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L1/Local Discontinuous Galerkin Method for the Time-Fractional Stokes Equation

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Abstract. In this paper, L1/local discontinuous Galerkin method seeking the numerical solution to the time-fractional Stokes equation is displayed, where the time-fractional derivative is in the sense of Caputo with derivative order $\alpha \in (0,1)$. Although the time-fractional derivative is used, its solution may be smooth since such examples can be easily constructed. In this case, we use the uniform L1 scheme to approach the temporal derivative and use the local discontinuous Galerkin (LDG) method to approximate the spatial derivative. If the solution has a certain weak regularity at the initial time, we use the non-uniform L1 scheme to discretize the time derivative and still apply LDG method to discretizing the spatial derivative. The numerical stability and error analysis for both situations are studied. Numerical experiments are also presented which support the theoretical analysis.

AMS subject classifications: 35R11, 65M60

Key words: L1 scheme, local discontinuous Galerkin method (LDG), time-fractional Stokes equation, Caputo derivative.

1. Introduction

The time-fractional Navier-Stokes equation was considered in [6]. Shortly after, Momani and Odibat [21] used the Adomian decomposition method for solving time-fractional Navier-Stokes equation in a tube. In 2015, the well-posedness of its mild solution was investigated [1]. Very recently, the mixed finite element method and the spectral method for this time-fractional equation were explored [17,30,31]. In this paper, we numerically study a special case of the time-fractional Navier-Stokes equation, i.e., the non-stationary 2D time-fractional Stokes equations. This model characterizes the long memory processes, thus it can be used to model anomalous diffusion in fractal media [32]. Recently, Xu *et al.* [19] proved the existence and uniqueness of the

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weak solution with the help of classical saddle-point theory. Then they constructed an efficient spectral method for numerical approximations of the weak solution. Li *et al.* [16] proposed two marker and cell schemes for the time fractional Stokes equation on non-uniform grids. Based on the fixed-point theorem, Xu *et al.* [28] studied the well-posedness of a time fractional stochastic Stokes model with finite and infinite delay and multiplicative Brownian motion. Motivated by above considerations, the objective in this paper is to present an L1/LDG method for the two-dimensional time-fractional Stokes equations. LDG method is a special class of discontinuous Galerkin (DG) methods, introduced firstly by Cockburn and Shu [4]. The main technique of LDG method is to rewrite higher-order derivative equation into an equivalent system containing only the first derivative, and then discretize it by the standard DG method. This method has received increasing interest due to its features: 1) easily handling meshes with hanging nodes, elements of general shapes, and local spaces of different types [29]; 2) freely designing numerical fluxes in herited conservation laws [3]. Till now, LDG methods have been successfully applied to fractional differential equations [5,10,11,14,27] for one space dimension, [12] for both one and two space dimensions, [13] for two space dimensions. Inspired by the analysis techniques in [13, 24, 26], we study the time-fractional Stokes equation in two spatial dimensions using the L1/LDG method. The research contents include two parts:

1. If the solution is smooth enough with respect to time, for example, belonging to $C^2[0,T]$ for a given time T. This case can exist since one can construct such a solution. In this case, we use the uniform L1 scheme to compute the time-fractional derivative and apply the LDG method to approximating the spatial derivative. We will show the constructed uniform L1/LDG scheme is numerically stable for $\mathcal{Q}^k$ elements on cartesian meshes. Furthermore, by the aid of the so-called Stokes projection (see [26]), we obtain the error estimates in L^2 norm for the velocity, the stress (gradient of velocity), and the pressure.

2. If the solution has weak (or low) regularity at the starting time, adopting the non-uniform L1 scheme is one of the best choices. In this situation, we use the non-uniform L1 scheme in time discretization. We still use the LDG method to approximate the spatial derivative. We prove that the established non-uniform L1/LDG scheme is numerically stable for the velocity and the stress. And we obtain the L^2 optimal error estimate for the velocity and L^2 suboptimal error estimate for the stress.

The remainder of the paper is organized as follows. In Section 2, we present some notations adopted throughout the paper. In Section 3, we build up the uniform L1/LDG and non-uniform L1/LDG schemes for time-fractional Stokes model (3.1). The L^2-stability and error estimates for the fully discrete schemes are presented. Numerical experiments are provided in Section 4, which support the theoretical analysis. Concluding remarks are given in the last section.

2. Notations

For a bounded rectangular domain $\Omega = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$, we divide it into a Cartesian grid $\mathcal{T}_h = \{K\}$ consisting of $N_x \times N_y$ rectangular elements

$$K := I_i \times J_j = \big(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\big) \times \big(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\big), \quad i = 1, \ldots, N_x, \quad j = 1, \ldots, N_y,$$

where

$$\begin{aligned} a_1 &= x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = b_1, \\ a_2 &= y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = b_2. \end{aligned}$$

Denoting $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta y_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, respectively. Then the maximal length of all edges is defined by $h = \max_{1\le i\le N_x, 1\le j\le N_y}(\Delta x_i, \Delta y_j)$. We assume that the mesh $\mathcal{T}_h$ is quasi-uniform in the sense that there exist constants $C_1, C_2 > 0$ such that $h \le C_1 \Delta x_i$ and $h \le C_2 \Delta y_j$ for all $K \in \mathcal{T}_h$. The associated finite element spaces are defined as

$$\begin{aligned} \underline{\Sigma}_h &= \left\{ \underline{\sigma} \in L^2(\Omega)^{2\times 2} : \underline{\sigma}|_K \in \mathcal{Q}^k(K)^{2\times 2}, \, \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{V}_h &= \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \mathcal{Q}^k(K)^2, \, \forall K \in \mathcal{T}_h \right\}, \\ Q_h &= \left\{ q \in L^2(\Omega) : q|_K \in \mathcal{Q}^k(K), \, \forall K \in \mathcal{T}_h, \, \iint_\Omega q \, \mathrm{d}x\mathrm{d}y = 0 \right\}, \end{aligned} \tag{2.1}$$

where $\mathcal{Q}^k(K)$ denotes the space of polynomials of the degree at most $k \ge 0$ defined on K.

For any vector-valued function $\mathbf{u} = (u_1, u_2)^\intercal$ and matrix-valued function $\underline{\sigma} = (\sigma_{ij})_{2\times 2}$, we use the standard notation $(\nabla \mathbf{u})_{ij} = \partial_j u_i$ and $(\nabla \cdot \underline{\sigma})_i = \sum_{j=1}^2 \partial_j \sigma_{ij}$. We also denote by $\mathbf{u} \otimes \mathbf{v}$ the matrix whose ij-th component is $u_i v_j$ and write

$$\underline{\sigma} : \underline{r} = \sum_{i,j=1}^{2} \sigma_{ij} r_{ij}, \quad \mathbf{u} \cdot \underline{\sigma} \cdot \vec{\mathbf{n}} = \sum_{i,j=1}^{2} u_i \sigma_{ij} n_j = \underline{\sigma} : (\mathbf{u} \otimes \vec{\mathbf{n}}).$$

We use a fixed vector $\mathbf{I}_0 = (1, 1)^\intercal$ to uniquely define the inflow and outflow boundaries of Ω, namely,

$$\begin{aligned} \partial\Omega^- &= \{(x, y) \in \partial\Omega : \mathbf{I}_0 \cdot \vec{\mathbf{n}} < 0\}, \\ \partial\Omega^+ &= \{(x, y) \in \partial\Omega : \mathbf{I}_0 \cdot \vec{\mathbf{n}} > 0\}, \end{aligned}$$

where $\vec{\mathbf{n}}$ is the outward unit normal vector of Ω. Similarly, we denote ∂K^- and ∂K^+ the inflow and outflow boundaries of K, respectively, i.e.,

$$\begin{aligned} \partial K^- &= \{(x, y) \in \partial K : \mathbf{I}_0 \cdot \vec{\mathbf{n}} < 0\}, \\ \partial K^+ &= \{(x, y) \in \partial K : \mathbf{I}_0 \cdot \vec{\mathbf{n}} > 0\}. \end{aligned}$$

If two elements K_1 and K_2 are neighbours and share one common side e, i.e., $e = \partial K_1 \cap \partial K_2$, then there are two traces for any function defined on e. We denote

$$\begin{aligned} p^+ &= p|_{\partial K_2^- \cap e}, \quad [\![p]\!]_e = p^+ - p^-, \\ p^- &= p|_{\partial K_1^+ \cap e}, \quad [\![p]\!]_{\partial\Omega} = p|_{\partial\Omega} \end{aligned}$$

for scalar functions

$$\begin{aligned} \mathbf{u}^+ &= \left(u_1^+, u_2^+\right)^\intercal, \quad [\![\mathbf{u}]\!]_e = \left([\![u_1]\!]_e, [\![u_2]\!]_e\right)^\intercal, \\ \mathbf{u}^- &= \left(u_1^-, u_2^-\right)^\intercal, \quad [\![\mathbf{u}]\!]_{\partial\Omega} = \left([\![u_1]\!]_{\partial\Omega}, [\![u_2]\!]_{\partial\Omega}\right)^\intercal \end{aligned}$$

for vector-valued functions, and

$$\begin{aligned} \underline{\sigma}^+ &= \left(\sigma_{ij}^+\right)_{2\times 2}, \quad [\![\underline{\sigma}]\!]_e = \left([\![\sigma_{ij}]\!]_e\right)_{2\times 2}, \\ \underline{\sigma}^- &= \left(\sigma_{ij}^-\right)_{2\times 2}, \quad [\![\underline{\sigma}]\!]_{\partial\Omega} = \left([\![\sigma_{ij}]\!]_{\partial\Omega}\right)_{2\times 2} \end{aligned}$$

for matrix-valued functions.

For each $h > 0$, $\mathcal{E}_B$ denotes the set of all boundary edges of the mesh $\mathcal{T}_h$ on $\partial\Omega$, $\mathcal{E}_I$ denotes the set of all interior edges of the mesh $\mathcal{T}_h$ in Ω, and $\mathcal{E}$ denotes the union of all edges, i.e., $\mathcal{E} = \mathcal{E}_B \cup \mathcal{E}_I$.

We define the inner products over the element K and the associated norms by

$$\begin{aligned} \|p\|_K^2 &= (p,p)_K, \quad (p,q)_K = \iint_K pq \,\mathrm{d}x\mathrm{d}y, \\ \|\mathbf{u}\|_K^2 &= (\mathbf{u},\mathbf{u})_K, \quad (\mathbf{u},\mathbf{v})_K = \iint_K \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x\mathrm{d}y, \\ \|\underline{\sigma}\|_K^2 &= (\underline{\sigma},\underline{\sigma})_K, \quad (\underline{\sigma},\underline{r})_K = \iint_K \underline{\sigma} : \underline{r}\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Summing over all the elements, we denote

$$\begin{aligned} \|p\|_\Omega^2 &= \sum_{K\in\mathcal{T}_h} \|p\|_K^2, \quad (p,q)_\Omega = \sum_{K\in\mathcal{T}_h} (p,q)_K, \\ \|\mathbf{u}\|_\Omega^2 &= \sum_{K\in\mathcal{T}_h} \|\mathbf{u}\|_K^2, \quad (\mathbf{u},\mathbf{v})_\Omega = \sum_{K\in\mathcal{T}_h} (\mathbf{u},\mathbf{v})_K, \\ \|\underline{\sigma}\|_\Omega^2 &= \sum_{K\in\mathcal{T}_h} \|\underline{\sigma}\|_K^2, \quad (\underline{\sigma},\underline{r})_\Omega = \sum_{K\in\mathcal{T}_h} (\underline{\sigma},\underline{r})_K. \end{aligned}$$

Furthermore, the L^2 norms and L^2 inner products on the edges $\partial K^\pm$ are given by

$$\begin{aligned} \|p\|_{\partial K^\pm}^2 &= (p,p)_{\partial K^\pm}, \qquad (p,q)_{\partial K^\pm} = \int_{\partial K^\pm} p^\mp(s)\, q^\mp(s)\mathrm{d}s, \\ \|\mathbf{u}\|_{\partial K^\pm}^2 &= \sum_{i=1}^2 \|u_i\|_{\partial K^\pm}^2, \quad \|\underline{\sigma}\|_{\partial K^\pm}^2 = \sum_{i,j=1}^2 \|\sigma_{ij}\|_{\partial K^\pm}^2. \end{aligned}$$

The norms on the whole outflow and inflow boundaries $\mathcal{E}$ are defined by

$$\|p\|_{\mathcal{E}}^2 = \sum_{e\in\mathcal{E}} \|p\|_e^2, \quad \|\mathbf{u}\|_{\mathcal{E}}^2 = \sum_{e\in\mathcal{E}} \|\mathbf{u}\|_e^2, \quad \|\underline{\sigma}\|_{\mathcal{E}}^2 = \sum_{e\in\mathcal{E}} \|\underline{\sigma}\|_e^2.$$

For any nonnegative integer m, denote by $\|\cdot\|_{H^m(K)}$ the standard Sobolev norm on the cell K. Then we define the broken Sobolev space $\underline{\Sigma} \times \mathbf{V} \times Q$ on $\mathcal{T}_h$ by [26]

$$\begin{aligned}
&\underline{\Sigma} = \Big\{\underline{\sigma} \in L^2(\Omega)^{2\times 2} : \underline{\sigma}|_K \in H^1(K)^{2\times 2},\ \forall K \in \mathcal{T}_h\Big\},\\
&\mathbf{V} = \Big\{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in H^1(K)^2,\ \forall K \in \mathcal{T}_h\Big\},\\
&Q = \left\{q \in L^2(\Omega) : q|_K \in H^1(K),\ \forall K \in \mathcal{T}_h,\ \iint_\Omega q\,\mathrm{d}x\mathrm{d}y = 0\right\}.
\end{aligned} \tag{2.2}$$

For any vector-valued function $\mathbf{u}$ and matrix-valued function $\underline{\sigma}$, we define the $H^m(K)$ and $H^m(\mathcal{T}_h)$ norms as

$$\begin{aligned}
&\|\mathbf{u}\|_{H^m(K)}^2 = \sum_{i=1}^2 \|u_i\|_{H^m(K)}^2, \qquad \|\mathbf{u}\|_{H^m(\mathcal{T}_h)}^2 = \sum_{K\in\mathcal{T}_h} \|\mathbf{u}\|_{H^m(K)}^2,\\
&\|\underline{\sigma}\|_{H^m(K)}^2 = \sum_{i,j=1}^2 \|\sigma_{ij}\|_{H^m(K)}^2, \quad \|\underline{\sigma}\|_{H^m(\mathcal{T}_h)}^2 = \sum_{K\in\mathcal{T}_h} \|\underline{\sigma}\|_{H^m(K)}^2.
\end{aligned}$$

3. L1/LDG method for the time-fractional Stokes equation

In this section, we establish the L1/LDG schemes for the following two-dimensional Caputo-type Stokes equation with periodic boundary value conditions [16],

$$\begin{cases}
{}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T],\\
\nabla \cdot \mathbf{u} = 0, & (\mathbf{x},t) \in \Omega \times (0,T],\\
\mathbf{u}|_{t=0} = \mathbf{u_0}(\mathbf{x}), & \mathbf{x} \in \Omega,
\end{cases} \tag{3.1}$$

where $\mathbf{x} = (x,y), \Omega = (a_1,b_1) \times (a_2,b_2) \subset \mathbb{R}^2$ is a bounded rectangular domain, $\mathbf{u} = \mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t))^{\intercal}$ the velocity vector, $\nu > 0$ the kinematic viscosity, $p = p(\mathbf{x},t)$ the pressure, $\mathbf{f} = (f_1(\mathbf{x},t), f_2(\mathbf{x},t))^{\intercal}$ the prescribed body force, $\mathbf{u_0}(\mathbf{x})$ represents the initial velocity vector, $T > 0$ represents a finite time, and where the time-fractional Caputo derivative operator is defined as follows [9]:

$${}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial \mathbf{u}(\mathbf{x},s)}{\partial s}\mathrm{d}s, \quad 0 < \alpha < 1, \tag{3.2}$$

in which $\Gamma(\cdot)$ denotes the Euler Gamma function. Since p is uniquely defined up to an additive constant, we assume that $\iint_\Omega p\,\mathrm{d}x\mathrm{d}y = 0$.

3.1. The semi-discrete LDG scheme

In order to get the LDG formulation, we firstly rewrite Eq. (3.1) into the following equivalent first-order system by introducing an auxiliary variable $\underline{\sigma} = \sqrt{\nu}\,\nabla\mathbf{u} = \sqrt{\nu}(\nabla u_1, \nabla u_2)^\intercal$:

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u} - \sqrt{\nu}\,\nabla\cdot\underline{\sigma} + \nabla p = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times(0,T], \\ \underline{\sigma} = \sqrt{\nu}\,\nabla\mathbf{u}, & (\mathbf{x},t)\in\Omega\times(0,T], \\ \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t)\in\Omega\times(0,T], \\ \mathbf{u}|_{t=0} = \mathbf{u_0}(\mathbf{x}), & \mathbf{x}\in\Omega. \end{cases} \tag{3.3}$$

Let $(\mathbf{u}_h, \underline{\sigma}_h, p_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ be the approximation of $(\mathbf{u}, \underline{\sigma}, p)$. Then we can define the semi-discrete LDG scheme for time-fractional Stokes equation (3.1) as follows: Find $(\mathbf{u}_h, \underline{\sigma}_h, p_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ such that for all the test functions $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$,

$$\begin{cases} \left({}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}_h, \mathbf{v}_h\right)_K = \mathcal{L}_K(\underline{\sigma}_h, \mathbf{v}_h) + \mathcal{P}_K(p_h, \mathbf{v}_h) + (\mathbf{f}, \mathbf{v}_h)_K, \\ (\underline{\sigma}_h, \underline{r}_h)_K = \mathcal{K}_K(\mathbf{u}_h, \underline{r}_h), \\ \mathcal{Q}_K(\mathbf{u}_h, w_h) = 0, \end{cases} \tag{3.4}$$

where

$$\mathcal{L}_K(\underline{\sigma}_h, \mathbf{v}_h) = -\sqrt{\nu}\left(\iint_K \underline{\sigma}_h : \nabla\mathbf{v}_h \,\mathrm{d}x\mathrm{d}y - \int_{\partial K} \mathbf{v}_h \cdot \widehat{\underline{\sigma}}_h \cdot \vec{\mathbf{n}}\,\mathrm{d}s\right), \tag{3.5}$$

$$\mathcal{P}_K(p_h, \mathbf{v}_h) = \iint_K p_h \nabla\cdot\mathbf{v}_h \,\mathrm{d}x\mathrm{d}y - \int_{\partial K} \widehat{p}_h \mathbf{v}_h \cdot \vec{\mathbf{n}}\,\mathrm{d}s, \tag{3.6}$$

$$\mathcal{K}_K(\mathbf{u}_h, \underline{r}_h) = -\sqrt{\nu}\left(\iint_K \mathbf{u}_h \cdot (\nabla\cdot\underline{r}_h)\,\mathrm{d}x\mathrm{d}y - \int_{\partial K} \widehat{\mathbf{u}}_{h,\sigma} \cdot \underline{r}_h \cdot \vec{\mathbf{n}}\,\mathrm{d}s\right), \tag{3.7}$$

$$\mathcal{Q}_K(\mathbf{u}_h, w_h) = \iint_K \mathbf{u}_h \cdot \nabla w_h \,\mathrm{d}x\mathrm{d}y - \int_{\partial K} \widehat{\mathbf{u}}_{h,p} \cdot \vec{\mathbf{n}} w_h \,\mathrm{d}s. \tag{3.8}$$

The "hat" terms are the so-called "numerical fluxes" that are yet to be determined. The freedom in choosing numerical fluxes can be utilized for designing a scheme that enjoys certain numerical stability properties. It turns out that we can take the following choices simply:

$$\widehat{\mathbf{u}}_{h,\sigma} = \mathbf{u}_h^-, \quad \widehat{\underline{\sigma}}_h = \underline{\sigma}_h^+, \quad \widehat{\mathbf{u}}_{h,p} = \mathbf{u}_h^-, \quad \widehat{p}_h = p_h^+. \tag{3.9}$$

Simple integration by parts yields

$$\mathcal{L}_K(\underline{\sigma}_h, \mathbf{v}_h) = \sqrt{\nu}\left(\iint_K \mathbf{v}_h \cdot (\nabla\cdot\underline{\sigma}_h)\,\mathrm{d}x\mathrm{d}y + \int_{\partial K^+} \mathbf{v}_h \cdot [\![\underline{\sigma}_h]\!] \cdot \vec{\mathbf{n}}\,\mathrm{d}s\right), \tag{3.10}$$

$$\mathcal{P}_K(p_h, \mathbf{v}_h) = -\iint_K \mathbf{v}_h \cdot \nabla p_h \,\mathrm{d}x\mathrm{d}y - \int_{\partial K^+} [\![p_h]\!] \mathbf{v}_h \cdot \vec{\mathbf{n}}\,\mathrm{d}s, \tag{3.11}$$

$$\mathcal{K}_K(\mathbf{u}_h, \underline{r}_h) = \sqrt{\nu}\left(\iint_K \underline{r}_h : \nabla \mathbf{u}_h \,\mathrm{d}x\mathrm{d}y - \int_{\partial K^-} [\![\mathbf{u}_h]\!] \cdot \underline{r}_h \cdot \vec{\mathbf{n}} \,\mathrm{d}s\right), \tag{3.12}$$

$$\mathcal{Q}_K(\mathbf{u}_h, w_h) = -\iint_K w_h \nabla \cdot \mathbf{u}_h \,\mathrm{d}x\mathrm{d}y + \int_{\partial K^-} [\![\mathbf{u}_h]\!] \cdot \vec{\mathbf{n}}\, w_h \,\mathrm{d}s. \tag{3.13}$$

Summing Eq. (3.4) over all elements yields

$$\begin{cases} \left({}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}_h, \mathbf{v}_h\right)_\Omega = \mathcal{L}(\underline{\sigma}_h, \mathbf{v}_h) + \mathcal{P}(p_h, \mathbf{v}_h) + (\mathbf{f}, \mathbf{v}_h)_\Omega, \\ (\underline{\sigma}_h, \underline{r}_h)_\Omega = \mathcal{K}(\mathbf{u}_h, \underline{r}_h), \\ \mathcal{Q}(\mathbf{u}_h, w_h) = 0. \end{cases} \tag{3.14}$$

Here $\Psi(\cdot,\cdot) = \sum_{K\in\mathcal{T}_h} \Psi_K(\cdot,\cdot)$ for $\Psi = \mathcal{L}, \mathcal{P}, \mathcal{K}, \mathcal{Q}$.

3.2. Projections

We first recall a projection $\mathbb{P}_h$ in the following: For any $\mathbf{q} \in H_0^1(\Omega)^2, \mathbb{P}_h\mathbf{q}|_K \in \mathcal{Q}^k(K)^2$ satisfies

$$\begin{aligned} &\iint_K (\mathbb{P}_h\mathbf{q} - \mathbf{q}) \cdot \nabla v \,\mathrm{d}x\mathrm{d}y = 0, \quad \forall v \in \mathcal{Q}^k(K), \\ &\int_e (\mathbb{P}_h\mathbf{q} - \mathbf{q}) \cdot \vec{\mathbf{n}}\, v \mathrm{d}s = 0, \qquad\quad \forall v \in \mathcal{Q}^k(e), \quad \forall e \in \partial K^-. \end{aligned} \tag{3.15}$$

Then by a standard scaling argument [2], there hold

$$\|\mathbf{q} - \mathbb{P}_h\mathbf{q}\|_\Omega + h\|\mathbf{q} - \mathbb{P}_h\mathbf{q}\|_{H^1(\Omega)} + h^{\frac{1}{2}}\|\mathbf{q} - \mathbb{P}_h\mathbf{q}\|_{\mathcal{E}} \le Ch\|\mathbf{q}\|_{H^1(\Omega)}, \tag{3.16}$$

and

$$\|\mathbb{P}_h\mathbf{q}\|_{H^1(\Omega)} \le C\|\mathbf{q}\|_{H^1(\Omega)}. \tag{3.17}$$

Lemma 3.1 ([7]). *For any*

$$q \in L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \iint_\Omega v \,\mathrm{d}x\mathrm{d}y = 0 \right\},$$

there exists a $\mathbf{w}^\star \in H_0^1(\Omega)^2$ *such that*

$$-\iint_\Omega q\nabla \cdot \mathbf{w}^\star \,\mathrm{d}x\mathrm{d}y \ge C_1\|q\|_\Omega^2, \quad \|\mathbf{w}^\star\|_{H^1(\Omega)} \le C_2\|q\|_\Omega, \tag{3.18}$$

where C_1 *and* C_2 *are two positive constants independent of* q*. Besides,*

$$\|q\|_\Omega^2 \le C\Big(\mathcal{K}\left(\mathbb{P}_h\mathbf{w}^\star, \underline{r}_h\right) + \mathcal{Q}\left(\mathbb{P}_h\mathbf{w}^\star, q\right) + \|\underline{r}_h\|_\Omega^2\Big) \tag{3.19}$$

for any $\underline{r}_h \in \underline{\Sigma}_h$*.*

We now introduce the Stokes projection. Let $\mathbf{u} \in \mathbf{V}$ satisfy $\nabla \cdot \mathbf{u} = 0$, $\underline{\sigma} = \sqrt{\nu}\nabla \mathbf{u}$ and $p \in Q$, define the Stokes projection $(\underline{\Pi_h \sigma}, \mathbf{\Pi_h u}, \Pi_h p) \in \underline{\Sigma}_h \times \mathbf{V}_h \times Q_h$ as: For any $(\underline{r}_h, \mathbf{v}_h, w_h) \in \underline{\Sigma}_h \times \mathbf{V}_h \times Q_h$, it holds that

$$\mathcal{L}(\underline{\Pi_h \sigma}, \mathbf{v}_h) + \mathcal{P}(\Pi_h p, \mathbf{v}_h) = \mathcal{L}(\underline{\sigma}, \mathbf{v}_h) + \mathcal{P}(p, \mathbf{v}_h), \tag{3.20}$$

$$(\underline{\Pi_h \sigma}, \underline{r}_h)_\Omega = \mathcal{K}(\mathbf{\Pi_h u}, \underline{r}_h), \tag{3.21}$$

$$\mathcal{Q}(\mathbf{\Pi_h u}, w_h) = \mathcal{Q}(\mathbf{u}, w_h). \tag{3.22}$$

Besides,

$$\iint_\Omega (\mathbf{\Pi_h u} - \mathbf{u}) \cdot e_i \mathrm{d}x\mathrm{d}y = 0, \quad i = 1, 2, \tag{3.23}$$

where $e_1 = (1, 0)^\intercal$ and $e_2 = (0, 1)^\intercal$, and

$$\iint_\Omega (\Pi_h p - p)\mathrm{d}x\mathrm{d}y = 0. \tag{3.24}$$

The Stokes projection defined above uniquely exists and satisfies the following approximation properties.

Lemma 3.2 ([26]). *If* $\mathbf{u} \in H^{k+2}(\Omega)^2$, $\underline{\sigma} = \sqrt{\nu}\nabla \mathbf{u} \in H^{k+1}(\Omega)^{2\times 2}$ *and* $p \in H^{k+2}(\Omega)$ *satisfies* $\iint_\Omega p \, \mathrm{d}x\mathrm{d}y = 0$, *then there exists a constant* C *depending on the regularity of* $(\mathbf{u}, \underline{\sigma}, p)$ *such that*

$$\|\mathbf{u} - \mathbf{\Pi_h u}\|_\Omega + \|\underline{\sigma} - \underline{\Pi_h \sigma}\|_\Omega + \|p - \Pi_h p\|_\Omega \leq C h^{k+1}. \tag{3.25}$$

3.3. Numerical analysis on uniform time meshes

In the present subsection, we would like to apply the L1 scheme on uniform meshes (called "uniform L1 scheme") as time discretization for $\mathbf{u}(\cdot, t) \in C^2([0, T])^2$ and LDG method as space discretization. For a given $T > 0$, let $\tau = T/M$ be the time mesh size, $t_n = n\tau, n = 0, 1, \ldots, M$ be the mesh points, $M \in \mathbb{Z}^+$. Then the well-known L1 discretisation at time t_n on uniform meshes is based on the following lemma.

Lemma 3.3 ([9, 20, 25]). *Suppose* $0 < \alpha < 1$, $y(t) \in C^2[0, t_n]$. *It holds that*

$$\begin{aligned}&\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{y'(s)}{(t_n - s)^\alpha} \mathrm{d}s - \frac{1}{\beta}\left[b_0 y(t_n) - \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i}) y(t_i) - b_{n-1} y(0)\right] \right| \\ &\leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})\right] \max_{0 \leq t \leq t_n} |y''(t)| \tau^{2-\alpha},\end{aligned} \tag{3.26}$$

where $\beta = \tau^\alpha \Gamma(2-\alpha)$ *and* $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$. *The coefficients* b_i *have the following properties:*

(i) $b_i > 0, i = 0, 1, \ldots, n$,

(ii) $1 = b_0 > b_1 > b_2 > \cdots > b_n,\ b_n \to 0$ *as* $n \to \infty$.

The conclusion (3.26) is a corollary in [25] which is for the case with $1 < \alpha < 2$. One can also refer to [9] whose proof follows the idea in [25]. Another proof for Lemma 3.3 was presented in [20].

Denote

$$\mathbf{u}^n = \mathbf{u}(\mathbf{x}, t_n) = \big(u_1(\mathbf{x}, t_n), u_2(\mathbf{x}, t_n)\big)^\intercal$$

and

$$\delta_t^\alpha \mathbf{u}^n = \frac{1}{\beta}\left[\mathbf{u}^n - \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\mathbf{u}^i - b_{n-1}\mathbf{u}^0\right]. \tag{3.27}$$

Let $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ be the approximation of $(\mathbf{u}(\mathbf{x}, t_n), \underline{\sigma}(\mathbf{x}, t_n), p(\mathbf{x}, t_n))$. Then we define the fully discrete uniform L1/LDG scheme as follows: Find $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ such that for all test functions $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$,

$$\begin{cases} (\delta_t^\alpha \mathbf{u}_h^n, \mathbf{v}_h)_\Omega = \mathcal{L}\left(\underline{\sigma}_h^n, \mathbf{v}_h\right) + \mathcal{P}\left(p_h^n, \mathbf{v}_h\right) + (\mathbf{f}^n, \mathbf{v}_h)_\Omega, & \text{(3.28a)} \\ (\underline{\sigma}_h^n, \underline{r}_h)_\Omega = \mathcal{K}\left(\mathbf{u}_h^n, \underline{r}_h\right), & \text{(3.28b)} \\ \mathcal{Q}\left(\mathbf{u}_h^n, w_h\right) = 0. & \text{(3.28c)} \end{cases}$$

3.3.1. Numerical stability analysis

In the present subsection, we study the numerical stability of the scheme (3.28) for solving the time-fractional Stokes equation. Firstly, we introduce a lemma that will be used later on.

Lemma 3.4 ([26]). *For any* $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$, *we have*

$$\mathcal{L}(\underline{r}_h, \mathbf{v}_h) + \mathcal{K}(\mathbf{v}_h, \underline{r}_h) = 0, \tag{3.29}$$

$$\mathcal{P}(w_h, \mathbf{v}_h) + \mathcal{Q}(\mathbf{v}_h, w_h) = 0. \tag{3.30}$$

Theorem 3.1. *The numerical solution* $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n)$ *of uniform L1/LDG scheme* (3.28) *for* (3.1) *is numerically stable, i.e., for* $n = 1, 2, \ldots, M$, *one has*

$$\|\mathbf{u}_h^n\|_\Omega^2 \le 2\|\mathbf{u}_h^0\|_\Omega^2 + 2\big(T^\alpha\Gamma(1-\alpha)\big)^2 \max_{1\le j\le n}\|\mathbf{f}^j\|_\Omega^2, \tag{3.31}$$

$$\|\underline{\sigma}_h^n\|_\Omega^2 \le 2\|\underline{\sigma}_h^0\|_\Omega^2 + 2T^\alpha\Gamma(1-\alpha)\max_{1\le j\le n}\|\mathbf{f}^j\|_\Omega^2, \tag{3.32}$$

$$\|p_h^n\|_\Omega^2 \le C\left(\frac{\|\mathbf{u}_h^1 - \mathbf{u}_h^0\|_\Omega^2 + \max\limits_{1\le j\le n}\|\mathbf{f}^{j+1} - \mathbf{f}^j\|_\Omega^2}{n^{2\alpha-2}\tau^{2\alpha}} + \|\underline{\sigma}_h^0\|_\Omega^2 + \max_{1\le j\le n}\|\mathbf{f}^j\|_\Omega^2\right), \tag{3.33}$$

where C *is a positive constant independent of* τ *and* h.

Proof. We start with the inequality (3.31). Taking the test function $(\mathbf{v}_h, \underline{r}_h, w_h) = (\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n)$ in (3.28) and adding them together yield

$$(\delta_t^\alpha \mathbf{u}_h^n, \mathbf{u}_h^n)_\Omega + \|\underline{\sigma}_h^n\|_\Omega^2 = (\mathbf{f}^n, \mathbf{u}_h^n)_\Omega\,, \tag{3.34}$$

where Lemma 3.4 is used. From the definition of operator δ_t^α, we obtain

$$\begin{aligned}\|\mathbf{u}_h^n\|_\Omega^2 + \beta\|\underline{\sigma}_h^n\|_\Omega^2 = &\sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\left(\mathbf{u}_h^i, \mathbf{u}_h^n\right)_\Omega \\ &+ b_{n-1}\left(\mathbf{u}_h^0, \mathbf{u}_h^n\right)_\Omega + \beta\left(\mathbf{f}^n, \mathbf{u}_h^n\right)_\Omega.\end{aligned}$$

Using Cauchy-Schwarz inequality, one gets

$$\begin{aligned}\|\mathbf{u}_h^n\|_\Omega^2 + \beta\|\underline{\sigma}_h^n\|_\Omega^2 \le &\frac{1}{2}\sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\|\mathbf{u}_h^i\|_\Omega^2 \\ &+ \frac{1}{2}(b_0 - b_{n-1})\|\mathbf{u}_h^n\|_\Omega^2 + b_{n-1}\|\mathbf{u}_h^0\|_\Omega^2 \\ &+ \frac{b_{n-1}}{4}\|\mathbf{u}_h^n\|_\Omega^2 + \frac{\beta^2}{b_{n-1}}\|\mathbf{f}^n\|_\Omega^2 + \frac{b_{n-1}}{4}\|\mathbf{u}_h^n\|_\Omega^2.\end{aligned} \tag{3.35}$$

Noticing the fact that

$$\tau^\alpha\Gamma(2-\alpha) = \beta < \Gamma(1-\alpha)T^\alpha b_{n-1}, \tag{3.36}$$

one has

$$\begin{aligned}\|\mathbf{u}_h^n\|_\Omega^2 \le &\sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\|\mathbf{u}_h^i\|_\Omega^2 \\ &+ 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 b_{n-1}\|\mathbf{f}^n\|_\Omega^2 + 2b_{n-1}\|\mathbf{u}_h^0\|_\Omega^2.\end{aligned} \tag{3.37}$$

Next, we prove (3.31) by mathematical induction. For $n = 1$ the estimate (3.37) has the form

$$\begin{aligned}\|\mathbf{u}_h^1\|_\Omega^2 &\le 2\|\mathbf{u}_h^0\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2\|\mathbf{f}^1\|_\Omega^2 \\ &\le 2\|\mathbf{u}_h^0\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 \max_{1\le k\le M}\|\mathbf{f}^k\|_\Omega^2,\end{aligned}$$

which is true. Supposing that the following inequality holds:

$$\|\mathbf{u}_h^s\|_\Omega^2 \le 2\|\mathbf{u}_h^0\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 \max_{1\le k\le M}\|\mathbf{f}^k\|_\Omega^2, \quad s = 2, 3, \ldots, m, \tag{3.38}$$

we need to prove that

$$\|\mathbf{u}_h^{m+1}\|_\Omega^2 \le 2\|\mathbf{u}_h^0\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 \max_{1\le k\le M}\|\mathbf{f}^k\|_\Omega^2$$

holds. Letting $n=m+1$ in (3.37) and using (3.38), we have

$$\begin{aligned}\left\|\mathbf{u}_h^{m+1}\right\|_\Omega^2 &\leq \sum_{i=1}^{m}(b_{m-i}-b_{m+1-i})\left\|\mathbf{u}_h^i\right\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 b_m\|\mathbf{f}^{m+1}\|_\Omega^2 + 2b_m\left\|\mathbf{u}_h^0\right\|_\Omega^2\\ &\leq 2\left\|\mathbf{u}_h^0\right\|_\Omega^2 + 2\big(\Gamma(1-\alpha)T^\alpha\big)^2 \max_{1\leq k\leq M}\|\mathbf{f}^k\|_\Omega^2,\end{aligned}$$

and the inequality (3.31) is proved.

In order to show (3.32) we take the test function $\mathbf{v}_h=\delta_t^\alpha\mathbf{u}_h^n$ in Eq. (3.28a), thus obtaining

$$\left\|\delta_t^\alpha\mathbf{u}_h^n\right\|_\Omega^2 = \mathcal{L}\left(\underline{\sigma}_h^n,\delta_t^\alpha\mathbf{u}_h^n\right) + \mathcal{P}\left(p_h^n,\delta_t^\alpha\mathbf{u}_h^n\right) + \left(\mathbf{f}^n,\delta_t^\alpha\mathbf{u}_h^n\right)_\Omega. \tag{3.39}$$

From Eqs. (3.28b) and (3.28c), we have

$$(\delta_t^\alpha\underline{\sigma}_h^n,\underline{\sigma}_h^n)_\Omega = \mathcal{K}\left(\delta_t^\alpha\mathbf{u}_h^n,\underline{\sigma}_h^n\right), \tag{3.40}$$

$$\mathcal{Q}\left(\delta_t^\alpha\mathbf{u}_h^n,p_h^n\right)=0, \tag{3.41}$$

by taking $(\underline{r}_h,w_h)=(\underline{\sigma}_h^n,p_h^n)$. Then, adding (3.39)-(3.41) and using Lemma 3.4, we obtain

$$(\delta_t^\alpha\underline{\sigma}_h^n,\underline{\sigma}_h^n)_\Omega + \left\|\delta_t^\alpha\mathbf{u}_h^n\right\|_\Omega^2 = \left(\mathbf{f}^n,\delta_t^\alpha\mathbf{u}_h^n\right)_\Omega. \tag{3.42}$$

Employing the definition of δ_t^α, it is easy to see that

$$\begin{aligned}\left\|\underline{\sigma}_h^n\right\|_\Omega^2 + \beta\left\|\delta_t^\alpha\mathbf{u}_h^n\right\|_\Omega^2 &= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left(\underline{\sigma}_h^i,\underline{\sigma}_h^n\right) + b_{n-1}\left(\underline{\sigma}_h^0,\underline{\sigma}_h^n\right) + \beta\left(\mathbf{f}^n,\delta_t^\alpha\mathbf{u}_h^n\right)_\Omega\\ &\leq \frac{1}{2}\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left\|\underline{\sigma}_h^i\right\|_\Omega^2 + \frac{1}{2}(b_0-b_{n-1})\left\|\underline{\sigma}_h^n\right\|_\Omega^2\\ &\quad + b_{n-1}\left\|\underline{\sigma}_h^0\right\|_\Omega^2 + \frac{b_{n-1}}{4}\left\|\underline{\sigma}_h^n\right\|_\Omega^2 + \beta\|\mathbf{f}^n\|_\Omega^2 + \frac{\beta}{4}\left\|\delta_t^\alpha\mathbf{u}_h^n\right\|_\Omega^2.\end{aligned}$$

It immediately follows that

$$\begin{aligned}\left\|\underline{\sigma}_h^n\right\|_\Omega^2 &\leq \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left\|\underline{\sigma}_h^i\right\|_\Omega^2 + 2b_{n-1}\left\|\underline{\sigma}_h^0\right\|_\Omega^2 + 2\beta\|\mathbf{f}^n\|_\Omega^2\\ &\leq \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left\|\underline{\sigma}_h^i\right\|_\Omega^2 + 2b_{n-1}\left\|\underline{\sigma}_h^0\right\|_\Omega^2 + 2\Gamma(1-\alpha)T^\alpha b_{n-1}\|\mathbf{f}^n\|_\Omega^2,\end{aligned}$$

where (3.36) is utilized. Thus, by using the mathematical induction, we complete the proof of (3.32).

Finally, we show the inequality (3.33). According to Lemma 3.1, we have, for $p_h^n\in L_0^2(\Omega)$, there exists a $\mathbf{u}^\star\in H_0^1(\Omega)^2$ such that

$$\|\mathbf{u}^\star\|_{H^1(\Omega)} \leq C_2\|p_h^n\|_\Omega, \tag{3.43}$$

and

$$\|p_h^n\|_\Omega^2 \leq C_*\Big(\mathcal{K}\left(\mathbb{P}_h\mathbf{u}^\star,\underline{\sigma}_h^n\right) + \mathcal{Q}\left(\mathbb{P}_h\mathbf{u}^\star,p_h^n\right) + \left\|\underline{\sigma}_h^n\right\|_\Omega^2\Big), \tag{3.44}$$

where we choose $\underline{r}_h=\underline{\sigma}_h^n$ in (3.19).

Using Lemma 3.4 and Eq. (3.28a) gives

$$
\begin{aligned}
&\mathcal{K}\left(\mathbb{P}_h\mathbf{u}^\star, \underline{\sigma}_h^n\right) + \mathcal{Q}\left(\mathbb{P}_h\mathbf{u}^\star, p_h^n\right) \\
=& -\mathcal{L}\left(\underline{\sigma}_h^n, \mathbb{P}_h\mathbf{u}^\star\right) - \mathcal{P}\left(p_h^n, \mathbb{P}_h\mathbf{u}^\star\right) \\
=& -\left(\delta_t^\alpha \mathbf{u}_h^n, \mathbb{P}_h\mathbf{u}^\star\right)_\Omega + \left(\mathbf{f}^n, \mathbb{P}_h\mathbf{u}^\star\right)_\Omega.
\end{aligned} \tag{3.45}
$$

Denote

$$
\mathbf{U}_h^n = \mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \quad \underline{\Xi}_h^n = \underline{\sigma}_h^{n+1} - \underline{\sigma}_h^n, \quad P_h^n = p_h^{n+1} - p_h^n, \quad \mathbf{F}^n = \mathbf{f}^{n+1} - \mathbf{f}^n.
$$

Then we have from (3.28) that

$$
\begin{cases}
\left(\delta_t^\alpha \mathbf{U}_h^n, \mathbf{v}_h\right)_\Omega = \mathcal{L}\left(\underline{\Xi}_h^n, \mathbf{v}_h\right) + \mathcal{P}\left(P_h^n, \mathbf{v}_h\right) + \left(\mathbf{F}^n, \mathbf{v}_h\right)_\Omega - \dfrac{b_n}{\beta}\left(\mathbf{U}_h^0, \mathbf{v}_h\right)_\Omega, \\
\left(\underline{\Xi}_h^n, \underline{r}_h\right)_\Omega = \mathcal{K}\left(\mathbf{U}_h^n, \underline{r}_h\right), \\
\mathcal{Q}\left(\mathbf{U}_h^n, w_h\right) = 0.
\end{cases} \tag{3.46}
$$

Let $(\mathbf{v}_h, \underline{r}_h, w_h) = (\mathbf{U}_h^n, \underline{\Xi}_h^n, P_h^n)$ in (3.46). Similar to the proof of (3.31), the following inequality can be reached:

$$
\left\|\mathbf{U}_h^n\right\|_\Omega^2 \le 2\left\|\mathbf{U}_h^0\right\|_\Omega^2 + 2\big(T^\alpha\Gamma(1-\alpha)\big)^2 \max_{1\le j\le n}\left\|\mathbf{F}^j\right\|_\Omega^2. \tag{3.47}
$$

Therefore,

$$
\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|_\Omega^2 \le 2\left\|\mathbf{u}_h^1 - \mathbf{u}_h^0\right\|_\Omega^2 + 2\big(T^\alpha\Gamma(1-\alpha)\big)^2 \max_{1\le j\le n}\|\mathbf{f}^{j+1} - \mathbf{f}^j\|_\Omega^2. \tag{3.48}
$$

By using Cauchy-Schwarz inequality, (3.17), (3.43), and (3.48), we get

$$
\begin{aligned}
&-\left(\delta_t^\alpha \mathbf{u}_h^n, \mathbb{P}_h\mathbf{u}^\star\right)_\Omega = \left(-\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1} b_j \frac{\mathbf{u}_h^{n-j} - \mathbf{u}_h^{n-j-1}}{\tau^\alpha}, \mathbb{P}_h\mathbf{u}^\star\right)_\Omega \\
\le& \frac{\left[2\left\|\mathbf{u}_h^1 - \mathbf{u}_h^0\right\|_\Omega^2 + 2\big(T^\alpha\Gamma(1-\alpha)\big)^2 \max\limits_{1\le j\le n}\|\mathbf{f}^{j+1} - \mathbf{f}^j\|_\Omega^2\right]^{\frac{1}{2}} n^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha}\|\mathbb{P}_h\mathbf{u}^\star\|_\Omega \\
\le& \frac{\left[2\left\|\mathbf{u}_h^1 - \mathbf{u}_h^0\right\|_\Omega^2 + 2\big(T^\alpha\Gamma(1-\alpha)\big)^2 \max\limits_{1\le j\le n}\|\mathbf{f}^{j+1} - \mathbf{f}^j\|_\Omega^2\right] n^{2-2\alpha}}{\big(\Gamma(2-\alpha)\tau^\alpha\big)^2} + \frac{\|p_h^n\|_\Omega^2}{4C_*}
\end{aligned} \tag{3.49}
$$

and

$$
\left(\mathbf{f}^n, \mathbb{P}_h\mathbf{u}^\star\right)_\Omega \le \|\mathbf{f}^n\|_\Omega \|\mathbb{P}_h\mathbf{u}^\star\|_\Omega \le C_* C_2^2 \|\mathbf{f}^n\|_\Omega^2 + \frac{\|p_h^n\|_\Omega^2}{4C_*}. \tag{3.50}
$$

Substituting (3.49) and (3.50) into (3.45) yields

$$\begin{aligned}&\mathcal{K}\left(\mathbb{P}_h\mathbf{u}^\star,\underline{\sigma}_h^n\right)+\mathcal{Q}\left(\mathbb{P}_h\mathbf{u}^\star,p_h^n\right)\\ \leq&\frac{\left[2\left\|\mathbf{u}_h^1-\mathbf{u}_h^0\right\|_\Omega^2+2\big(T^\alpha\Gamma(1-\alpha)\big)^2\max\limits_{1\le j\le n}\|\mathbf{f}^{j+1}-\mathbf{f}^j\|_\Omega^2\right]n^{2-2\alpha}}{\big(\Gamma(2-\alpha)\tau^\alpha\big)^2}\\&+C_*C_2^2\|\mathbf{f}^n\|_\Omega^2+\frac{\|p_h^n\|_\Omega^2}{2C_*}.\end{aligned}$$

Then from (3.32) and (3.44), we have

$$\left\|p_h^n\right\|_\Omega^2\le C\left(\frac{\left\|\mathbf{u}_h^1-\mathbf{u}_h^0\right\|_\Omega^2+\max\limits_{1\le j\le n}\|\mathbf{f}^{j+1}-\mathbf{f}^j\|_\Omega^2}{n^{2\alpha-2}\tau^{2\alpha}}+\left\|\underline{\sigma}_h^0\right\|_\Omega^2+\max_{1\le j\le n}\left\|\mathbf{f}^j\right\|_\Omega^2\right).$$

The proof is thus completed. □

The numerical stability of the stress and the pressure (see (3.32) and (3.33)) has never been obtained before for the time-fractional Stokes equation. Besides, from (3.32) and (3.33), one can see that the H^1 numerical stability of the velocity in scheme (3.28) is also derived.

3.3.2. Error estimate

In this subsection, we study the error estimate for the pressure, the stress (gradient of velocity), and the velocity of fully discrete L1/LDG scheme (3.28) for Eq. (3.1). We assume that the exact solution has the following smoothness properties:

$$\mathbf{u}(\mathbf{x},t)\in C^2\big([0,T];H^{k+2}(\Omega)\big)^2,\quad p(\mathbf{x},t)\in L^\infty\big(0,T;H^{k+2}(\Omega)\big). \tag{3.51}$$

Theorem 3.2. *Suppose that $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ satisfy the regularity assumption* (3.51)*. Let $(\mathbf{u}(\mathbf{x},t_n),\underline{\sigma}(\mathbf{x},t_n),p(\mathbf{x},t_n))$ be the exact solution of* (3.3) *and $(\mathbf{u}_h^n,\underline{\sigma}_h^n,p_h^n)\in\mathbf{V}_h\times\underline{\Sigma}_h\times Q_h$ be the numerical solution of the fully discrete uniform L1/LDG scheme* (3.28)*, respectively. Then for $n=1,2,\ldots,M$, the following error estimate holds:*

$$\left\|\mathbf{u}(\mathbf{x},t_n)-\mathbf{u}_h^n\right\|_\Omega+\left\|\underline{\sigma}(\mathbf{x},t_n)-\underline{\sigma}_h^n\right\|_\Omega+\tau^{\frac{\alpha}{2}}\left\|p(\mathbf{x},t_n)-p_h^n\right\|_\Omega\le C\big(\tau^{2-\alpha}+h^{k+1}\big), \tag{3.52}$$

where C is a positive constant which does not depend on time step length τ and space mesh h.

Proof. Let us start by denoting

$$\begin{aligned}\mathbf{e}^n=\big(\mathbf{e}_\mathbf{u}^n,\underline{e}_\sigma^n,e_p^n\big)&=\big(\mathbf{u}(\mathbf{x},t_n)-\mathbf{u}_h^n,\underline{\sigma}(\mathbf{x},t_n)-\underline{\sigma}_h^n,p(\mathbf{x},t_n)-p_h^n\big)\\&:=\big(\mathbf{u}^n-\mathbf{u}_h^n,\underline{\sigma}^n-\underline{\sigma}_h^n,p^n-p_h^n\big).\end{aligned} \tag{3.53}$$

With the help of the Stokes projection defined in (3.20)-(3.24), we split the error of e^n into two parts, i.e., $\mathbf{e}^n = \boldsymbol{\xi}^n - \boldsymbol{\eta}^n$, where

$$\boldsymbol{\xi}^n = \left(\boldsymbol{\xi}_{\mathbf{u}}^n, \underline{\xi}_{\sigma}^n, \xi_p^n\right) = \left(\boldsymbol{\Pi_h}\mathbf{u}^n - \mathbf{u}_h^n, \underline{\Pi_h}\underline{\sigma}^n - \underline{\sigma}_h^n, \Pi_h p^n - p_h^n\right), \tag{3.54}$$

$$\boldsymbol{\eta}^n = \left(\boldsymbol{\eta}_{\mathbf{u}}^n, \underline{\eta}_{\sigma}^n, \eta_p^n\right) = \left(\boldsymbol{\Pi_h}\mathbf{u}^n - \mathbf{u}^n, \underline{\Pi_h}\underline{\sigma}^n - \underline{\sigma}^n, \Pi_h p^n - p^n\right). \tag{3.55}$$

By Lemma 3.2, we have

$$\left\|\boldsymbol{\eta}_{\mathbf{u}}^n\right\|_{\Omega} + \left\|\underline{\eta}_{\sigma}^n\right\|_{\Omega} + \left\|\eta_p^n\right\|_{\Omega} \leq Ch^{k+1}. \tag{3.56}$$

In what follows, we will focus on the estimate for $\boldsymbol{\xi}^n$.

Using (3.3), we obtain

$$\begin{cases} \left({}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}^n, \mathbf{v}_h\right)_{\Omega} = \mathcal{L}\left(\underline{\sigma}^n, \mathbf{v}_h\right) + \mathcal{P}\left(p^n, \mathbf{v}_h\right) + \left(\mathbf{f}^n, \mathbf{v}_h\right)_{\Omega}, \\ \left(\underline{\sigma}^n, \underline{r}_h\right)_{\Omega} = \mathcal{K}\left(\mathbf{u}^n, \underline{r}_h\right), \\ \mathcal{Q}\left(\mathbf{u}^n, w_h\right) = 0, \end{cases} \tag{3.57}$$

where $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ is the test function and ${}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}^n := {}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}|_{t=t_n}$. Applying the property of Stokes projection (3.20)-(3.22), it holds that

$$\begin{cases} \mathcal{L}\left(\underline{\eta}_{\sigma}^n, \mathbf{v}_h\right) + \mathcal{P}\left(\eta_p^n, \mathbf{v}_h\right) = 0, \\ \mathcal{K}\left(\boldsymbol{\eta}_{\mathbf{u}}^n, \underline{r}_h\right) = \left(\underline{\eta}_{\sigma}^n, \underline{r}_h\right)_{\Omega}, \\ \mathcal{Q}\left(\boldsymbol{\eta}_{\mathbf{u}}^n, w_h\right) = 0. \end{cases} \tag{3.58}$$

Subtracting (3.28) from (3.57) with $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ and using (3.58), we obtain the following error equation:

$$\begin{cases} \left({}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}^n - \delta_t^{\alpha}\mathbf{u}_h^n, \mathbf{v}_h\right)_{\Omega} = \mathcal{L}\left(\underline{\xi}_{\sigma}^n, \mathbf{v}_h\right) + \mathcal{P}\left(\xi_p^n, \mathbf{v}_h\right), \\ \left(\underline{\xi}_{\sigma}^n, \underline{r}_h\right)_{\Omega} = \mathcal{K}\left(\boldsymbol{\xi}_{\mathbf{u}}^n, \underline{r}_h\right), \\ \mathcal{Q}\left(\boldsymbol{\xi}_{\mathbf{u}}^n, w_h\right) = 0. \end{cases} \tag{3.59}$$

Taking the test function $(\mathbf{v}_h, \underline{r}_h, w_h) = (\boldsymbol{\xi}_{\mathbf{u}}^n, \underline{\xi}_{\sigma}^n, \xi_p^n)$ in (3.59) and adding them together yield

$$\left({}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}^n - \delta_t^{\alpha}\mathbf{u}_h^n, \boldsymbol{\xi}_{\mathbf{u}}^n\right)_{\Omega} + \left\|\underline{\xi}_{\sigma}^n\right\|_{\Omega}^2 = 0, \tag{3.60}$$

where Lemma 3.4 is utilized. Denoting $\mathbf{R}^n = \delta_t^{\alpha}\mathbf{u}^n - {}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}^n$, we obtain that

$$\left(\delta_t^{\alpha}\boldsymbol{\xi}_{\mathbf{u}}^n, \boldsymbol{\xi}_{\mathbf{u}}^n\right) \leq \left(\mathbf{R}^n, \boldsymbol{\xi}_{\mathbf{u}}^n\right) + \left(\delta_t^{\alpha}\boldsymbol{\eta}_{\mathbf{u}}^n, \boldsymbol{\xi}_{\mathbf{u}}^n\right). \tag{3.61}$$

Based on the definition of operator δ_t^{α} and Cauchy-Schwarz inequality, the above equation can be further estimated as

$$\begin{aligned} \left\|\boldsymbol{\xi}_{\mathbf{u}}^n\right\|_{\Omega}^2 \leq & \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\left(\boldsymbol{\xi}_{\mathbf{u}}^i, \boldsymbol{\xi}_{\mathbf{u}}^n\right) + \frac{b_{n-1}}{4}\left\|\boldsymbol{\xi}_{\mathbf{u}}^n\right\|_{\Omega}^2 + \frac{b_{n-1}}{4}\left\|\boldsymbol{\xi}_{\mathbf{u}}^n\right\|_{\Omega}^2 \\ & + Cb_{n-1}\left(h^{2k+2} + \tau^{4-2\alpha}\right) \\ \leq & \frac{1}{2}\sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\left\|\boldsymbol{\xi}_{\mathbf{u}}^i\right\|_{\Omega}^2 + \frac{1}{2}(b_0 - b_{n-1})\left\|\boldsymbol{\xi}_{\mathbf{u}}^n\right\|_{\Omega}^2 + \frac{b_{n-1}}{2}\left\|\boldsymbol{\xi}_{\mathbf{u}}^n\right\|_{\Omega}^2 \\ & + Cb_{n-1}\left(h^{2k+2} + \tau^{4-2\alpha}\right). \end{aligned} \tag{3.62}$$

Therefore,

$$\|\boldsymbol{\xi}_{\mathbf{u}}^n\|_\Omega^2 \le \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\|\boldsymbol{\xi}_{\mathbf{u}}^i\|_\Omega^2 + Cb_{n-1}\big(h^{2k+2}+\tau^{4-2\alpha}\big). \tag{3.63}$$

Now we prove that $\boldsymbol{\xi}_{\mathbf{u}}^n$ satisfies

$$\|\boldsymbol{\xi}_{\mathbf{u}}^n\|_\Omega^2 \le C\big(h^{2k+2}+\tau^{4-2\alpha}\big), \tag{3.64}$$

where C is the constant in (3.63). We prove the above inequality by the mathematical induction. When $n=1$, (3.63) becomes

$$\|\boldsymbol{\xi}_{\mathbf{u}}^1\|_\Omega^2 \le C\big(h^{2k+2}+\tau^{4-2\alpha}\big), \tag{3.65}$$

which is true. Assuming the following estimates hold:

$$\|\boldsymbol{\xi}_{\mathbf{u}}^m\|_\Omega^2 \le C\big(h^{2k+2}+\tau^{4-2\alpha}\big), \quad m=2,3,\ldots,s, \tag{3.66}$$

we only need to prove

$$\|\boldsymbol{\xi}_{\mathbf{u}}^{s+1}\|^2 \le C\big(h^{2k+2}+\tau^{4-2\alpha}\big). \tag{3.67}$$

Let $n=s+1$ in (3.63), and then by the induction hypothesis (3.66), we can get the desired estimate. For the estimate of $\underline{\xi}_\sigma^n$, we can use (3.59), Lemma 3.4, and the mathematical induction to obtain

$$\|\underline{\xi}_\sigma^n\|_\Omega \le C\big(h^{k+1}+\tau^{2-\alpha}\big). \tag{3.68}$$

By Lemma 3.1, for any $\xi_p^n \in L_0^2(\Omega)$, there exists a $\mathbf{p}^\star \in H_0^1(\Omega)^2$ such that

$$\|\mathbf{p}^\star\|_{H^1(\Omega)} \le C_2\|\xi_p^n\|_\Omega \tag{3.69}$$

and

$$\|\xi_p^n\|_\Omega^2 \le C_{**}\left(\mathcal{K}\left(\mathbb{P}_h\mathbf{p}^\star,\underline{\xi}_\sigma^n\right)+\mathcal{Q}\left(\mathbb{P}_h\mathbf{p}^\star,\xi_p^n\right)+\|\underline{\xi}_\sigma^n\|_\Omega^2\right). \tag{3.70}$$

It follows from Lemma 3.4, and the relations (3.17), (3.59), (3.69) that

$$\begin{aligned}
&\mathcal{K}\big(\mathbb{P}_h\mathbf{p}^\star,\underline{\xi}_\sigma^n\big)+\mathcal{Q}\left(\mathbb{P}_h\mathbf{p}^\star,\xi_p^n\right)\\
&=-\mathcal{L}\big(\underline{\xi}_\sigma^n,\mathbb{P}_h\mathbf{p}^\star\big)-\mathcal{P}\left(\xi_p^n,\mathbb{P}_h\mathbf{p}^\star\right)=-\big({}_C\mathrm{D}_{0,t}^\alpha\mathbf{u}^n-\delta_t^\alpha\mathbf{u}_h^n,\mathbb{P}_h\mathbf{p}^\star\big)_\Omega\\
&\le\big\|{}_C\mathrm{D}_{0,t}^\alpha\mathbf{u}^n-\delta_t^\alpha\mathbf{u}_h^n\big\|_\Omega\|\mathbb{P}_h\mathbf{p}^\star\|_\Omega\le C\big(h^{k+1}+\tau^{2-\alpha}+\tau^{2-\frac{3\alpha}{2}}+\tau^{-\frac{\alpha}{2}}h^{k+1}\big)\|\xi_p^n\|_\Omega\\
&\le C\big(h^{k+1}+\tau^{-\frac{\alpha}{2}}h^{k+1}+\tau^{2-\frac{3\alpha}{2}}\big)^2+\frac{1}{2C_{**}}\|\xi_p^n\|_\Omega^2.
\end{aligned} \tag{3.71}$$

Substituting (3.71) into (3.70) and applying (3.68) give

$$\tau^{\frac{\alpha}{2}}\|\xi_p^n\|_\Omega \le C\big(h^{k+1}+\tau^{2-\alpha}\big). \tag{3.72}$$

Finally, combining the results (3.64), (3.68) and (3.72) with (3.56) and using triangle inequality, we complete the proof of (3.52). □

Remark 3.1. The error estimates of the stress and the pressure which have never been derived are obtained here. In addition, one can see that the H^1 estimate of the velocity of scheme (3.28) is derived from (3.52).

Generally speaking, the solution to the time-fractional differential equation may have weak regularity at the initial time since the (left) fractional derivative is used. If the uniform meshes are used, then numerical results have low computational efficiency. In order to overcome this, the non-uniform meshes have to be used. In the following subsection, we study this case.

3.4. Numerical analysis on non-uniform time meshes

In general, $\mathbf{u}(\mathbf{x},t)$ likely behaves weakly regular at the starting time $t=0$ due to the use of the (left) fractional Caputo derivative, i.e., $|\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}|$ and/or $|\frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2}|$ will blow up when $t\to 0^+$ albeit $\mathbf{u}(\mathbf{x},t)$ is continuous on $[0,T]$. In order to guarantee the computational efficiency, we use L1 method on non-uniform meshes for time fractional derivative, and still use the LDG method for spatial derivative.

For a given $T>0$, let $t_n=T(n/M)^r$, $n=0,1,\ldots,M$ be the mesh points, $r\geq 1$. Let $\tau_n=t_n-t_{n-1}$, $n=1,\ldots,M$ be the time mesh size. If $r=1$, then the mesh is uniform.

The Caputo fractional derivative

$$ {}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u}(\mathbf{x},t_n)=\big({}_C\mathrm{D}_{0,t}^{\alpha}u_1(\mathbf{x},t_n),{}_C\mathrm{D}_{0,t}^{\alpha}u_2(\mathbf{x},t_n)\big)^{\intercal} $$

for $n\geq 1$ can be written as

$$ {}_C\mathrm{D}_{0,t}^{\alpha}u_j(\mathbf{x},t_n)=\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{-\alpha}\frac{\partial u_j(\mathbf{x},s)}{\partial s}\mathrm{d}s,\quad j=1,2, $$

which is approximated by the classical L1 formula on the non-uniform meshes (or non-uniform L1 scheme)

$$ \begin{aligned} \Upsilon_t^{\alpha}u_j^n&=\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n-1}\frac{u_j^{i+1}-u_j^i}{\tau_{i+1}}\int_{t_i}^{t_{i+1}}(t_n-s)^{-\alpha}\mathrm{d}s\\ &=\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n-1}\frac{u_j^{i+1}-u_j^i}{\tau_{i+1}}\left[(t_n-t_i)^{1-\alpha}-(t_n-t_{i+1})^{1-\alpha}\right]\\ &=\frac{1}{\Gamma(2-\alpha)}\left[d_{n,1}u_j^n+\sum_{i=1}^{n-1}(d_{n,i+1}-d_{n,i})u_j^{n-i}-d_{n,n}u_j^0\right] \end{aligned} \tag{3.73} $$

for $n=1,\ldots,M$, and $j=1,2$. Here

$$ d_{n,1}=\tau_n^{-\alpha},\quad d_{n,i}=\frac{(t_n-t_{n-i})^{1-\alpha}-(t_n-t_{n-i+1})^{1-\alpha}}{\tau_{n-i+1}},\quad i=2,3,\ldots,n. $$

Note that the coefficients $d_{n,i}$ can be estimated as follows:

$$d_{n,i+1} \leq d_{n,i}, \tag{3.74}$$

and

$$(1-\alpha)(t_n - t_{n-i})^{-\alpha} < d_{n,i} < (1-\alpha)(t_n - t_{n-i+1})^{-\alpha} \tag{3.75}$$

for $1 \leq i \leq n-1$.

The truncation error $\mathbf{Q}^n$ at time $t = t_n$ is defined by

$$\mathbf{Q}^n = (Q_1^n, Q_2^n)^\intercal := \left({}_C\mathrm{D}_{0,t}^\alpha u_1 - \Upsilon_t^\alpha u_1^n, {}_C\mathrm{D}_{0,t}^\alpha u_2 - \Upsilon_t^\alpha u_2^n\right)^\intercal. \tag{3.76}$$

Denote

$$P_{n-k}^{(n)} = \frac{1}{a_0^{(k)}} \begin{cases} 1, & k = n, \\ \sum\limits_{j=k+1}^{n} \left(a_{j-k-1}^{(j)} - a_{j-k}^{(j)}\right) P_{n-j}^{(n)}, & 1 \leq k \leq n-1, \end{cases}$$

where

$$a_{j-k}^{(j)} = \frac{d_{j,j-k+1}}{\Gamma(2-\alpha)}.$$

Throughout this subsection, we assume that there exists a unique solution $\mathbf{u}(\mathbf{x}, t)$ of Eq. (3.1) such that

$$\left|\frac{\partial^l \mathbf{u}(\mathbf{x},t)}{\partial t^l}\right| \leq C\left(1 + t^{\alpha - l}\right), \quad l = 0, 1, 2, \tag{3.77}$$

where

$$\left|\frac{\partial^l \mathbf{u}(\mathbf{x},t)}{\partial t^l}\right| = \sqrt{\left(\frac{\partial^l u_1(\mathbf{x},t)}{\partial t^l}\right)^2 + \left(\frac{\partial^l u_2(\mathbf{x},t)}{\partial t^l}\right)^2}.$$

Such a regularity assumption with time t is often used, see e.g., [8, 15, 22]. Then with the above assumptions, we have the following lemma.

Lemma 3.5 ([24])**.** *Assume that* (3.77) *holds for any fixed* $\mathbf{x}$. *Then there exists a constant* C *such that for all* t_n

$$|\mathbf{Q}^n| \leq C n^{-\min\{2-\alpha, r\alpha\}}, \quad n = 1, \ldots, M,$$

where $|\mathbf{Q}^n| = \sqrt{(Q_1^n)^2 + (Q_2^n)^2}$.

The weak form of the time-fractional Stokes equation at t_n is formulated as

$$\begin{cases} \left({}_C\mathrm{D}_{0,t}^\alpha \mathbf{u}^n, \mathbf{v}_h\right)_\Omega = \mathcal{L}\left(\underline{\sigma}^n, \mathbf{v}_h\right) + \mathcal{P}(p^n, \mathbf{v}_h) + (\mathbf{f}^n, \mathbf{v}_h)_\Omega, \\ (\underline{\sigma}^n, \underline{r}_h)_\Omega = \mathcal{K}\left(\mathbf{u}^n, \underline{r}_h\right), \\ \mathcal{Q}(\mathbf{u}^n, w_h) = 0. \end{cases} \tag{3.78}$$

Here $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ is an arbitrary test function and the notations $\mathcal{L}, \mathcal{P}, \mathcal{K}$, and $\mathcal{Q}$ are defined in equations (3.5)-(3.8).

Let $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ be an approximation of $(\mathbf{u}(\mathbf{x}, t_n), \underline{\sigma}(\mathbf{x}, t_n), p(\mathbf{x}, t_n))$. We define the fully discrete non-uniform L1/LDG scheme as follows: Find $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ such that for all test functions $(\mathbf{v}_h, \underline{r}_h, w_h) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$,

$$\begin{cases} (\Upsilon_t^\alpha \mathbf{u}_h^n, \mathbf{v}_h)_\Omega = \mathcal{L}(\underline{\sigma}_h^n, \mathbf{v}_h) + \mathcal{P}(p_h^n, \mathbf{v}_h) + (\mathbf{f}^n, \mathbf{v}_h)_\Omega, \\ (\underline{\sigma}_h^n, \underline{r}_h)_\Omega = \mathcal{K}(\mathbf{u}_h^n, \underline{r}_h), \\ \mathcal{Q}(\mathbf{u}_h^n, w_h) = 0. \end{cases} \tag{3.79}$$

3.4.1. Numerical stability analysis

The non-uniform L1/LDG scheme (3.79) using the numerical flux (3.9) for the two-dimensional time-fractional Stokes equation satisfies the following L^2-stability. Let us start by introducing the following lemma.

Lemma 3.6 ([18]). *For any finite time $t_M = T > 0$ and a given nonnegative sequence $(\lambda_l)_{l=0}^{M-1}$, assume that there exists a constant λ, independent of time-steps such that $\lambda \geq \sum_{l=0}^{M-1} \lambda_l$. Suppose that $\{v^n | n \geq 0\}$ is a grid function such that*

$$\Upsilon_t^\alpha v^n \leq \sum_{l=1}^{n} \lambda_{n-l} v^l + \xi^n + \eta^n, \quad 1 \leq n \leq M,$$

where $\{\xi^n, \eta^n \,|\, 1 \leq n \leq M\}$ are nonnegative sequences. If the maximum time-step τ_M satisfies the estimate

$$\tau_M \leq \left(2\Gamma(2-\alpha)\lambda\right)^{-\frac{1}{\alpha}},$$

then for $1 \leq n \leq M$,

$$v^n \leq 2E_{\alpha,1}(2\lambda t_n^\alpha)\left(v^0 + \max_{1\leq k\leq n} \sum_{j=1}^{k} P_{k-j}^{(k)} \xi^j + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{1\leq k\leq n} \eta^k\right).$$

Here $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ is the Mittag-Leffler function.

Theorem 3.3. *The numerical solution $(\mathbf{u}_h^n, \underline{\sigma}_h^n)$ of non-uniform L1/LDG scheme* (3.79) *for* (3.1) *is stable, i.e., for $n = 1, \ldots, M$, one has*

$$\|\mathbf{u}_h^n\|_\Omega \leq 2\|\mathbf{u}_h^0\|_\Omega + \frac{2t_n^\alpha}{\Gamma(1+\alpha)} \max_{1\leq j\leq n} \|\mathbf{f}^j\|_\Omega, \tag{3.80}$$

$$\|\underline{\sigma}_h^n\|_\Omega^2 \leq 2\|\underline{\sigma}_h^0\|_\Omega^2 + \frac{2t_n^\alpha}{\Gamma(1+\alpha)} \max_{1\leq j\leq n} \|\mathbf{f}^j\|_\Omega^2. \tag{3.81}$$

Proof. We start with the inequality (3.80). Taking the test functions $(\mathbf{v}_h, \underline{r}_h, w_h) = (\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n)$ in scheme (3.79) and adding them together, we arrive at

$$(\Upsilon_t^\alpha \mathbf{u}_h^n, \mathbf{u}_h^n)_\Omega + \|\underline{\sigma}_h^n\|_\Omega^2 \leq (\mathbf{f}^n, \mathbf{u}_h^n)_\Omega, \tag{3.82}$$

by using Lemma 3.4. By the definition of operator Υ_t^α and Cauchy-Schwarz inequality, there holds

$$\begin{aligned}(\Upsilon_t^\alpha \mathbf{u}_h^n, \mathbf{u}_h^n)_\Omega = & \frac{d_{n,1}}{\Gamma(2-\alpha)} (\mathbf{u}_h^n, \mathbf{u}_h^n)_\Omega \\ & + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} (d_{n,i+1} - d_{n,i}) \left(\mathbf{u}_h^{n-i}, \mathbf{u}_h^n\right)_\Omega \\ & - \frac{d_{n,n}}{\Gamma(2-\alpha)} \left(\mathbf{u}_h^0, \mathbf{u}_h^n\right)_\Omega \\ \geq & \left(\Upsilon_t^\alpha \|\mathbf{u}_h^n\|_\Omega\right) \|\mathbf{u}_h^n\|_\Omega. \end{aligned} \tag{3.83}$$

Substituting (3.83) into (3.82) yields

$$\Upsilon_t^\alpha \|\mathbf{u}_h^n\|_\Omega \leq \|\mathbf{f}^n\|_\Omega. \tag{3.84}$$

By using Lemma 3.6 with $v^n = \|\mathbf{u}_h^n\|_\Omega$ and $\eta^n = \|f^n\|_\Omega$, we get (3.80).

In order to show the estimate (3.81), we take $(\mathbf{v}_h, \underline{r}_h, w_h) = (\Upsilon_t^\alpha \mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n)$ in (3.79), thus obtaining

$$\begin{cases} (\Upsilon_t^\alpha \mathbf{u}_h^n, \Upsilon_t^\alpha \mathbf{u}_h^n)_\Omega = \mathcal{L}\left(\underline{\sigma}_h^n, \Upsilon_t^\alpha \mathbf{u}_h^n\right) + \mathcal{P}\left(p_h^n, \Upsilon_t^\alpha \mathbf{u}_h^n\right) + \left(\mathbf{f}^n, \Upsilon_t^\alpha \mathbf{u}_h^n\right)_\Omega, \\ (\Upsilon_t^\alpha \underline{\sigma}_h^n, \underline{\sigma}_h^n)_\Omega = \mathcal{K}\left(\Upsilon_t^\alpha \mathbf{u}_h^n, \underline{\sigma}_h^n\right), \\ \mathcal{Q}\left(\Upsilon_t^\alpha \mathbf{u}_h^n, p_h^n\right) = 0. \end{cases} \tag{3.85}$$

Applying Lemma 3.4 and (3.85) and using the Cauchy-Schwarz inequality give

$$(\Upsilon_t^\alpha \underline{\sigma}_h^n, \underline{\sigma}_h^n)_\Omega + \left\|\Upsilon_t^\alpha \mathbf{u}_h^n\right\|_\Omega^2 = (\mathbf{f}^n, \Upsilon_t^\alpha \mathbf{u}_h^n)_\Omega \leq \frac{1}{2}\|f^n\|_\Omega^2 + \frac{1}{2}\left\|\Upsilon_t^\alpha \mathbf{u}_h^n\right\|_\Omega^2,$$

which leads to

$$\Upsilon_t^\alpha \left\|\underline{\sigma}_h^n\right\|_\Omega^2 \leq \|f^n\|_\Omega^2. \tag{3.86}$$

Combining (3.86) and Lemma 3.6 with $v^n = \|\underline{\sigma}_h^n\|_\Omega^2$, $\eta^n = \|f^n\|_\Omega^2$ yields the proof. $\square$

Although the numerical stability of the pressure is not obtained, we have shown the numerical stability of the velocity and the stress (i.e., the H^1 numerical stability for the velocity) for the case with weak regularity at the initial time.

3.4.2. Error estimate

In the subsection, we consider the error estimates for the velocity and the stress of fully discrete non-uniform L1/LDG scheme (3.79) for Eq. (3.1). The main results are given in the following theorem.

Theorem 3.4. *Suppose that* $\mathbf{u}(\mathbf{x}, t)$ *satisfies the temporal regularity assumption* (3.77), $\mathbf{u}(\cdot, t) \in H^{k+2}(\Omega)^2$ *and* $p(\mathbf{x}, t) \in L^\infty(0, T; H^{k+2}(\Omega))$. *Let* $(\mathbf{u}(\mathbf{x}, t_n), \underline{\sigma}(\mathbf{x}, t_n), p(\mathbf{x}, t_n))$ *be the exact solution of* (3.3) *and* $(\mathbf{u}_h^n, \underline{\sigma}_h^n, p_h^n) \in \mathbf{V}_h \times \underline{\Sigma}_h \times Q_h$ *be the numerical solution of* *the fully discrete non-uniform L1/LDG scheme* (3.79)*, respectively. Then for* $n=1,\dots,M$*, the following error estimates hold:*

$$\left\|\mathbf{u}(\mathbf{x},t_n)-\mathbf{u}_h^n\right\|_\Omega \le C\big(M^{-\min\{2-\alpha,r\alpha\}}+h^{k+1}\big), \tag{3.87}$$
$$\left\|\underline{\sigma}(\mathbf{x},t_n)-\underline{\sigma}_h^n\right\|_\Omega \le C\big(M^{-\min\{2-\alpha,\frac{r\alpha}{2}\}}+h^{k+1}\big), \tag{3.88}$$

where C *is a positive constant independent of* M *and* h*.*

Proof. Let us show (3.87). We begin by denoting

$$\begin{aligned}\mathbf{e}^n=\big(\mathbf{e}_\mathbf{u}^n,\underline{e}_\sigma^n,e_p^n\big)&=\big(\mathbf{u}(\mathbf{x},t_n)-\mathbf{u}_h^n,\underline{\sigma}(\mathbf{x},t_n)-\underline{\sigma}_h^n,p(\mathbf{x},t_n)-p_h^n\big)\\&:=\big(\mathbf{u}^n-\mathbf{u}_h^n,\underline{\sigma}^n-\underline{\sigma}_h^n,p^n-p_h^n\big).\end{aligned} \tag{3.89}$$

By the Stokes projection defined in (3.20)-(3.24), we split the error of $\mathbf{e}^n$ into two parts, i.e., $\mathbf{e}^n=\boldsymbol{\xi}^n-\boldsymbol{\eta}^n$, where

$$\boldsymbol{\xi}^n=\big(\boldsymbol{\xi}_\mathbf{u}^n,\underline{\xi}_\sigma^n,\xi_p^n\big)=\big(\boldsymbol{\Pi_h}\mathbf{u}^n-\mathbf{u}_h^n,\underline{\Pi_h}\underline{\sigma}^n-\underline{\sigma}_h^n,\Pi_hp^n-p_h^n\big)\,, \tag{3.90}$$
$$\boldsymbol{\eta}^n=\big(\boldsymbol{\eta}_\mathbf{u}^n,\underline{\eta}_\sigma^n,\eta_p^n\big)=\big(\boldsymbol{\Pi_h}\mathbf{u}^n-\mathbf{u}^n,\underline{\Pi_h}\underline{\sigma}^n-\underline{\sigma}^n,\Pi_hp^n-p^n\big). \tag{3.91}$$

Applying Lemma 3.2, we have

$$\left\|\boldsymbol{\eta}_\mathbf{u}^n\right\|_\Omega+\left\|\underline{\eta}_\sigma^n\right\|_\Omega+\left\|\eta_p^n\right\|_\Omega\le Ch^{k+1}. \tag{3.92}$$

It remains to estimate $\boldsymbol{\xi}^n$. Subtracting (3.79) from (3.78) with $(\mathbf{v}_h,\underline{r}_h,w_h)\in\mathbf{V}_h\times\underline{\Sigma}_h\times Q_h$ and using (3.58), we obtain the following error equation:

$$\begin{cases}\big({}_C\mathrm{D}_{0,t}^\alpha\mathbf{u}^n-\Upsilon_t^\alpha\mathbf{u}_h^n,\mathbf{v}_h\big)_\Omega=\mathcal{L}\big(\underline{\xi}_\sigma^n,\mathbf{v}_h\big)+\mathcal{P}\left(\xi_p^n,\mathbf{v}_h\right),\\ \left(\underline{\xi}_\sigma^n,\underline{r}_h\right)_\Omega=\mathcal{K}\left(\boldsymbol{\xi}_\mathbf{u}^n,\underline{r}_h\right),\\ \mathcal{Q}\left(\boldsymbol{\xi}_\mathbf{u}^n,w_h\right)=0.\end{cases} \tag{3.93}$$

Taking the test function $(\mathbf{v}_h,\underline{r}_h,w_h)=(\boldsymbol{\xi}_\mathbf{u}^n,\underline{\xi}_\sigma^n,\xi_p^n)$ in (3.93) and adding them together lead to

$$\big({}_C\mathrm{D}_{0,t}^\alpha\mathbf{u}^n-\Upsilon_t^\alpha\mathbf{u}_h^n,\boldsymbol{\xi}_\mathbf{u}^n\big)_\Omega+\left\|\underline{\xi}_\sigma^n\right\|_\Omega^2=0,$$

where we have used Lemma 3.4. Substituting (3.89)-(3.91) into the above equation implies that

$$(\Upsilon_t^\alpha\boldsymbol{\xi}_\mathbf{u}^n,\boldsymbol{\xi}_\mathbf{u}^n)_\Omega\le-(\mathbf{Q}^n,\boldsymbol{\xi}_\mathbf{u}^n)_\Omega+(\Upsilon_t^\alpha\boldsymbol{\eta}_\mathbf{u}^n,\boldsymbol{\xi}_\mathbf{u}^n)_\Omega\,. \tag{3.94}$$

Then applying (3.83), (3.92), Lemma 3.5, and the Cauchy-Schwarz inequality, we obtain

$$\Upsilon_t^\alpha\left\|\boldsymbol{\xi}_\mathbf{u}^n\right\|_\Omega\le\|\mathbf{Q}^n\|_\Omega+\left\|\Upsilon_t^\alpha\boldsymbol{\eta}_\mathbf{u}^n\right\|_\Omega\le C\big(n^{-\min\{2-\alpha,r\alpha\}}+h^{k+1}\big). \tag{3.95}$$

It follows from [18] that

$$\sum_{j=1}^n P_{n-j}^{(n)}|\mathbf{Q}^j|\le C\bigg(\alpha^{-1}T^\alpha M^{-r\alpha}+\frac{r^2}{1-\alpha}4^{r-1}T^\alpha M^{-\min\{r\alpha,2-\alpha\}}\bigg),\quad n\ge1. \tag{3.96}$$

Using Lemma 3.6 and (3.95), (3.96), one has

$$\|\boldsymbol{\xi}_{\mathbf{u}}^n\|_\Omega \le C\big(h^{k+1} + M^{-\min\{2-\alpha,r\alpha\}}\big),$$

which, together with (3.92) and the triangle inequality, yields the optimal error estimate (3.87).

In order to prove (3.88), we take $(\mathbf{v}_h, \underline{r}_h, w_h) = (\Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n, \underline{\xi}_\sigma^n, \xi_p^n)$ into (3.93), thus obtaining

$$\begin{cases} \big({}_C\mathrm{D}_{0,t}^\alpha \mathbf{u}^n - \Upsilon_t^\alpha \mathbf{u}_h^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\big)_\Omega = \mathcal{L}\big(\underline{\xi}_\sigma^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\big) + \mathcal{P}\left(\xi_p^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\right), \\ \big(\Upsilon_t^\alpha \underline{\xi}_\sigma^n, \underline{\xi}_\sigma^n\big)_\Omega = \mathcal{K}\big(\Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n, \underline{\xi}_\sigma^n\big), \\ \mathcal{Q}\left(\boldsymbol{\xi}_{\mathbf{u}}^n, \xi_p^n\right) = 0. \end{cases}$$

By Lemma 3.4, we get

$$\big({}_C\mathrm{D}_{0,t}^\alpha \mathbf{u}^n - \Upsilon_t^\alpha \mathbf{u}_h^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\big)_\Omega + \big(\Upsilon_t^\alpha \underline{\xi}_\sigma^n, \underline{\xi}_\sigma^n\big)_\Omega = 0. \tag{3.97}$$

Applying (3.73), (3.76), and the Cauchy-Schwarz inequality, we deduce from (3.97) that

$$\begin{aligned} \Upsilon_t^\alpha \big\|\underline{\xi}_\sigma^n\big\|_\Omega^2 + 2\big\|\Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\big\|_\Omega^2 &\le 2\,(\Upsilon_t^\alpha \boldsymbol{\eta}_{\mathbf{u}}^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n)_\Omega - 2\,(\mathbf{Q}^n, \Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n)_\Omega \\ &\le \big\|\Upsilon_t^\alpha \boldsymbol{\eta}_{\mathbf{u}}^n\big\|_\Omega^2 + \|\mathbf{Q}^n\|_\Omega^2 + 2\big\|\Upsilon_t^\alpha \boldsymbol{\xi}_{\mathbf{u}}^n\big\|_\Omega^2, \end{aligned}$$

which, together with (3.92), leads to

$$\Upsilon_t^\alpha \big\|\underline{\xi}_\sigma^n\big\|_\Omega^2 \le C\|\mathbf{Q}^n\|_\Omega^2 + Ch^{2k+2}. \tag{3.98}$$

From [23, Lemma 3.3], we know that

$$\sum_{j=1}^n P_{n-j}^{(n)} \left\|\mathbf{Q}^j\right\|_\Omega^2 \le CM^{-2\min\{2-\alpha,\frac{r\alpha}{2}\}}. \tag{3.99}$$

Then, combining (3.98), (3.99), and Lemma 3.6, one has

$$\|\underline{\xi}_\sigma^n\|_\Omega \le C\big(M^{-\min\{2-\alpha,\frac{r\alpha}{2}\}} + h^{k+1}\big). \tag{3.100}$$

Thus we get (3.88) by using (3.92), (3.100), and the triangle inequality. This completes the proof. □

Remark 3.2. Note that the error estimate for the stress in Theorem 3.4 is suboptimal because of a loss of theoretical accuracy $\mathcal{O}(M^{-r\alpha/2})$ in time. The result indicates that the $(2-\alpha)$ order accuracy in time can be reached when we choose the graded parameter $r \ge \frac{4-2\alpha}{\alpha}$. The reason why the L^2-norm error estimate for the stress is suboptimal is that the global consistency error $\sum_{j=1}^n P_{n-j}^{(n)}\|\mathbf{Q}^n\|_\Omega^2 \le CM^{-2\min\{2-\alpha,\frac{r\alpha}{2}\}}$ has a loss of time accuracy. In order to check the actual temporal convergence rate for the stress, a numerical example is given in Section 4 (see details of Example 4.2). Tables 9-11 present the error estimates and the convergence rates for different α and r on the rectangular meshes. It is observed that scheme (3.79) achieves optimal convergence of order $M^{-\min\{2-\alpha,r\alpha\}}$ for the stress in time.

Remark 3.3. It could be interesting to check the convergence rates of the pressure for the model (3.1) with weaker regularity solution, although we cannot give the theoretical analysis at present. From Example 4.2, we conclude that the optimal order of convergence (i.e., $(2-\alpha)$-order accurate in time and $(k+1)$-order accurate in space) for the pressure can be obtained if we use non-uniform L1 formula in the temporal direction and the LDG method in space; see Tables 13 and 14.

4. Numerical experiments

In the current section, two numerical examples are shown to demonstrate the efficiency of the present numerical schemes for solving the time-fractional Stokes equation (3.1).

Example 4.1. Consider the following time-fractional Stokes equation:

$$\begin{cases} {}_C\mathrm{D}^{\alpha}_{0,t}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,y,t), & (x,y,t) \in (-1,1)\times(-1,1)\times(0,1], \\ \nabla\cdot\mathbf{u} = 0, & (x,y,t) \in (-1,1)\times(-1,1)\times(0,1], \\ \mathbf{u}|_{t=0} = \mathbf{0}, & (x,y) \in (-1,1)\times(-1,1) \end{cases} \tag{4.1}$$

with periodic boundary value conditions. Here $0<\alpha<1, \nu = 1\times 10^{-2}$, and

$$\mathbf{f}(x,y,t) = \begin{pmatrix} \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin(\pi x)\cos(\pi y) + 2\nu\pi^2 t^2\sin(\pi x)\cos(\pi y) + 2t^2xy^2 \\ \dfrac{-2t^{2-\alpha}}{\Gamma(3-\alpha)}\cos(\pi x)\sin(\pi y) - 2\nu\pi^2 t^2\cos(\pi x)\sin(\pi y) + 2t^2x^2y \end{pmatrix}.$$

The exact solution of the above equation is

$$\mathbf{u}(x,y,t) = \begin{pmatrix} t^2\sin(\pi x)\cos(\pi y) \\ -t^2\cos(\pi x)\sin(\pi y) \end{pmatrix}, \quad p(x,y,t) = t^2xy.$$

We solve this example by using scheme (3.28). In Tables 1 and 2, we list the L^2-norm errors and temporal convergence orders of the velocity and pressure for problem (4.1) at $T=1$ with different parameters α ($\alpha = 0.4, 0.6, 0.8$). The L^2-norm errors and spatial convergence orders of the velocity and pressure for problem (4.1) at $T=1$ with different α are presented in Tables 3 and 4. We can clearly see from Tables 1 and 3 that the numerical convergence orders for velocity in the temporal and spatial directions are $(2-\alpha)$ and $(k+1)$, respectively, which is in agreement with the theoretical results. From Table 2, the numerical convergence orders for pressure in the temporal direction show much better results than theoretical analysis. And Table 4 shows $(k+1)$-th order accuracy for pressure in the spatial direction as expected.

Table 1: The L^2-norm errors and temporal convergence orders of velocity for Example 4.1, $M = N_x = N_y$, $k = 1$, $T = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	2.38E-01	-	2.49E-01	-	2.87E-01	-
8	6.94E-02	1.78	7.61E-02	1.71	1.01E-01	1.51
16	1.89E-02	1.88	2.27E-02	1.74	3.78E-02	1.42
32	4.96E-03	1.93	7.03E-03	1.69	1.53E-02	1.30
64	1.30E-03	1.93	2.33E-03	1.59	6.49E-03	1.24

Table 2: The L^2-norm errors and temporal convergence orders of pressure for Example 4.1, $M = N_x = N_y$, $k = 1$, $T = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	2.49E-02	-	2.81E-02	-	3.14E-02	-
8	4.14E-03	2.59	4.56E-03	2.62	5.47E-03	2.52
16	8.14E-04	2.35	9.76E-04	2.22	1.74E-03	1.65
32	2.01E-04	2.02	3.10E-04	1.65	7.66E-04	1.19
64	5.72E-05	1.81	1.13E-04	1.46	3.39E-04	1.17

Table 3: The L^2-norm errors and spatial convergence orders of velocity for Example 4.1 using $\mathcal{Q}^k$ polynomials with different α. $T = 1$, $r = (2-\alpha)/\alpha$.

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$N_x \times N_y$	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\mathcal{Q}^1$	4×4	2.34E-01	-	2.33E-01	-	2.34E-01	-
	8×8	6.77E-02	1.79	6.72E-02	1.79	6.70E-02	1.80
	16×16	1.82E-02	1.90	1.82E-02	1.89	1.81E-02	1.89
	32×32	4.66E-03	1.97	4.65E-03	1.96	4.65E-03	1.96
$\mathcal{Q}^2$	4×4	2.58E-02	-	2.55E-02	-	2.59E-02	-
	8×8	3.40E-03	2.93	3.36E-03	2.93	3.37E-03	2.94
	16×16	4.42E-04	2.94	4.40E-04	2.93	4.41E-04	2.93

Table 4: The L^2-norm errors and spatial convergence orders of pressure for Example 4.1 using $\mathcal{Q}^k$ polynomials with different α. $T = 1$, $r = (2-\alpha)/\alpha$.

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$N_x \times N_y$	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\mathcal{Q}^1$	4×4	2.48E-02	-	2.78E-02	-	3.07E-02	-
	8×8	4.20E-03	2.56	4.55E-03	2.61	4.89E-03	2.65
	16×16	8.32E-04	2.34	8.65E-04	2.39	8.93E-04	2.45
	32×32	1.95E-04	2.09	1.98E-04	2.13	1.99E-04	2.16
$\mathcal{Q}^2$	4×4	3.08E-03	-	3.33E-03	-	3.58E-03	-
	8×8	3.31E-04	3.22	3.44E-04	3.28	3.58E-04	3.32
	16×16	3.94E-05	3.07	3.99E-05	3.11	4.05E-05	3.14

Example 4.2. Consider the following time-fractional Stokes equation:

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,y,t), & (x,y,t) \in (-1,1) \times (-1,1) \times (0,1], \\ \nabla \cdot \mathbf{u} = 0, & (x,y,t) \in (-1,1) \times (-1,1) \times (0,1], \\ \mathbf{u}|_{t=0} = \mathbf{0}, & (x,y) \in (-1,1) \times (-1,1) \end{cases} \tag{4.2}$$

with periodic boundary value conditions. Here $0 < \alpha < 1$, $\nu = 1 \times 10^{-2}$, and

$$\begin{aligned} &\mathbf{f}(x,y,t) \\ =& \begin{pmatrix} \left(\Gamma(\alpha+1) + \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\sin(\pi x)\cos(\pi y) + 2\nu\pi^2(t^2+t^\alpha)\sin(\pi x)\cos(\pi y) + 2t^2xy^2 \\ -\left(\Gamma(\alpha+1) + \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\cos(\pi x)\sin(\pi y) - 2\nu\pi^2(t^2+t^\alpha)\cos(\pi x)\sin(\pi y) + 2t^2x^2y \end{pmatrix}. \end{aligned}$$

Its exact solution is

$$\mathbf{u}(x,y,t) = \begin{pmatrix} (t^2+t^\alpha)\sin(\pi x)\cos(\pi y) \\ -(t^2+t^\alpha)\cos(\pi x)\sin(\pi y) \end{pmatrix}, \quad p(x,y,t) = t^2xy.$$

At first, we check the L^2-norm convergence rate for the velocity. Since the exact solution $\mathbf{u}$ of problem (4.2) has weak regularity at the initial time, we calculate this example by using the non-uniform L1/LDG scheme (3.79). Tables 5-7 display the L^2-norm errors and temporal convergence orders of the velocity at $T = 1$ for different

Table 5: The L^2-norm errors and temporal convergence orders of velocity for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	4.71E-01	-	4.70E-01	-	4.80E-01	-
8	1.37E-01	1.78	1.36E-01	1.79	1.43E-01	1.75
16	8.63E-02	0.67	5.00E-02	1.45	4.00E-02	1.83
32	6.71E-02	0.36	3.43E-02	0.54	1.17E-02	1.78
64	5.18E-02	0.37	2.32E-02	0.57	7.35E-03	0.67

Table 6: The L^2-norm errors and temporal convergence orders of velocity for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = (2-\alpha)/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	4.80E-01	-	4.81E-01	-	4.90E-01	-
8	1.44E-01	1.74	1.44E-01	1.74	1.51E-01	1.70
16	4.10E-02	1.81	4.12E-02	1.81	4.64E-02	1.70
32	1.17E-02	1.81	1.20E-02	1.77	1.55E-02	1.58
64	3.46E-03	1.75	3.82E-03	1.66	5.81E-03	1.41

values of α ($\alpha = 0.4, 0.6, 0.8$). It is clear that for the time-fractional Stokes equation with weak regular solution $\mathbf{u}$ at the starting time, if the uniform L1 formula is used as time discretization, the scheme for velocity can not reach the optimal convergence order (i.e., $(2-\alpha)$-th order) in time direction (see Table 5). In contrast, if we apply the non-uniform L1 method to approximate the time fractional derivative, the optimal convergence orders can be observed (see Tables 6 and 7). We also show accuracy test in space of scheme (3.79) for Eq. (4.2). The numerical results can be found in Table 8, which coincides with theoretical results obtained in Theorem 3.4.

Table 7: The L^2-norm errors and temporal convergence orders of velocity for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = (2(2-\alpha))/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	4.89E-01	-	5.09E-01	-	5.39E-01	-
8	1.64E-01	1.58	1.70E-01	1.58	1.88E-01	1.52
16	5.69E-01	1.53	5.82E-02	1.54	6.99E-02	1.43
32	2.04E-02	1.48	2.13E-02	1.45	2.85E-02	1.30
64	7.30E-03	1.48	8.10E-03	1.39	1.22E-02	1.22

Table 8: The L^2-norm errors and spatial convergence orders of velocity for Example 4.2 using $\mathcal{Q}^k$ polynomials with different α. $T = 1$, $r = (2-\alpha)/\alpha$.

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$N_x \times N_y$	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\mathcal{Q}^1$	4×4	4.72E-01	-	4.69E-01	-	4.67E-01	-
	8×8	1.37E-01	1.79	1.36E-01	1.79	1.35E-01	1.79
	16×16	3.65E-02	1.90	3.65E-02	1.90	3.64E-02	1.89
	32×32	9.32E-03	1.97	9.32E-03	1.97	9.31E-03	1.97
$\mathcal{Q}^2$	4×4	5.24E-02	-	5.20E-02	-	5.16E-02	-
	8×8	6.87E-03	2.93	6.83E-03	2.93	6.79E-03	2.93
	16×16	8.87E-04	2.95	8.86E-04	2.95	8.84E-04	2.94

Table 9: The L^2-norm errors and temporal convergence orders of stress for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	2.59E-01	-	2.63E-01	-	2.73E-01	-
8	7.61E-02	1.77	7.68E-02	1.77	8.10E-02	1.75
16	4.99E-02	0.61	3.04E-02	1.34	2.25E-02	1.85
32	3.91E-02	0.35	2.15E-02	0.50	7.90E-03	1.51
64	3.07E-02	0.35	1.50E-02	0.52	5.11E-03	0.63

Table 10: The L^2-norm errors and temporal convergence orders of stress for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = (2-\alpha)/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	2.63E-01	-	2.67E-01	-	2.77E-01	-
8	7.87E-02	1.74	7.98E-02	1.74	8.48E-02	1.71
16	2.27E-02	1.79	2.29E-02	1.80	2.58E-02	1.72
32	6.98E-03	1.70	7.21E-03	1.67	8.97E-03	1.52
64	2.26E-03	1.63	2.44E-03	1.56	3.44E-03	1.38

Table 11: The L^2-norm errors and temporal convergence orders of stress for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = (2(2-\alpha))/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	2.67E-01	-	2.80E-01	-	2.98E-01	-
8	8.80E-02	1.60	9.23E-02	1.60	1.03E-01	1.54
16	3.09E-02	1.51	3.19E-02	1.53	3.84E-02	1.42
32	1.14E-02	1.45	1.19E-02	1.42	1.58E-02	1.28
64	4.13E-03	1.46	4.59E-03	1.38	6.79E-03	1.22

Table 12: The L^2-norm errors and spatial convergence orders of stress for Example 4.2 using $\mathcal{Q}^k$ polynomials with different α. $T = 1$, $r = (2-\alpha)/\alpha$.

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$N_x \times N_y$	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\mathcal{Q}^1$	4×4	2.60E-01	-	2.63E-01	-	2.68E-01	-
	8×8	7.59E-02	1.77	7.71E-02	1.77	7.86E-02	1.77
	16×16	2.06E-02	1.88	2.07E-02	1.90	2.08E-02	1.92
	32×32	5.90E-03	1.80	5.90E-03	1.81	5.91E-03	1.82
$\mathcal{Q}^2$	4×4	5.11E-02	-	5.19E-02	-	5.29E-02	-
	8×8	7.06E-03	2.86	7.11E-03	2.87	7.19E-03	2.88
	16×16	9.31E-04	2.92	9.34E-04	2.93	9.37E-04	2.94

Table 13: The L^2-norm errors and temporal convergence orders of pressure for Example 4.2, $M = N_x = N_y$, $k = 1$, $T = 1$, $r = (2(2-\alpha))/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2 error	Order	L^2 error	Order	L^2 error	Order
8	7.87E-03	-	8.40E-03	-	9.34E-03	-
16	2.65E-03	1.57	2.85E-03	1.56	3.64E-03	1.36
32	1.11E-03	1.25	1.24E-03	1.20	1.74E-03	1.07
64	4.30E-04	1.38	5.09E-04	1.28	7.90E-04	1.14

Table 14: The L^2-norm errors and spatial convergence orders of pressure for Example 4.2 using $\mathcal{Q}^k$ polynomials with different α. $T=1$, $r=(2-\alpha)/\alpha$.

		$\alpha=0.4$		$\alpha=0.6$		$\alpha=0.8$	
	$N_x\times N_y$	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\mathcal{Q}^1$	4×4	4.27E-02	-	4.57E-02	-	4.91E-02	-
	8×8	7.58E-03	2.49	7.93E-03	2.53	8.31E-03	2.56
	16×16	1.58E-03	2.26	1.62E-03	2.29	1.65E-03	2.33
	32×32	3.83E-04	2.05	3.86E-04	2.07	3.88E-04	2.09
$\mathcal{Q}^2$	4×4	5.55E-03	-	5.81E-03	-	6.09E-03	-
	8×8	6.31E-04	3.14	6.44E-04	3.17	6.58E-04	3.21
	16×16	7.77E-05	3.02	7.82E-05	3.04	7.87E-05	3.06

Then, for $\alpha=0.4,0.6,0.8$, we check the L^2-norm convergence rate for the stress. The error and error order are listed in Tables 9-12. The results clearly demonstrate that the accuracy is $\mathcal{O}(M^{-\min\{2-\alpha,r\alpha\}}+h^{k+1})$, which shows that Theorem 3.4 gives a suboptimal error estimate of stress in the time direction.

Next, it could be interesting to test the convergence of scheme (3.79) for the pressure, although we cannot give the theoretical analysis in this paper. Table 13 lists some computational results and the convergence orders with a fixed graded parameter $r=\frac{2(2-\alpha)}{\alpha}$. From the data of Table 13, the convergence order $(2-\alpha)$ of the scheme (3.79) in the temporal direction is confirmed. From Table 14, we can see that the orders of convergence for the pressure in L^2-norms are close to $(k+1)$.

5. Concluding remarks

In this paper, the time-fractional Stokes equation is numerically studied. We apply the finite difference methods on uniform and non-uniform meshes to approximate the time fractional derivative, and apply the LDG method on uniform meshes to approximate the space derivative. Then we prove that the established schemes are numerically stable and convergent. Finally, several numerical examples are presented which are in line with the theoretical analysis.

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