} \widehat{\mathcal{S}}_{\alpha,\beta}^\nu(t-s,x,y)P_{\bar{h}}f\left(s, u_n^{\bar{h}}(s,y)\right)dyds \\ &+ \int_0^t\int_{\mathcal{D}} \widehat{\mathcal{S}}_{\alpha,\beta}^\nu(t-s,x,y)P_{\bar{h}}h(s)d\mathbb{W}^H(s,y) \\ &+ \int_0^t\int_{\mathcal{D}} \widehat{\mathcal{S}}_{\alpha,\beta}^\nu(t-s,x,y)P_{\bar{h}}g\left(s, u_n^{\bar{h}}(s,y)\right)d\mathbb{W}(s,y). \end{aligned} \tag{4.8}$$

Also, on the finite element space $\mathcal{S}_{\bar{h}}$, we introduce the discrete norm $|||\cdot|||_p$ for any $p \in \mathbb{R}$ defined by

$$|||\psi|||_p^2 = \sum_{k=1}^M \left(\lambda_k^{\bar{h},\beta}\right)^{\frac{p}{\beta}}\left(\psi, \varphi_k^{\bar{h}}\right)^2, \quad \psi \in \mathcal{S}_{\bar{h}}. \tag{4.9}$$

It is clear that the norm $|||\cdot|||_p$ is well defined for all real p. From the definition of the discrete fractional Laplacian$(-\Delta_{\bar{h}})^\beta$ we have $|||\psi|||_p = \|\psi\|_{\mathbb{H}^p}$ for $p = 0, \beta$ and for all

$\psi \in \mathcal{S}_{\bar{h}}$. Therefore there is no confusion in using $\|\psi\|_{\mathbb{H}^p}$ instead of $\|\!|\psi|\!\|_p$ for $p = 0, \beta$ and for all $\psi \in \mathcal{S}_{\bar{h}}$. Further, we need the following inverse inequality, see [21, Lemma 3.2] for details.

Lemma 4.2. *For any $l > s$, there exists a constant C independent of $\bar{h}$ such that*

$$\|\!|\chi|\!\|_l \le C\bar{h}^{s-l}\|\!|\chi|\!\|_s, \quad \forall \chi \in \mathcal{S}_{\bar{h}}. \tag{4.10}$$

As the discrete analogues of Lemma 2.4, we have the following estimates.

Lemma 4.3. *Let $\widehat{\mathcal{T}}^{\nu}_{\alpha,\beta}(t,x,y)$ be defined by* (4.5) *and $a_{\bar{h}} \in \mathcal{S}_{\bar{h}}$. Then, for all $t > 0$,*

$$\left|\!\left|\!\left| \int_{\mathcal{D}} \widehat{\mathcal{T}}^{\nu}_{\alpha,\beta}(t,x,y) a_{\bar{h}}(y) dy \right|\!\right|\!\right|_p \le \begin{cases} C(1+\nu t)e^{-\nu t} t^{\frac{\alpha(q-p)}{2\beta}} \|\!|a_{\bar{h}}|\!\|_q, & 0 \le q \le p \le 2\beta, \\ C(1+\nu t)e^{-\nu t} t^{-\alpha} \|\!|a_{\bar{h}}|\!\|_q, & q > p. \end{cases}$$

Lemma 4.4. *Let $\widehat{\mathcal{R}}^{\nu}_{\alpha,\beta}(t,x,y)$ be defined by* (4.6) *and $b_{\bar{h}} \in \mathcal{S}_{\bar{h}}$. Then, for all $t > 0$,*

$$\left|\!\left|\!\left| \int_{\mathcal{D}} \widehat{\mathcal{R}}^{\nu}_{\alpha,\beta}(t,x,y) b_{\bar{h}}(y) dy \right|\!\right|\!\right|_p \le \begin{cases} Ce^{-\nu t} t^{1-\frac{\alpha(p-q)}{2\beta}} \|\!|b_{\bar{h}}|\!\|_q, & 0 \le q \le p \le 2\beta, \\ Ce^{-\nu t} t^{1-\alpha} \|\!|b_{\bar{h}}|\!\|_q, & q > p. \end{cases}$$

By slightly modifying the proof of [21, Lemma 3.5], we have

Lemma 4.5. *Let $\widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t,x,y)$ be defined by* (4.7) *and $\psi \in \mathcal{S}_{\bar{h}}$. Then, for all $t > 0$,*

$$\left|\!\left|\!\left| \int_{\mathcal{D}} \widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t,x,y) \psi(y) dy \right|\!\right|\!\right|_p \le \begin{cases} Ce^{-\nu t} t^{-1+\alpha+\frac{\alpha(q-p)}{2\beta}} \|\!|\psi|\!\|_q, & p - 2\beta \le q \le p, \\ Ce^{-\nu t} t^{-1} \|\!|\psi|\!\|_q, & q > p. \end{cases}$$

4.2. Mean-square convergence analysis

In this subsection, we derive an error estimate for the problem (3.1). First, we need the following Lipschitz assumption:

($\mathbf{A_4}$) There exists a positive constant l' such that for any $\gamma \ge 0$,

$$\|f(t,u_1) - f(t,u_2)\| + \|g(t,u_1) - g(t,u_2)\|_{\mathcal{L}_2^0} \le l'\|u_1 - u_2\|$$

for all $t \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{H}$.

Theorem 4.1. *Assume that the functions f, g, and h satisfy ($\boldsymbol{A}_1$) for some $\gamma \ge 0$, ($\boldsymbol{A}_3$)-($\boldsymbol{A}_4$) hold, $3/2 < \alpha < 2$, $1/2 < \beta \le 1$, $\nu > 0$, $1/2 < H < 1$, and that the $\mathcal{F}_0$-adapted random initial values satisfy $a \in L^2(\Omega; \mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))$, $b \in L^2(\Omega; \mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}(\mathcal{D}))$, with $\widetilde{\gamma} = \max(\gamma, \beta/\alpha)$. Let u_n and u_n^h be the solutions of* (3.1) *and* (4.4)*, respectively. Then, with $\ell_{\bar{h}} = |\ln \bar{h}|$,*

$$\begin{aligned} &\mathbb{E}\left\|u_n(t) - u_n^{\bar{h}}(t)\right\|^2 + \bar{h}^{2\beta}\mathbb{E}\left\|(-\Delta)^{\frac{\beta}{2}}\left(u_n^{\bar{h}}(t) - u_n(t)\right)\right\|^2 \\ &\le C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}} + C\bar{h}^{4\widetilde{\gamma}}, \quad \forall t \in [0,T], \end{aligned}$$

where C is a positive constant independent of τ and $\bar{h}$.

Proof. We represent the error $u_n^{\bar{h}} - u_n$ as

$$u_n^{\bar{h}} - u_n = \left(u_n^{\bar{h}} - P_{\bar{h}} u_n\right) + (P_{\bar{h}} u_n - u_n) := \Pi_1 + \Pi_2.$$

By (A.5) and (4.1) we have

$$\mathbb{E}\|\Pi_2(t)\|^2 + \bar{h}^{2\beta}\mathbb{E}\left\|(-\Delta)^{\frac{\beta}{2}}\Pi_2(t)\right\|^2 \le C\bar{h}^{4\beta}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\beta}}. \tag{4.11}$$

Moreover, we consider the equation

$$\begin{aligned}
&{}_0^c\partial_t^{\alpha,\nu} v(t,x) + (-\Delta_{\bar{h}})^\beta v(t,x) \\
&= P_{\bar{h}}\left(f\big(t,u_n(t,x)\big) + g\big(t,u_n(t,x)\big)\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t\partial x} + h(t)\frac{\partial^2 \mathbb{W}_n^H(t,x)}{\partial t \partial x}\right)
\end{aligned} \tag{4.12}$$

with $0 < t \le T$, $v(0) = a_{\bar{h}} = P_{\bar{h}} a$, and $\partial_t v_n^{\bar{h}}(0) = b_{\bar{h}}(x) = P_{\bar{h}} b$. Let $\Pi_1^1 = v - P_{\bar{h}} u_n$, $\Pi_1^2 = u_n^{\bar{h}} - v$. Then $\Pi_1 = \Pi_1^1 + \Pi_1^2$. It follows from (3.1) and (4.12) that

$${}_0^c\partial_t^{\alpha,\nu}\Pi_1^1 + (-\Delta_{\bar{h}})^\beta \Pi_1^1 = (-\Delta_{\bar{h}})^\beta (R_{\bar{h}} u_n - P_{\bar{h}} u_n) \quad \text{with} \quad \Pi_1^1(0) = \partial_t \Pi_1^1(0) = 0,$$

where we have used the identity $(-\Delta_{\bar{h}})^\beta R_{\bar{h}} = P_{\bar{h}}(-\Delta)^\beta$. By (4.8), we obtain that

$$\Pi_1^1(t,x) = \int_0^t \int_{\mathcal{D}} \widehat{\mathcal{S}}^\nu_{\alpha,\beta}(t-s,x,y)(-\Delta_{\bar{h}})^\beta \big(R_{\bar{h}} u_n(s,y) - P_{\bar{h}} u_n(s,y)\big)dyds.$$

For any $0 < \varepsilon < 2\beta$, by Hölder's inequality, and using Lemmas 4.1, 4.2, and 4.5, we deduce that, for $p = 0, \beta$,

$$\begin{aligned}
\mathbb{E}\left\|\Pi_1^1(t)\right\|^2_{\mathbb{H}^p} &\le C\mathbb{E}\left(\int_0^t (t-s)^{\frac{\alpha\varepsilon}{2\beta}-1} e^{-\nu(t-s)} \left\|(-\Delta_{\bar{h}})^\beta (R_{\bar{h}} u_n - P_{\bar{h}} u_n)(s)\right\|_{\varepsilon-2\beta+p} ds\right)^2 \\
&\le C\bar{h}^{4\widetilde{\gamma}-2p-2\varepsilon}\mathbb{E}\left(\int_0^t (t-s)^{\frac{\alpha\varepsilon}{2\beta}-1} e^{-\nu(t-s)}\|u_n(s)\|_{\mathbb{H}^{2\widetilde{\gamma}}} ds\right)^2 \\
&\le C\bar{h}^{4\widetilde{\gamma}-2p-2\varepsilon}\int_0^t (t-s)^{\frac{\alpha\varepsilon}{2\beta}-1} e^{-2\nu(t-s)} ds \int_0^t (t-s)^{\frac{\alpha\varepsilon}{2\beta}-1}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} ds \\
&\le C\varepsilon^{-1}\bar{h}^{4\widetilde{\gamma}-2p-2\varepsilon} \le C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}-2p}, \quad \forall t \in [0,T].
\end{aligned} \tag{4.13}$$

The last two inequalities follow from the fact $\Gamma(\alpha\varepsilon/(2\beta)) \sim 2\beta/(\alpha\varepsilon)$ as $\varepsilon \to 0^+$ and by choosing $\varepsilon = 1/\ell_{\bar{h}}$. Noticing that Π_1^2 satisfies the following equation:

$$\begin{aligned}
&{}_0^c\partial_t^{\alpha,\nu}\Pi_1^2(t,x) + (-\Delta_{\bar{h}})^\beta \Pi_1^2(t,x) \\
&= P_{\bar{h}} f\left(t, u_n^{\bar{h}}(t,x)\right) - P_{\bar{h}} f\big(t,u_n(t,x)\big) + P_{\bar{h}} g\left(t, u_n^{\bar{h}}(t,x)\right)\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t\partial x} \\
&\quad - P_{\bar{h}} g\big(t,u_n(t,x)\big)\frac{\partial^2 \mathbb{W}_n(t,x)}{\partial t\partial x}
\end{aligned}$$

with $\Pi_1^2(0)=\partial_t\Pi_1^2(0)=0$, by (4.8) we have

$$\begin{aligned}\Pi_1^2(t,x)=&\int_0^t\int_{\mathcal{D}}\widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t-s,x,y)\Big(P_{\bar h}f\left(s,u_n^{\bar h}(s,y)\right)-P_{\bar h}f\big(s,u_n(s,y)\big)\Big)dyds\\&+\int_0^t\int_{\mathcal{D}}\widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t-s,x,y)\Big(P_{\bar h}g\left(s,u_n^{\bar h}(s,y)\right)d\mathbb{W}_n(s,y)\\&\qquad-P_{\bar h}g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\Big).\end{aligned}$$

For $p=0,\beta$, by using the fact that $\{\varphi_k^{\bar h}\}_{k=1}^M$ is an orthonormal basis in $S_{\bar h}$, the assumption on f given in $(\mathbf{A_4})$, Lemma 2.1, Hölder's inequality, (4.7),(4.11), and (4.13), we deduce that

$$\begin{aligned}&C\mathbb{E}\left\|\int_0^t\int_{\mathcal{D}}\widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t-s,x,y)\left(P_{\bar h}f\left(s,u_n^{\bar h}(s,y)\right)-P_{\bar h}f\big(s,u_n(s,y)\big)\right)dyds\right\|^2_{\mathbb{H}^p}\\
&=C\mathbb{E}\sum_{k=1}^M\left(\lambda_k^{\bar h,\beta}\right)^{\frac{p}{\beta}}\Bigg(\int_0^t\int_{\mathcal{D}}(t-s)^{\alpha-1}e^{-\nu(t-s)}\sum_{l=1}^M E_{\alpha,\alpha}\left(-\lambda_l^{\bar h,\beta}(t-s)^{\alpha}\right)\\
&\qquad\times\varphi_l^{\bar h}(\cdot)\varphi_l^{\bar h}(y)\left(P_{\bar h}f\left(s,u_n^{\bar h}(s,y)\right)-P_{\bar h}f\big(s,u_n(s,y)\big)\right)dyds,\varphi_k^{\bar h}\Bigg)^2\\
&\le C\mathbb{E}\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\left\|P_{\bar h}f\left(s,u_n^{\bar h}(s)\right)-P_{\bar h}f\big(s,u_n(s)\big)\right\|^2ds\\
&\le C\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\mathbb{E}\left\|u_n^{\bar h}(s)-u_n(s)\right\|^2ds\\
&\le C\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\left(\mathbb{E}\left\|\Pi_1^2(s)\right\|^2_{\mathbb{H}^p}+\mathbb{E}\left\|\Pi_1^1(s)\right\|^2_{\mathbb{H}^p}+\mathbb{E}\|\Pi_2(s)\|^2_{\mathbb{H}^p}\right)ds\\
&\le C\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\mathbb{E}\left\|\Pi_1^2(s)\right\|^2_{\mathbb{H}^p}ds+C\ell_{\bar h}\bar h^{4\widetilde{\gamma}-2p}+C\bar h^{4\widetilde{\gamma}-2p},\end{aligned}\tag{4.14}$$

where

$$\frac{\left(\lambda_k^{\bar h,\beta}(t-s)^{\alpha}\right)^{\frac{p}{\beta}}}{\left(1+\lambda_k^{\bar h,\beta}(t-s)^{\alpha}\right)^2}\le C,$$
$$\mathbb{E}\left\|u_n^{\bar h}(s)-u_n(s)\right\|^2\le C\mathbb{E}\left\|u_n^{\bar h}(s)-u_n(s)\right\|^2_{\mathbb{H}^p}.$$

Without loss of generality, we assume that there exists a positive integer N_t such that $t=t_{N_t+1}$. Since $\{\varphi_k^{\bar h}\}_{k=1}^M$ is an orthonormal basis in $S_{\bar h}$, $\{\xi_l\}_{l=1}^{\infty}$ is a family of mutually independent one-dimensional standard Brownian motions with independent increments, then by Lemma 2.1, $(\mathbf{A_4})$, Hölder's inequality, the Itô isometry, the boundedness assumption on $\varsigma_k^n(t)$,(4.7),(4.11), and (4.13), we obtain that for $p=0,\beta$,

$$C\mathbb{E}\left\|\int_0^t\int_{\mathcal{D}}\widehat{\mathcal{S}}^{\nu}_{\alpha,\beta}(t-s,x,y)\Big(P_{\bar h}g\left(s,u_n^{\bar h}(s,y)\right)d\mathbb{W}_n(s,y)-P_{\bar h}g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\Big)\right\|^2_{\mathbb{H}^p}$$

$$
\begin{aligned}
&= C\mathbb{E}\sum_{k=1}^{M}\left(\lambda_k^{\bar{h},\beta}\right)^{\frac{p}{\beta}}\Bigg|\int_0^t\int_{\mathcal{D}}(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\bar{h},\beta}(t-s)^{\alpha}\right)\varphi_k^{\bar{h}}(y)\\
&\qquad\qquad\times\sum_{j=1}^{M}\sum_{l=1}^{\infty}\left(P_{\bar{h}}\left(g\big(s,u_n^{\bar{h}}(s)\big)-g\big(s,u_n(s)\big)\right)\cdot e_l,\varphi_j^{\bar{h}}\right)\varphi_j^{\bar{h}}(y)\varsigma_l^n(s)\\
&\qquad\qquad\times\left(\sum_{i=1}^{N_t}\frac{1}{\sqrt{\tau}}\xi_{li}\chi_i(s)\right)dyds\Bigg|^2\\
&= \frac{C}{\tau^2}\mathbb{E}\sum_{k=1}^{M}\left(\lambda_k^{\bar{h},\beta}\right)^{\frac{p}{\beta}}\Bigg|\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\bar{h},\beta}(t-s)^{\alpha}\right)\\
&\qquad\qquad\times\sum_{l=1}^{\infty}\left(P_{\bar{h}}\left(g\left(s,u_n^{\bar{h}}(s)\right)-g\big(s,u_n(s)\big)\right)\cdot e_l,\varphi_k^{\bar{h}}\right)\varsigma_l^n(s)dsd\xi_l(r)\Bigg|^2\\
&\le \frac{C}{\tau}\mathbb{E}\sum_{k=1}^{M}\sum_{l=1}^{\infty}\left(\lambda_k^{\bar{h},\beta}\right)^{\frac{p}{\beta}}\sum_{i=1}^{N_t}\Bigg|\int_{t_i}^{t_{i+1}}(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}(-\lambda_k^{\bar{h},\beta}(t-s)^{\alpha})\\
&\qquad\qquad\times(P_{\bar{h}}(g(s,u_n^{\bar{h}}(s))-g(s,u_n(s)))\cdot e_l,\varphi_k^{\bar{h}})\varsigma_l^n(s)ds\Bigg|^2\\
&\le C\left(\mu_1^n\right)^2\mathbb{E}\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\left\|P_{\bar{h}}g\left(s,u_n^{\bar{h}}(s)\right)-P_{\bar{h}}g\big(s,u_n(s)\big)\right\|_{\mathcal{L}_2^0}^2ds\\
&\le C\left(\mu_1^n\right)^2\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\mathbb{E}\left\|\Pi_1^2(s)\right\|_{\mathbb{H}^p}^2ds\\
&\quad+C\left(\mu_1^n\right)^2\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}-2p}+C\left(\mu_1^n\right)^2\bar{h}^{4\widetilde{\gamma}-2p}.
\end{aligned}
\tag{4.15}
$$

Combining (4.14)-(4.15) together, we obtain for $p=0,\beta$ that

$$
\begin{aligned}
\mathbb{E}\left\|\Pi_1^2(t)\right\|_{\mathbb{H}^p}^2\le{}& C\int_0^t(t-s)^{2\alpha-2-\frac{\alpha p}{\beta}}e^{-2\nu(t-s)}\mathbb{E}\left\|\Pi_1^2(s)\right\|_{\mathbb{H}^p}^2ds\\
&+C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}-2p}+C\bar{h}^{4\widetilde{\gamma}-2p}.
\end{aligned}
$$

The Grönwall-Bellman inequality [2] conduces us to

$$
\mathbb{E}\left\|\Pi_1^2(t)\right\|^2\le C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}}+C\bar{h}^{4\widetilde{\gamma}},\quad\forall t\in[0,T],
$$

and by the Grönwall's lemma for singular kernels [16], we have

$$
\mathbb{E}\left\|\Pi_1^2(t)\right\|_{\mathbb{H}^\beta}^2\le C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}-2\beta}+C\bar{h}^{4\widetilde{\gamma}-2\beta},\quad t\in[0,T].
$$

Therefore,

$$
\mathbb{E}\left\|\Pi_1^2(t)\right\|_{\mathbb{H}^p}^2\le C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}-2p}+C\bar{h}^{4\widetilde{\gamma}-2p},\quad t\in[0,T],
$$

and consequently, the desired assertion follows by the triangle inequality. □

Furthermore, thanks to Theorems 3.2 and 4.1, a space-time mean-square error estimate for problem (1.1) follows from the triangle inequality.

Theorem 4.2. *Suppose that the assumptions of Theorem* 4.1 *hold. Let* u *be the solution of* (1.1) *with* $a \in L^2(\Omega; \mathbb{H}^{2\widetilde{\gamma}}(\mathcal{D}))$, $b \in L^2(\Omega; \mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}(\mathcal{D}))$, *and let* $u_n^{\bar{h}}$ *be the solution of* (4.4) *with* $a_{\bar{h}} = P_{\bar{h}} a, b_h = P_{\bar{h}} b$. *Then, with* $\ell_{\bar{h}} = |\ln \bar{h}|$,

$$\mathbb{E}\|u(t) - u_n^{\bar{h}}(t)\|^2 \le C\tau^2 + C\sum_{l=1}^{\infty}(\eta_l^n)^2 + C\sum_{l=1}^{\infty}(\widetilde{\eta}_l^n)^2 + C\ell_{\bar{h}}\bar{h}^{4\widetilde{\gamma}} + C\bar{h}^{4\widetilde{\gamma}}, \quad \forall t \in [0,T],$$

provided that the infinite series are convergent, where C *is a positive constant independent of* τ *and* $\bar{h}$.

5. Numerical experiments

In this section, we present two examples to show the temporal and spatial convergence rates. Since the exact solution is unknown, the temporal and spatial convergence rates can be tested by calculating

$$E_\tau = \left(\mathbb{E}\left\|u_\tau^n - u_{\frac{\tau}{2}}^n\right\|^2\right)^{\frac{1}{2}}, \quad E_h = \left(\mathbb{E}\left\|u_h^n - u_{\frac{h}{2}}^n\right\|^2\right)^{\frac{1}{2}},$$

where u_τ^n and u_h^n mean the numerical solution of u at time t_n with time step size τ and the one with mesh size h, respectively. The corresponding convergence rates can be presented by

$$\text{Rate} = \frac{\ln(E_h/E_{h/2})}{\ln(2)}, \quad \text{Rate} = \frac{\ln(E_\tau/E_{\tau/2})}{\ln(2)}.$$

Example 5.1. In this example, we take

$$\begin{aligned} &a(x) = x(1-x), && b(x) = \sin(\pi x), \\ &f(t,u) = \sin(u)t, && g(t,u) = ut, \\ &\varsigma_k(t) = \varrho_k(t) = k^{-2}, && h(t) = 1, \quad \mathcal{D} = (0,1) \end{aligned}$$

to validate the temporal convergence. We choose $\nu = 1$, $T = 0.05$, $h = 1/128$, 100 samples; and the numerical results are shown in Table 1, which validate Theorem 3.2.

Example 5.2. In this example, we verify the spatial convergence rates. Let

$$\begin{aligned} &a_1(x) = x(1-x), && a_2(x) = |x - 1/2|\chi_{(0,1)}, \\ &b(x) = \sin(\pi x), && f(t,u) = \sin(u)t, \\ &g(t,u) = ut, && h(t) = 1, \\ &\varsigma_k(t) = \varrho_k(t) = k^{-2}, && \mathcal{D} = (0,1), \end{aligned}$$

where $\chi_{(a,b)}$ means characteristic function on (a,b).

Table 1: Temporal errors and convergence rates.

N			16	32	64	128
$\alpha = 1.6$	$\beta = 0.75$	$H = 0.6$	3.2659e-04	1.4578e-04	7.7576e-05	3.7069e-05
			Rate	1.1637	0.9101	1.0654
$\alpha = 1.6$	$\beta = 0.75$	$H = 0.8$	6.0097e-05	2.5470e-05	1.2400e-05	4.1909e-06
			Rate	1.2385	1.0385	1.5650
$\alpha = 1.6$	$\beta = 1$	$H = 0.6$	3.2436e-04	1.4489e-04	7.7088e-05	3.6833e-05
			Rate	1.1626	0.9104	1.0655
$\alpha = 1.8$	$\beta = 0.75$	$H = 0.6$	1.4896e-04	6.4596e-05	3.4498e-05	1.4122e-05
			Rate	1.2054	0.9049	1.2886
$\alpha = 1.8$	$\beta = 0.75$	$H = 0.8$	2.3444e-05	1.1450e-05	4.9354e-06	1.9361e-06
			Rate	1.0339	1.2141	1.3500
$\alpha = 1.8$	$\beta = 1$	$H = 0.6$	1.2385e-04	6.7051e-05	3.2700e-05	1.4993e-05
			Rate	0.8853	1.0360	1.1250

In Table 2, we take $\nu = 1, T = 0.05, \tau = T/128, H = 0.6, 0.8$, and 100 as the number of simulation trajectories. When $a(x) = a_1(x)$ and $\alpha = 1.6, 1.8$, $\beta = 0.9, 0.7$, by the definition of $\tilde{\gamma}$, we have $\tilde{\gamma} = \gamma = 1 > \beta/\alpha$ and the convergence rates in space should be $\mathcal{O}(h^2)$. These results in Table 2 agree with Theorem 4.1.

In Table 3, we take $\nu = 1, T = 0.1, \tau = T/256, H = 0.8$, and 100 as the number of simulation trajectories. When $a(x) = a_2(x)$ and $\alpha = 1.8$, $\beta = 0.6, 0.9$, we have $a(x) \in \mathbb{H}^1(\Omega)$ with $\epsilon > 0$ arbitrarily small, $\tilde{\gamma} = \max\{\gamma, \beta/\alpha\} = 1/2$ and the convergence rates in space should be $\mathcal{O}(h^1)$. These results in Table 3 agree with Theorem 4.1.

Table 2: Spatial errors and convergence rates.

$1/h$			64	128	256
$\alpha = 1.6$	$\beta = 0.9$	$H = 0.6$	6.4434e-05	1.6117e-05	4.0296e-06
			Rate	1.9992	1.9999
$\alpha = 1.6$	$\beta = 0.7$	$H = 0.6$	6.6595e-05	1.6663e-05	4.1668e-06
			Rate	1.9988	1.9996
$\alpha = 1.6$	$\beta = 0.7$	$H = 0.8$	6.5916e-05	1.6493e-05	4.1243e-06
			Rate	1.9988	1.9996
$\alpha = 1.8$	$\beta = 0.7$	$H = 0.8$	6.7634e-05	1.6932e-05	4.2352e-06
			Rate	1.9980	1.9993

Table 3: Spatial errors and convergence rates.

$1/h$			128	256	512	
$\alpha = 1.8$	$\beta = 0.6$	$H = 0.8$	1.7747e-02	9.9015e-03	5.1483e-03	
			Rate	0.8419	0.9569	0.9436
$\alpha = 1.8$	$\beta = 0.9$	$H = 0.8$	8.5858e-03	4.3190e-03	2.1626e-03	
			Rate	0.9913	0.9979	

6. Conclusions

This paper first introduces the semilinear stochastic space-time fractional wave equations with external infinite dimensional multiplicative Gaussian noise and fractional Gaussian noise. It is modeling the wave propagation with frequency-dependent power-law attenuation under the influences of the fluctuations of the external noises, the striking features of which are multiscale and usually happening in inhomogeneous media. Then we establish the theory of finite element approximations. Finally, numerical experiments are performed to confirm the theoretical analyses.

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Appendix A. Proof of Theorem 3.1.

We divide the proof into four steps:

Step 1. Estimate of $\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}$. By Lemma 2.4 and Remark 2.4, from (3.2) we obtain that

$$\begin{aligned}
\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \leq\, & C\mathbb{E}\bigg\| \int_{\mathcal{D}} \mathcal{T}^{\nu}_{\alpha,\beta}(t,\cdot,y)a(y)dy\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + C\mathbb{E}\bigg\| \int_{\mathcal{D}} \mathcal{R}^{\nu}_{\alpha,\beta}(t,\cdot,y)b(y)dy\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \\
& + C\mathbb{E}\bigg\| \int_0^t \int_{\mathcal{D}} \mathcal{S}^{\nu}_{\alpha,\beta}(t-s,\cdot,y)f\big(s,u_n(s,y)\big)dyds\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \\
& + C\mathbb{E}\bigg\| \int_0^t \int_{\mathcal{D}} \mathcal{S}^{\nu}_{\alpha,\beta}(t-s,\cdot,y)g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \\
& + C\mathbb{E}\bigg\| \int_0^t \int_{\mathcal{D}} \mathcal{S}^{\nu}_{\alpha,\beta}(t-s,\cdot,y)h(s)d\mathbb{W}^H_n(s,y)\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \\
\leq\, & C\left(1+\nu^2t^2\right)e^{-2\nu t}\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + Ce^{-2\nu t}\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \\
& + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \qquad\qquad (A.1)
\end{aligned}$$

To estimate $\mathcal{J}_1$, we use the fact that $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$, the assumption on f given in $(\mathbf{A_1})$, Lemma 2.1, and Hölder's inequality, which leads to

$$\mathcal{J}_1 \leq CT\mathbb{E}\sum_{k=1}^{\infty}\int_0^t (t-s)^{2\alpha-4}e^{-2\nu(t-s)}\frac{\left(\lambda_k^{\beta}(t-s)^{\alpha}\right)^{\frac{2}{\alpha}}}{\left(1+\lambda_k^{\beta}(t-s)^{\alpha}\right)^2}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(f(s,u_n(s)),\varphi_k\big)^2ds$$

$$\leq C+C\int_0^t (t-s)^{2\alpha-4}e^{-2\nu(t-s)}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}ds, \tag{A.2}$$

where

$$\frac{\left(\lambda_k^{\beta}(t-s)^{\alpha}\right)^{\frac{2}{\alpha}}}{\left(1+\lambda_k^{\beta}(t-s)^{\alpha}\right)^2}\leq C,$$
$$\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\leq C\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}.$$

Without loss of generality, we assume that there exists a positive integer N_t such that $t=t_{N_t+1}$. Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$, the Brownian motion has independent increments and $\{\xi_l\}_{l=1}^{\infty}$ is a family of mutually independent one-dimensional standard Brownain motions, then by (2.8), $(\mathbf{A_1})$, Hölder's inequality, the Itô isometry, and the boundedness assumption of $\varsigma_k^n(t)$, we obtain

$$\begin{aligned}\mathcal{J}_2 &= \frac{C}{\tau^2}\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}}\Bigg(\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\\ &\qquad\times\sum_{l=1}^{\infty}\big(g\big(s,u_n(s)\big)\cdot e_l,\varphi_k\big)\varsigma_l^n(s)dsd\xi_l(r)\Bigg)^2\\ &\leq C\left(\mu_1^n\right)^2\mathbb{E}\int_0^t(t-s)^{2\alpha-4}e^{-2\nu(t-s)}\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}g\big(s,u_n(s)\big)\right\|^2_{\mathcal{L}_2^0}ds\\ &\leq C\left(\mu_1^n\right)^2+C\left(\mu_1^n\right)^2\int_0^t(t-s)^{2\alpha-4}e^{-2\nu(t-s)}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}ds.\end{aligned}\tag{A.3}$$

Note that $\{\xi_l^H\}_{l=1}^{\infty}$ is a family of mutally independent one-dimensional fractional Brownain motions. By slightly modifying the arguments in (A.3), in view of (2.9), $(\mathbf{A_1})$, Lemma 2.2, Hölder's inequality, and the boundedness assumption of $\varrho_k^n(t)$, we deduce that

$$\begin{aligned}\mathcal{J}_3 &\leq \frac{C}{\tau^2}t^{2H-1}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}}\int_0^t\Bigg(\int_0^t(t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\\ &\qquad\times\big(h(s)\cdot e_l,\varphi_k\big)\varrho_l^n(s)\sum_{i=1}^{N_t}\chi_i(r)\chi_i(s)ds\Bigg)^2dr\\ &\leq\frac{C}{\tau}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}(t-s)^{2\alpha-2}e^{-2\nu(t-s)}\left|E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\right|^2\\ &\quad\times\left|\left(h(s)\cdot e_l,\varphi_k\right)\right|^2\left|\varrho_l^n(s)\right|^2dsdr\\ &\leq C\left(\widetilde{\mu}_1^n\right)^2\int_0^t(t-s)^{2\alpha-4}e^{-2\nu(t-s)}\left\|(-\Delta)^{\widetilde{\gamma}-\frac{\beta}{\alpha}}h(s)\right\|^2_{\mathcal{L}_2^0}ds\leq C\left(\widetilde{\mu}_1^n\right)^2.\end{aligned}\tag{A.4}$$

Collecting the above estimates, we obtain that for all $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \le & C\left(1+\nu^2t^2\right)e^{-2\nu t}\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + Ce^{-2\nu t}\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \\ & + C + C\left(\mu_1^n\right)^2 + C\left(\widetilde{\mu}_1^n\right)^2 + \left(C + C(\mu_1^n)^2 + C(\widetilde{\mu}_1^n)^2\right) \\ & \times \int_0^t (t-s)^{2\alpha-4}e^{-2\nu(t-s)}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}ds.\end{aligned}$$

Then an application of Grönwall's lemma for singular kernels [16] gives

$$\begin{aligned}\sup_{0\le t\le T}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} \le & \left(C + C(\mu_1^n)^2\right)\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + \left(C + C(\mu_1^n)^2\right)\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \\ & + \left(C + C(\mu_1^n)^2 + C(\widetilde{\mu}_1^n)^2\right)\left(1+(\mu_1^n)^2\right). \end{aligned} \tag{A.5}$$

Step 2. Estimate of $\mathbb{E}\|\partial_t u_n(t)\|_{\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}}$. By slightly modifying the arguments in (A.3) and (A.4), we conclude that

$$\begin{aligned}&\mathbb{E}\Bigg\|\int_0^t\int_{\mathcal{D}} e^{-\nu(t-s)}\sum_{k=1}^{\infty}\Big(-\nu(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \\ &\qquad + (t-s)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\Big)\varphi_k(x)\varphi_k(y)g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\Bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \\ &\le \frac{1}{\tau}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\sum_{i=1}^{N_t}\Bigg|\int_{t_i}^{t_{i+1}} e^{-\nu(t-s)}\Big(-\nu(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \\ &\qquad + (t-s)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\Big)\big(g(s,u_n(s))\cdot e_l,\varphi_k\big)\varsigma_l^n(s)ds\Bigg|^2 \\ &\le C\mathbb{E}\sum_{k,l=1}^{\infty}\int_0^t e^{-2\nu(t-s)}\left((t-s)^{2\alpha-2}+(t-s)^{2\alpha-4}\right)\frac{1}{\left(1+\lambda_k^{\beta}(t-s)^{\alpha}\right)^2} \\ &\quad \times\left|\lambda_k^{\widetilde{\gamma}-\frac{\beta}{\alpha}}\big(g(s,u_n(s))\cdot e_l,\varphi_k\big)\varsigma_l^n(s)\right|^2 ds \\ &\le C\left(\mu_1^n\right)^2\left(t^{2\alpha-1}+t^{2\alpha-3}\right)\sup_{0\le t\le T}\left(1+\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right),\end{aligned} \tag{A.6}$$

$$\begin{aligned}&\mathbb{E}\Bigg\|\int_0^t\int_{\mathcal{D}} e^{-\nu(t-s)}\sum_{k=1}^{\infty}\Big(-\nu(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right) \\ &\qquad + (t-s)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\Big)\varphi_k(x)\varphi_k(y)h(s)d\mathbb{W}_n^H(s,y)\Bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} \\ &= \frac{1}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\Bigg(\int_0^t\int_0^t e^{-\nu(t-s)}\Big(-\nu(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\end{aligned}$$

$$
\begin{aligned}
&\quad + (t-s)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}(t-s)^{\alpha}\right)\Big)\big(h(s)\cdot e_l,\varphi_k\big)\varrho_l^n(s) \\
&\quad \times \sum_{i=1}^{N_t}\chi_i(r)\chi_i(s)dsd\xi_l^H(r)\Bigg)^2 \\
&\le C\left(\widetilde{\mu}_1^n\right)^2\left(t^{2\alpha+2H-2}+(t-s)^{2\alpha+2H-4}\right). \qquad \text{(A.7)}
\end{aligned}
$$

Then by similar arguments as in Step 1, we obtain that

$$
\begin{aligned}
\sup_{0\le t\le T}\mathbb{E}\|\partial_t u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} &\le C\mathbb{E}\|a\|^2_{\mathbb{H}^{2\widetilde{\gamma}}} + C\mathbb{E}\|b\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}} + C\left(\widetilde{\mu}_1^n\right)^2 \\
&\quad + \left(C + C(\mu_1^n)^2\right)\left(1+\sup_{0\le t\le T}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right). \qquad \text{(A.8)}
\end{aligned}
$$

Step 3. Estimate of $\mathbb{E}\|{}_0^c\partial_t^{\alpha,\nu}u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-2\beta}}$. Note that $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$. By the definition of fractional Laplacian, there exists

$$
\begin{aligned}
&C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\left((-\Delta)^{\beta}u_n(t),\varphi_k\right)^2 \\
&= C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}}\big(u_n(t),\varphi_k\big)^2 = C\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}. \qquad \text{(A.9)}
\end{aligned}
$$

By using $(\mathbf{A_1})$, $\lambda_k^{2\beta/\alpha-2\beta}\le C$ and $\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}-2\beta/\alpha}}\le C\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}$, we have

$$
\begin{aligned}
&C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\big(f(t,u_n(t)),\varphi_k\big)^2 \\
&\le C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\big(f\big(t,u_n(t)\big),\varphi_k\big)^2 \le C\mathbb{E}\left(1+\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right).
\end{aligned}
$$

Note that $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{D})$, the Brownian motion has independent increments, and $\{\xi_l\}_{l=1}^{\infty}$ is a family of mutually independent one-dimensional standard Brownain motions. Hence, we deduce from (2.8), $(\mathbf{A_1})$, the Itô isometry, and the boundedness assumption on ς_k^n that

$$
\begin{aligned}
&C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\left(g\big(t,u_n(t)\big)\frac{\partial^2\mathbb{W}_n(t,\cdot)}{\partial t\partial\cdot},\varphi_k\right)^2 \\
&\le \frac{C}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\sum_{i=1}^{N_t}\left(\int_{t_i}^{t_{i+1}}\big(g\big(t,u_n(t)\big)\cdot e_l,\varphi_k\big)\varsigma_l^n(t)\chi_i(t)d\xi_l(r)\right)^2 \\
&\le \frac{C(\mu_1^n)^2}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\chi_i(t)\big(g\big(t,u_n(t)\big)\cdot e_l,\varphi_k\big)^2dr
\end{aligned}
$$

$$\leq \frac{Ct(\mu_1^n)^2}{\tau}\left(1+\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right). \tag{A.10}$$

Since $\{\xi_l^H\}_{l=1}^{\infty}$ is a family of mutually independent one-dimensional fractional Brownain motions, using (2.9), $(\mathbf{A_1})$, Lemma 2.2, and the boundedness assumption on ϱ_k^n, we get

$$\begin{aligned}
&C\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\left(h(t)\frac{\partial^2\mathbb{W}_n^H(t,\cdot)}{\partial t\partial\cdot},\varphi_k\right)^2\\
&=\frac{C}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-2\beta}\sum_{i=1}^{N_t}\left(\int_{t_i}^{t_{i+1}}\big(h(t)\cdot e_l,\varphi_k\big)\varrho_l^n(t)\chi_i(t)d\xi_l^H(r)\right)^2\\
&\leq\frac{C(\widetilde{\mu}_1^n)^2\tau^{2H-1}}{\tau^2}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\sum_{i=1}^{N_t}\int_{t_i}^{t_{i+1}}\chi_i(t)\big(h(t)\cdot e_l,\varphi_k\big)^2dr\\
&\leq Ct\left(\widetilde{\mu}_1^n\right)^2\tau^{2H-2}.
\end{aligned}\tag{A.11}$$

Therefore, it follows from (1.1) that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\mathbb{E}\left\|{}_0^c\partial_t^{\alpha,\nu}u_n(t)\right\|^2_{\mathbb{H}^{2\widetilde{\gamma}-2\beta}}\\
&\leq C\left(\widetilde{\mu}_1^n\right)^2\tau^{2H-2}+\left(C+C(\mu_1^n)^2\tau^{-1}\right)\left(1+\sup_{0\leq t\leq T}\mathbb{E}\|u_n(t)\|^2_{\mathbb{H}^{2\widetilde{\gamma}}}\right).
\end{aligned}$$

Step 4. We prove a Hölder regularity property of the solution u_n. Without loss of generality, we assume that there exist positive integers $N_{\theta_1}, N_{\theta_2}$ such that $\theta_1=t_{N_{\theta_1}+1},\theta_2=t_{N_{\theta_2}+1}$. In view of (2.8)-(2.9), $(\mathbf{A_1})$, Hölder's inequality, the Itô isometry, Lemma 2.2, the boundedness assumption on $\varsigma_k^n(t)$ and $\varrho_k^n(t)$, we deduce that

$$\begin{aligned}
&C\mathbb{E}\bigg\|\int_0^{\theta_2}\int_{\mathcal{D}}\mathcal{S}_{\alpha,\beta}^{\nu}(\theta_2-s,\cdot,y)g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\\
&\qquad-\int_0^{\theta_1}\int_{\mathcal{D}}\mathcal{S}_{\alpha,\beta}^{\nu}(\theta_1-s,\cdot,y)g\big(s,u_n(s,y)\big)d\mathbb{W}_n(s,y)\bigg\|^2_{\mathbb{H}^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}}\\
&\leq\frac{C}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\sum_{i=1}^{N_{\theta_1}}\bigg|\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\left((\theta_2-s)^{\alpha-1}e^{-\nu(\theta_2-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(\theta_2-s)^{\alpha}\right)\right.\\
&\qquad\left.-(\theta_1-s)^{\alpha-1}e^{-\nu(\theta_1-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(\theta_1-s)^{\alpha}\right)\right)\\
&\qquad\times\big(g(s,u_n(s))\cdot e_l,\varphi_k\big)\varsigma_l^n(s)dsd\xi_l(r)\bigg|^2\\
&+\frac{C}{\tau^2}\mathbb{E}\sum_{k,l=1}^{\infty}\lambda_k^{2\widetilde{\gamma}-\frac{2\beta}{\alpha}}\sum_{i=N_{\theta_1}+1}^{N_{\theta_2}}\bigg|\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}(\theta_2-s)^{\alpha-1}e^{-\nu(\theta_2-s)}E_{\alpha,\alpha}\left(-\lambda_k^{\beta}(\theta_2-s)^{\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \big(g\big(s,u_n(s)\big)\cdot e_l,\varphi_k\big)\varsigma_l^n(s)dsd\xi_l(r)\Big|^2 \\
&\le C\left(\mu_1^n\right)^2\left(|\theta_2-\theta_1|^2+|\theta_2-\theta_1|^{2\alpha-1}\right)\left(1+\sup_{0\le s\le T}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\tilde{\gamma}}}\right), \qquad \text{(A.12)}
\end{aligned}
$$

and

$$
\begin{aligned}
&C\mathbb{E}\bigg\|\int_0^{\theta_2}\int_{\mathcal{D}}\mathcal{S}^{\nu}_{\alpha,\beta}(\theta_2-s,\cdot,y)h(s)d\mathbb{W}_n^H(s,y) \\
&\qquad -\int_0^{\theta_1}\int_{\mathcal{D}}\mathcal{S}^{\nu}_{\alpha,\beta}(\theta_1-s,\cdot,y)h(s)d\mathbb{W}_n^H(s,y)\bigg\|^2_{\mathbb{H}^{2\tilde{\gamma}-\frac{2\beta}{\alpha}}} \\
&\le C\left(\tilde{\mu}_1^n\right)^2\left(|\theta_2-\theta_1|^2+|\theta_2-\theta_1|^{2\alpha-1}\right). \qquad \text{(A.13)}
\end{aligned}
$$

In a similar way, we have for $0\le\theta_1\le\theta_2\le T$,

$$
\begin{aligned}
&\mathbb{E}\big\|u_n(\theta_2)-u_n(\theta_1)\big\|^2_{\mathbb{H}^{2\tilde{\gamma}-\frac{2\beta}{\alpha}}} \\
&\le C|\theta_2-\theta_1|^2\Big(\mathbb{E}\|a\|^2_{\mathbb{H}^{2\tilde{\gamma}}}+\mathbb{E}\|b\|^2_{\mathbb{H}^{2\tilde{\gamma}-\frac{2\beta}{\alpha}}}+1+(\mu_1^n)^2+(\tilde{\mu}_1^n)^2 \\
&\qquad +\sup_{0\le s\le T}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\tilde{\gamma}}}+(\mu_1^n)^2\sup_{0\le s\le T}\mathbb{E}\|u_n(s)\|^2_{\mathbb{H}^{2\tilde{\gamma}}}\Big). \qquad \text{(A.14)}
\end{aligned}
$$

The proof is therefore complete. □

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