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Error Estimate of a New Conservative Finite Difference Scheme for the Klein-Gordon-Dirac System

Shasha Bian[1], Yue Cheng[2], Boling Guo[3] and Tingchun Wang[2,*]

[1] *The Graduate School of China Academy of Engineering Physics, Beijing 100088, China*
[2] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*
[3] *Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*

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Abstract. In this paper, we derive and analyze a conservative Crank-Nicolson-type finite difference scheme for the Klein-Gordon-Dirac (KGD) system. Differing from the derivation of the existing numerical methods given in literature where the numerical schemes are proposed by directly discretizing the KGD system, we translate the KGD equations into an equivalent system by introducing an auxiliary function, then derive a nonlinear Crank-Nicolson-type finite difference scheme for solving the equivalent system. The scheme perfectly inherits the mass and energy conservative properties possessed by the KGD, while the energy preserved by the existing conservative numerical schemes expressed by two-level's solution at each time step. By using energy method together with the 'cut-off' function technique, we establish the optimal error estimate of the numerical solution, and the convergence rate is $\mathcal{O}(\tau^2 + h^2)$ in l^∞-norm with time step τ and mesh size h. Numerical experiments are carried out to support our theoretical conclusions.

AMS subject classifications: 65M06, 65M12

Key words: Klein-Gordon-Dirac equation, nonlinear finite difference scheme, conservation, error analysis.

1. Introduction

In this paper, we consider the Klein-Gordon-Dirac (KGD) equation in d ($d = 1, 2$) dimensions [17]

*Corresponding author. *Email address:* `wangtingchun2010@gmail.com` (T. Wang)

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$$
\begin{aligned}
&\partial_{tt}\phi(t,\mathbf{x}) - \Delta\phi(t,\mathbf{x}) + \phi(t,\mathbf{x}) \\
&= g\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\
&i\partial_t\Psi(t,\mathbf{x}) + i\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\sigma_3\Psi(t,\mathbf{x}) \\
&= g\phi(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t>0.
\end{aligned} \tag{1.1}
$$

The Klein-Gordon-Dirac equation is a fundamental model in quantum electrodynamics and describes the dynamics of a complex-value Dirac vector field $\Psi(t,\mathbf{x}) := (\psi_1(t,\mathbf{x}),\psi_2(t,\mathbf{x}))\in\mathbb{C}^2$ interacting with a neutral real-valued meson field $\phi := \phi(t,\mathbf{x})\in\mathbb{R}$ through the Yukawa potential [13,25,27,33,36] with a coupling constant $0<g\in\mathbb{R}$. Note that $\omega>0$ is the radio between the mass of the electron and the mass of the meson, $i=\sqrt{-1}$ is the imaginary unit, t is time, $\mathbf{x}\in\mathbb{R}^d$ is the spatial coordinate vector. In 2D, $\mathbf{x}=(x_1,x_2)^T$, $\partial_j=\partial/\partial x_j$, $j=1,2$ and $\Delta=\sum_{j=1}^{2}\partial_j^2$. In addition, Ψ^* is the conjugate transpose of Ψ, σ_1, σ_2 and σ_3 are the Pauli matrices given as

$$
\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{1.2}
$$

The KGD system (1.1) is dispersive and conserves the total mass

$$
M(t) := \|\Psi(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^d}|\Psi(t,\mathbf{x})|^2 d\mathbf{x} \equiv \|\Psi(0,\cdot)\|_{L^2}^2, \tag{1.3}
$$

and energy

$$
\begin{aligned}
E(t) := &\frac{1}{2}\int_{\mathbb{R}^d}|\partial_t\phi(t,\mathbf{x})|^2dx + \frac{1}{2}\int_{\mathbb{R}^d}|\partial_x\phi(t,\mathbf{x})|^2d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d}|\phi(t,\mathbf{x})|^2d\mathbf{x} \\
&+ \int_{\mathbb{R}^d}\Bigg[i\Psi^*(t,\mathbf{x})\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}) \\
&\qquad - g\phi(t,\mathbf{x})\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x})\Bigg]d\mathbf{x} \equiv E(0), \quad t\ge 0.
\end{aligned} \tag{1.4}
$$

For the KGD system (1.1), we introduce an auxiliary function $u := \partial_t\phi$ to rewrite it into the following equivalent system:

$$
\begin{aligned}
&\partial_t u(t,\mathbf{x}) - \Delta\phi(t,\mathbf{x}) + \phi(t,\mathbf{x}) = g\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\
&\partial_t\phi(t,\mathbf{x}) = u(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\
&i\partial_t\Psi(t,\mathbf{x}) + i\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\sigma_3\Psi(t,\mathbf{x}) \\
&= g\phi(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0.
\end{aligned} \tag{1.5}
$$

Corresponding to (1.3)-(1.4), the system (1.5) still preserves the total mass

$$\overline{M}(t) := \|\Psi(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^d} |\Psi(t,\mathbf{x})|^2 d\mathbf{x} \equiv \|\Psi(0,\cdot)\|_{L^2}^2, \tag{1.6}$$

and energy

$$\begin{aligned}\overline{E}(t) := &\frac{1}{2}\int_{\mathbb{R}^d} |u(t,\mathbf{x})|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi(t,\mathbf{x})|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d} |\phi(t,\mathbf{x})|^2 d\mathbf{x} \\ &+ \int_{\mathbb{R}^d} \Big[i\Psi^*(t,\mathbf{x}) \sum_{j=1}^{d} \sigma_j \partial_j \Psi(t,\mathbf{x}) - \omega\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}) \\ &\qquad - g\phi(t,\mathbf{x})\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}) \Big] d\mathbf{x} \equiv \overline{E}(0), \quad t \geq 0. \end{aligned} \tag{1.7}$$

There have been extensive theoretical studies on the KGD (1.1) system, including the local and global well-posedness of the Cauchy problem and the existences of bound state solutions, for which we refer to [2, 14, 15, 19, 20, 23, 24, 32, 35] and references therein. For the numerical part, different kinds of numerical methods, including the finite difference time domain (FDTD) methods and spectral methods have been proposed and analyzed for efficient computations of wave propagation in classical quantum physics, i.e., dispersive waves in the Gross-Pitaevskii equation [3], the Klein-Gordon equation [7, 8, 12, 22, 29, 34, 40, 43], the Dirac equation [1, 4–6, 26, 28, 30, 39], the Klein-Gordon-Schrodinger equations [10, 21], the Klein-Gordon-Zakharov equations [11, 16, 37], the Maxwell-Dirac equations [9, 31], etc. However, the approaches for KGD (1.1) proposed in the literature [17, 41, 42] are limited. To the best of our knowledge, there has not been any numerical work on the KGD (1.5) which is equivalent to (1.1). As well as, in the construction of the finite difference scheme, the nonlinear term of Klein-Gordon part is treated as n and $n+1$ layers which is not mentioned in existed literatures. Thus, the aim of this paper is to design a new conservative Crank-Nicolson-type finite difference (CNFD) scheme for KGD (1.5) and analyze the effectiveness and accuracy of the scheme.

In this paper, we propose and analyze a conservative CNFD scheme for KGD (1.5) with periodic boundary conditions. We prove that the scheme perfectly inherits the conservative properties of KGD system in the discrete sense. In the process of error analysis, we apply the energy method and the 'cut-off' function technique to obtain the convergence order of the scheme under proper assumptions about the exact solutions. Moreover, in the numerical calculation, our scheme can be started directly, i.e. we do not need to calculate the value of the first layer time alone.

The remaining part of this paper is arranged as follows. In Section 2, we introduce a new conservative CNFD scheme and state our main results. Section 3 is devoted to the error analysis. Numerical results are shown in Section 4 to demonstrate the error behavior. Finally, some conclusions are drawn in Section 5. Throughout the paper, we adopt the standard notations for Sobolev spaces and write $p \lesssim q$ to represent that there exists a constant $C > 0$ independent of the discrete parameters, such that $|p| \leq Cq$.

2. A new conservative scheme and the main results

In this section, we shall derive a new finite difference scheme for the KGD system. For simplicity of notations, here we only illustrate the numerical method in one dimension, and extension to two-dimensional case is straightforward with some small modifications. We truncate the whole space onto a finite interval $\Omega = (a, b)$ with periodic boundary conditions in practical computation, which is large enough to ignore the truncation error. The initial-boundary value problem of the one-dimensional (1D) KGD system reads

$$\begin{aligned}
&\partial_t u(t,x) - \partial_{xx}\phi(t,x) + \phi(t,x) = g\Psi^*(t,x)\sigma_3\Psi(t,x), && x\in\Omega, \quad t>0,\\
&\partial_t \phi(t,x) = u(t,x), && x\in\Omega, \quad t>0,\\
&i\partial_t\Psi(t,x) + i\sigma_1\partial_x\Psi(t,x) - \omega\sigma_3\Psi = g\phi(t,x)\sigma_3\Psi(t,x), && x\in\Omega, \quad t>0,\\
&u(t,a) = u(t,b), \quad \partial_x u(t,a) = \partial_x u(t,b), && t\geq 0,\\
&\phi(t,a) = \phi(t,b), \quad \partial_x \phi(t,a) = \partial_x \phi(t,b), && t\geq 0,\\
&\Psi(t,a) = \Psi(t,b), \quad \partial_x \Psi(t,a) = \partial_x \Psi(t,b), && t\geq 0,\\
&\phi(0,x) = \phi_0(x), \quad u(0,x) = \phi_1(x), \quad \Psi(0,x) = \Psi_0(x), && x\in\overline{\Omega},
\end{aligned} \tag{2.1}$$

where $\phi := \phi(t,x) \in \mathbb{R}$ and $\Psi := \Psi(t,x) = (\psi_1(t,x), \psi_2(t,x))^T \in \mathbb{C}^2$. The periodic initial-boundary value problem (2.1) conserves the total mass

$$\overline{M}(t) := \|\Psi(t,\cdot)\|^2_{L^2(\Omega)} = \int_\Omega |\Psi(t,x)|^2 dx \equiv \|\Psi_0\|^2_{L^2(\Omega)}, \tag{2.2}$$

and energy

$$\begin{aligned}
\overline{E}(t) := &\frac{1}{2}\int_\Omega |u(t,x)|^2 dx + \frac{1}{2}\int_\Omega |\partial_x\phi(t,x)|^2 dx + \frac{1}{2}\int_\Omega |\phi(t,x)|^2 dx\\
&+ \int_\Omega \Big[i\Psi^*(t,x)\sigma_1\partial_x\Psi(t,x) - \omega\Psi^*(t,x)\sigma_3\Psi(t,x)\\
&\qquad - g\phi(t,x)\Psi^*(t,x)\sigma_3\Psi(t,x)\Big] dx \equiv \overline{E}(0), \quad t\geq 0.
\end{aligned} \tag{2.3}$$

2.1. A conservative CNFD scheme

Choose temporal step size $\tau := \Delta t > 0$ and denote time steps as $t_n := n\tau$ for $n = 0, 1, \ldots$. Choose mesh size $h := \Delta x = \frac{b-a}{M}$ with M being an even positive integer and denote grid points as $x_j := a + j\Delta x$, $j = 0, 1, \ldots, M$.

Denote

$$\begin{aligned}
\widetilde{X}_M &= \left\{ U = (U_0, U_1, U_2, \cdots, U_M)^T \mid U_j \in \mathbb{R},\ j = 0, 1, \ldots, M,\ U_0 = U_M \right\},\\
X_M &= \left\{ U = (U_0, U_1, U_2, \cdots, U_M)^T \mid U_j \in \mathbb{C}^2,\ j = 0, 1, \ldots, M,\ U_0 = U_M \right\},
\end{aligned}$$

and we always use $U_{-1} = U_{M-1}$ and $U_1 = U_{M+1}$ if they are involved. Introduce some finite difference discretization operators for $U \in X_M$ or $\widetilde{X}_M$

$$\delta_x^+ U_j^n = \frac{U_{j+1}^n - U_j^n}{h}, \quad \delta_x U_j^n = \frac{U_{j+1}^n - U_{j-1}^n}{2h}, \quad \delta_x^2 U_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2},$$
$$U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + U_j^n}{2}, \quad \delta_t^+ U_j^n = \frac{U_j^{n+1} - U_j^n}{\tau}, \quad \delta_t^2 U_j^n = \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\tau^2}.$$

The discrete inner product, standard l^2-norm, H^1-seminorm and l^∞-norm for $U, V \in X_M$ (or $U, V \in \widetilde{X}_M$) are given as

$$(U, V) := h \sum_{j=0}^{M-1} V_j^* U_j, \qquad \|U\| := \sqrt{h \sum_{j=0}^{M-1} |U_j|^2},$$
$$|U|_1 := \sqrt{h \sum_{j=0}^{M-1} |\delta_x^+ U_j|^2}, \quad \|U\|_\infty := \max_{0 \le j \le M-1} |U_j|.$$

Let ϕ_j^n, u_j^n and Ψ_j^n be numerical approximations of $\phi(t_n, x_j), u(t_n, x_j)$ and $\Psi(t_n, x_j)$, respectively, for $j = 0, 1, \ldots, M$ and $n = 0, 1, \ldots,$ and denote

$$\phi^n = \left(\phi_0^n, \phi_1^n, \cdots, \phi_M^n\right)^T \in \widetilde{X}_M,$$
$$u^n = \left(u_0^n, u_1^n, \cdots, u_M^n\right)^T \in \widetilde{X}_M,$$
$$\Psi^n = \left(\Psi_0^n, \Psi_1^n, \cdots, \Psi_M^n\right)^T \in X_M$$

as the solution vector at $t = t_n$. Based on the above basics, we propose the following nonlinear CNFD scheme to discrete the KGD system (2.1) for $j = 1, \ldots, M$:

$$\delta_t^+ u_j^n - \delta_x^2 \phi_j^{n+\frac{1}{2}} + \phi_j^{n+\frac{1}{2}} = \frac{g}{2}\left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} + \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n\right], \quad n \geqslant 0, \tag{2.4}$$
$$\delta_t^+ \phi_j^n = u_j^{n+\frac{1}{2}}, \qquad n \geqslant 0, \tag{2.5}$$
$$i\delta_t^+ \Psi_j^n + i\sigma_1 \delta_x \Psi_j^{n+\frac{1}{2}} - \omega\sigma_3 \Psi_j^{n+\frac{1}{2}} = g\phi_j^{n+\frac{1}{2}} \sigma_3 \Psi_j^{n+\frac{1}{2}}, \qquad n \geqslant 0. \tag{2.6}$$

Meanwhile, the periodic boundary and initial conditions in (2.1) are discretized as

$$\begin{aligned} &u_M^{n+1} = u_0^{n+1}, \quad u_{-1}^{n+1} = u_{M-1}^{n+1}, \quad u_j^0 = \phi_1(x_j), \quad n \ge 0, \quad j = 0, 1, \ldots, M, \\ &\phi_M^{n+1} = \phi_0^{n+1}, \quad \phi_{-1}^{n+1} = \phi_{M-1}^{n+1}, \quad \phi_j^0 = \phi_0(x_j), \quad n \ge 0, \quad j = 0, 1, \ldots, M, \\ &\Psi_M^{n+1} = \phi_0^{n+1}, \quad \Psi_{-1}^{n+1} = \Psi_{M-1}^{n+1}, \quad \Psi_j^0 = \Psi_0(x_j), \quad n \ge 0, \quad j = 0, 1, \ldots, M. \end{aligned} \tag{2.7}$$

The above scheme is time symmetric and time reversible, i.e. the scheme (2.4)-(2.6) is unchanged under $\tau \leftrightarrow -\tau$, $n+1 \leftrightarrow n$ and taking complex conjugates. In numerical calculation, eliminate the intermediate item u^n for $n \ge 0$ in the scheme (2.4)-(2.6),

we obtain the following equivalent nonlinear implicit finite difference scheme for $j = 1, \ldots, M$:

$$\begin{aligned}
&\delta_t^2 \phi_j^n - \frac{1}{4}\delta_x^2\big(\phi_j^{n-1} + 2\phi_j^n + \phi_j^{n+1}\big) + \frac{1}{4}\big(\phi_j^{n-1} + 2\phi_j^n + \phi_j^{n+1}\big) \\
&= \frac{g}{4}\left[\big(\Psi_j^{n-1}\big)^* \sigma_3 \Psi_j^{n-1} + 2\big(\Psi_j^n\big)^* \sigma_3 \Psi_j^n + \big(\Psi_j^{n+1}\big)^* \sigma_3 \Psi_j^{n+1}\right], \qquad n \geqslant 0, \quad (2.8) \\
&-\frac{4}{\tau}\big(\delta_t^+ \phi_j^0 - u_j^0\big) + 2\delta_x^2 \phi_j^{\frac{1}{2}} + g\Big[\big(\Psi_j^1\big)^* \sigma_3 \Psi_j^1 + \big(\Psi_j^0\big)^* \sigma_3 \Psi_j^0\Big] + \phi_j^0 = \phi_j^1, \quad n \geqslant 0, \quad (2.9) \\
&i\delta_t^+ \Psi_j^n + i\sigma_1 \delta_x \Psi_j^{n+\frac{1}{2}} - \omega\sigma_3 \Psi_j^{n+\frac{1}{2}} = g\phi_j^{n+\frac{1}{2}} \sigma_3 \Psi_j^{n+\frac{1}{2}}, \qquad n \geqslant 0. \quad (2.10)
\end{aligned}$$

The computations of the scheme (2.8)-(2.10) can be arranged as follows: ϕ^0, ψ^0 are given by the initial conditions, ϕ^1 is updated by (2.9), then Ψ^1 is obtained by (2.10), and then ϕ^2 is updated by (2.8), for $n \geq 2$, ϕ^n and Ψ^n are available, ϕ^{n+1} is updated by (2.8) and Ψ^{n+1} is updated by (2.10).

2.2. Mass and energy conservation

As the discrete version (or approximation) of the initial-boundary value problem (2.1) possessing the conservative properties (2.2)-(2.3), the proposed scheme (2.4)-(2.7) still possesses similar conservative properties which can be viewed as the discrete version of (2.2)-(2.3).

Theorem 2.1. *The nonlinear CNFD scheme* (2.4)-(2.7) *preserves the total mass and energy in the discrete sense, i.e.,*

$$M^n := \|\Psi^n\|^2 \equiv M^0, \quad n \geq 0 \tag{2.11}$$

and

$$\begin{aligned}
E^n := &\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\delta_x^+ \phi^n\|^2 + \frac{1}{2}\|\phi^n\|^2 + ih\sum_{j=0}^{M-1}\big(\Psi_j^n\big)^* \sigma_1 \delta_x \Psi_j^n \\
&- \omega h \sum_{j=0}^{M-1}\big(\Psi_j^n\big)^* \sigma_3 \Psi_j^n - gh\sum_{j=0}^{M-1}\phi_j^n\big(\Psi_j^n\big)^* \sigma_3 \Psi_j^n \equiv E^0, \quad n \geq 0.
\end{aligned} \tag{2.12}$$

Proof. Firstly, we prove the mass conservation (2.11). To do this, we make the inner product of $\tau(\Psi^{n+1} + \Psi^n)$ with (2.6), then take the imaginary part to obtain

$$\begin{aligned}
&\mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1} + \Psi_j^n\big)^*\left(i\delta_t^+ \Psi_j^n + i\sigma_1 \delta_x \Psi_j^{n+\frac{1}{2}} - \omega\sigma_3 \Psi_j^{n+\frac{1}{2}}\right)\right\} \\
&= \mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1} + \Psi_j^n\big)^* g\phi_j^{n+\frac{1}{2}} \sigma_3 \Psi_j^{n+\frac{1}{2}}\right\},
\end{aligned} \tag{2.13}$$

where $\mathrm{Im}\{a\}$ means taking the imaginary part of complex value a. Simple calculations gives the left three items of (2.13)

$$\mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1}+\Psi_j^n\big)^* i\delta_t^+\Psi_j^n\right\} \tag{2.14}$$

$$=\mathrm{Re}\left\{h\sum_{j=0}^{M-1}\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\big(\Psi_j^{n+1}-\Psi_j^n\big)\right\}=\|\Psi^{n+1}\|^2-\|\Psi^n\|^2,$$

$$\mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1}+\Psi_j^n\big)^* i\sigma_1\delta_x\Psi_j^{n+\frac{1}{2}}\right\} \tag{2.15}$$

$$=\mathrm{Re}\left\{\frac{\tau h}{2}\sum_{j=0}^{M-1}\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\sigma_1\delta_x\big(\Psi_j^{n+1}+\Psi_j^n\big)\right\}$$

$$=\frac{\tau h}{4}\sum_{j=0}^{M-1}\left[\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\sigma_1\delta_x\big(\Psi_j^{n+1}+\Psi_j^n\big)+\delta_x\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\sigma_1\big(\Psi_j^{n+1}+\Psi_j^n\big)\right]=0,$$

$$\mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\omega\sigma_3\Psi_j^{n+\frac{1}{2}}\right\} \tag{2.16}$$

$$=-\frac{h\tau\omega i}{4}\sum_{j=0}^{M-1}\left[\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\sigma_3\big(\Psi_j^{n+1}+\Psi_j^n\big)-\big(\Psi_j^{n+1}+\Psi_j^n\big)^*\sigma_3\big(\Psi_j^{n+1}+\Psi_j^n\big)\right]=0,$$

where $\mathrm{Re}\{a\}$ means taking the real part of complex value a. From (2.16), by using the fact that $(\Psi_j^{n+1}+\Psi_j^n)^*\sigma_3(\Psi_j^{n+1}+\Psi_j^n)$ is real, we get

$$\mathrm{Im}\left\{h\sum_{j=0}^{M-1}\tau\big(\Psi_j^{n+1}+\Psi_j^n\big)^* g\phi_j^{n+\frac{1}{2}}\sigma_3\Psi_j^{n+\frac{1}{2}}\right\}=0. \tag{2.17}$$

Inserting (2.14)-(2.17) into (2.13) gives

$$\|\Psi^{n+1}\|^2-\|\Psi^n\|^2=0, \tag{2.18}$$

which immediately gives (2.11).

Secondly, we prove the energy conservation (2.12). To do this, we compute the discrete inner products of (2.4) with $\delta_t^+\phi^n$, (2.5) with $\delta_t^+u^n$, (2.6) with $2\delta_t^+(\Psi^n)$, respectively, then take the real part of the third result to obtain

$$h\sum_{j=0}^{M-1}\delta_t^+\phi_j^n\delta_t^+u_j^n-h\sum_{j=0}^{M-1}\delta_t^+\phi_j^n\delta_x^2\phi_j^{n+\frac{1}{2}}+h\sum_{j=0}^{M-1}\delta_t^+\phi_j^n\phi_j^{n+\frac{1}{2}}$$

$$=h\sum_{j=0}^{M-1}\delta_t^+\phi_j^n\frac{g}{2}\left[\big(\Psi_j^{n+1}\big)^*\sigma_3\Psi_j^{n+1}+\big(\Psi_j^n\big)^*\sigma_3\Psi_j^n\right], \tag{2.19}$$

$$h\sum_{j=0}^{M-1}\delta_t^+u_j^n\delta_t^+\phi_j^n = h\sum_{j=0}^{M-1}\delta_t^+u_j^n u_j^{n+\frac{1}{2}}, \tag{2.20}$$

$$\begin{aligned}&\mathrm{Re}\Bigg\{h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^* i\delta_t^+\Psi_j^n + h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^* i\sigma_1\delta_x\Psi_j^{n+\frac{1}{2}}\\ &\qquad - h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^*\omega\sigma_3\Psi_j^{n+\frac{1}{2}}\Bigg\}\\ &= \mathrm{Re}\Bigg\{h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^* g\phi_j^{n+\frac{1}{2}}\sigma_3\Psi_j^{n+\frac{1}{2}}\Bigg\}.\end{aligned} \tag{2.21}$$

By simple calculations, (2.19) and (2.20) become

$$\begin{aligned}&h\sum_{j=0}^{M-1}\delta_t^+\phi_j^n\delta_t^+u_j^n + \frac{1}{2\tau}\left(\left\|\delta_x^+\phi^{n+1}\right\|^2 - \left\|\delta_x^+\phi^n\right\|^2\right) + \frac{1}{2\tau}\left(\|\phi^{n+1}\|^2 - \|\phi^n\|^2\right)\\ &= \frac{g}{2\tau}h\sum_{j=0}^{M-1}\left(\phi_j^{n+1}-\phi_j^n\right)\left[\left(\Psi_j^{n+1}\right)^*\sigma_3\Psi_j^{n+1} + \left(\Psi_j^n\right)^*\sigma_3\Psi_j^n\right],\end{aligned} \tag{2.22}$$

$$h\sum_{j=0}^{M-1}\delta_t^+u_j^n\delta_t^+\phi_j^n = \frac{1}{2\tau}\left(\|u^{n+1}\|^2 - \|u^n\|^2\right). \tag{2.23}$$

For the left three terms and the right one term of (2.21), we have

$$\mathrm{Re}\Bigg\{h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^* i\delta_t^+\Psi_j^n\Bigg\} = 2\mathrm{Im}\left\{\left\|\delta_t^+\Psi^n\right\|^2\right\} = 0, \tag{2.24}$$

$$\begin{aligned}&\mathrm{Re}\Bigg\{h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^* i\sigma_1\delta_x\Psi_j^{n+\frac{1}{2}}\Bigg\}\\ &= \mathrm{Im}\Bigg\{\frac{h}{\tau}\sum_{j=0}^{M-1}\left(\Psi_j^{n+1}-\Psi_j^n\right)^*\sigma_1\delta_x\left(\Psi_j^{n+1}+\Psi_j^n\right)\Bigg\}\\ &= -\frac{ih}{2\tau}\sum_{j=0}^{M-1}\left(\Psi_j^{n+1}-\Psi_j^n\right)^*\sigma_1\delta_x\left(\Psi_j^{n+1}+\Psi_j^n\right) - \delta_x\left(\Psi_j^{n+1}+\Psi_j^n\right)^*\sigma_1\left(\Psi_j^{n+1}-\Psi_j^n\right)\\ &= -\frac{ih}{\tau}\sum_{j=0}^{M-1}\left[\left(\Psi_j^{n+1}\right)^*\sigma_1\delta_x\Psi_j^{n+1} - \left(\Psi_j^n\right)^*\sigma_1\delta_x\Psi_j^n\right],\end{aligned} \tag{2.25}$$

$$\mathrm{Re}\Bigg\{h\sum_{j=0}^{M-1}2\delta_t^+\left(\Psi_j^n\right)^*\omega\sigma_3\Psi_j^{n+\frac{1}{2}}\Bigg\}$$

$$= \mathrm{Re}\left\{ \frac{\omega h}{\tau} \sum_{j=0}^{M-1} \left(\Psi_j^{n+1} - \Psi_j^n\right)^* \sigma_3 \left(\Psi_j^{n+1} + \Psi_j^n\right) \right\}$$

$$= \frac{\omega h}{2\tau} \sum_{j=0}^{M-1} \left[\left(\Psi_j^{n+1} - \Psi_j^n\right)^* \sigma_3 \left(\Psi_j^{n+1} + \Psi_j^n\right) + \left(\Psi_j^{n+1} + \Psi_j^n\right)^* \sigma_3 \left(\Psi_j^{n+1} - \Psi_j^n\right) \right]$$

$$= \frac{\omega h}{\tau} \sum_{j=0}^{M-1} \left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} - \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n \right], \tag{2.26}$$

$$\mathrm{Re}\left\{ h \sum_{j=0}^{M-1} 2\delta_t^+ \left(\Psi_j^n\right)^* g \phi_j^{n+\frac{1}{2}} \sigma_3 \Psi_j^{n+\frac{1}{2}} \right\}$$

$$= \frac{gh}{4\tau} \sum_{j=0}^{M-1} \left(\phi_j^{n+1} + \phi_j^n\right) \left[\left(\Psi_j^{n+1} - \Psi_j^n\right)^* \sigma_3 \left(\Psi_j^{n+1} + \Psi_j^n\right) + \left(\Psi_j^{n+1} + \Psi_j^n\right)^* \sigma_3 \left(\Psi_j^{n+1} - \Psi_j^n\right) \right]$$

$$= \frac{gh}{2\tau} \sum_{j=0}^{M-1} \left(\phi_j^{n+1} + \phi_j^n\right) \left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} - \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n \right]. \tag{2.27}$$

Inserting (2.24)-(2.27) into (2.21) yields

$$\begin{aligned} &\frac{ih}{\tau} \sum_{j=0}^{M-1} \left[\left(\Psi_j^{n+1}\right)^* \sigma_1 \delta_x \Psi_j^{n+1} - \left(\Psi_j^n\right)^* \sigma_1 \delta_x \Psi_j^n \right] \\ &\quad - \frac{\omega h}{\tau} \sum_{j=0}^{M-1} \left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} - \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n) \right] \\ &= \frac{gh}{2\tau} \sum_{j=0}^{M-1} \left(\phi_j^{n+1} + \phi_j^n\right) \left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} - \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n) \right]. \end{aligned} \tag{2.28}$$

Combining (2.22), (2.23) and (2.28) leads to $E^n \equiv E^{n-1}$, and the energy conservation (2.12) holds. □

2.3. Existence and uniqueness

The purpose of this section is to prove the existence of the numerical solution of (2.4)-(2.6).

Lemma 2.1. *For any given ϕ^n, u^n and Ψ^n with $n = 0, 1, \ldots,$ the solution ϕ^{n+1}, u^{n+1} and Ψ^{n+1} of the CNFD scheme* (2.4)-(2.6) *exists and is unique at each time step.*

Proof. For a fixed n, the scheme (2.4)-(2.6) can be written as

$$\frac{2}{\tau}\left(u_j^{n+\frac{1}{2}} - u_j^n\right) - \delta_x^2 \phi_j^{n+\frac{1}{2}} + \phi_j^{n+\frac{1}{2}} = \frac{g}{2} \left[\left(\Psi_j^{n+1}\right)^* \sigma_3 \Psi_j^{n+1} + \left(\Psi_j^n\right)^* \sigma_3 \Psi_j^n \right], \tag{2.29}$$

$$\frac{2}{\tau}\Big(\phi_j^{n+\frac{1}{2}}-\phi_j^n\Big)=u_j^{n+\frac{1}{2}}, \tag{2.30}$$

$$\Psi_j^{n+\frac{1}{2}}=\Psi_j^n-\frac{\tau}{2}\sigma_1\delta_x\Psi_j^{n+\frac{1}{2}}-i\frac{\omega\tau}{2}\sigma_3\Psi_j^{n+\frac{1}{2}}-i\frac{g\tau}{2}\phi_j^{n+\frac{1}{2}}\sigma_3\Psi_j^{n+\frac{1}{2}}. \tag{2.31}$$

Combining (2.29) and (2.30), we can get

$$\begin{aligned}\phi_j^{n+\frac{1}{2}}=&\left(\frac{4}{\tau^2}+1-\delta_x^2\right)^{-1}\\&\times\left(\frac{4}{\tau^2}\phi_j^n+\frac{2}{\tau}u_j^n+\frac{g}{2}\left[\big(\Psi_j^{n+1}\big)^*\sigma_3\Psi_j^{n+1}+\big(\Psi_j^n\big)^*\sigma_3\Psi_j^n\right]\right).\end{aligned} \tag{2.32}$$

Note

$$\phi_j^{n+\frac{1}{2}}:=\mathcal{F}\big(\phi_j^n,u_j^n,\Psi_j^{n+1},\Psi_j^n\big)\in\widetilde{X}_M. \tag{2.33}$$

As a consequence, the solvability of the scheme (2.4)-(2.6) is equivalent to the following finite difference equation:

$$\Psi_j^{n+\frac{1}{2}}=\Psi_j^n-\frac{\tau}{2}\sigma_1\delta_x\Psi_j^{n+\frac{1}{2}}-i\frac{\omega\tau}{2}\sigma_3\Psi_j^{n+\frac{1}{2}}-i\frac{g\tau}{2}\mathcal{F}\big(\phi_j^n,u_j^n,\Psi_j^{n+1},\Psi_j^n\big)\sigma_3\Psi_j^{n+\frac{1}{2}}. \tag{2.34}$$

Define a map $\mathcal{G}^n:X_M\to X_M$, for $W\in X_M$ as

$$\mathcal{G}^nW_j=W_j-\Psi_j^n+\frac{\tau}{2}\sigma_1\delta_xW_j+i\frac{\omega\tau}{2}\sigma_3W_j+i\frac{g\tau}{2}\mathcal{F}\big(\phi_j^n,u_j^n,\Psi_j^{n+1},\Psi_j^n\big)\sigma_3W_j. \tag{2.35}$$

It is obvious that $\mathcal{G}^n$ is continuous from $X_M\to X_M$. Moreover, the fact for $\|W\|^2>\|\Psi^n\|^2$

$$\mathrm{Re}\big\{(\mathcal{G}^nW,W)\big\}=\|W\|^2-\mathrm{Re}\big\{(\Psi^n,W)\big\}\geq\|W\|^2\big(\|W\|^2-\|\Psi^n\|^2\big)>0, \tag{2.36}$$

which implies that there exists a solution W^* such that $\mathcal{G}^nW^*=0$ by applying the Brouwer fixed point theorem [3]. In other words, the scheme (2.4)-(2.6) is solvable.

The uniqueness is a direct consequence of the fact that the solutions in (2.4)-(2.6) are updated as $\phi^1\to\Psi^1\to\phi^2\to\Psi^2\cdots$, where a linear system is solved at each step. □

2.4. Main results

Before giving our error estimate results of the proposed scheme, we make the following two type of assumptions on the exact solution ϕ and Ψ of the KGD equation (2.1):

Assumption (A)

$$\begin{aligned}&\phi\in C^4\big([0,T];L^\infty(\Omega)\big)\cap C^3\big([0,T];W_p^{1,\infty}(\Omega)\big)\cap C^2\big([0,T];W_p^{2,\infty}(\Omega)\big)\\&\qquad\cap C^1\big([0,T];W_p^{3,\infty}(\Omega)\big)\cap C\big([0,T];W_p^{4,\infty}(\Omega)\big),\\&\Psi\in C^3\big([0,T];[W_p^{1,\infty}(\Omega)]^2\big)\cap C^2\big([0,T];[W_p^{2,\infty}(\Omega)]^2\big)\cap C^1\big([0,T];[W_p^{4,\infty}(\Omega)]^2\big).\end{aligned}$$

Assumption (B)

$$\begin{aligned}
&\phi \in C^4\big([0,T];L^\infty(\Omega)\big) \cap C^3\big([0,T];W_p^{1,\infty}(\Omega)\big) \cap C^2\big([0,T];W_p^{2,\infty}(\Omega)\big)\\
&\qquad \cap C^1\big([0,T];W_p^{3,\infty}(\Omega)\big) \cap C\big([0,T];W_p^{4,\infty}(\Omega)\big),\\
&\Psi \in C^3\big([0,T];[L^\infty(\Omega)]^2\big) \cap C^2\big([0,T];[W_p^{1,\infty}(\Omega)]^2\big) \cap C^1\big([0,T];[W_p^{3,\infty}(\Omega)]^2\big),
\end{aligned}$$

where

$$W_p^{m,\infty}(\Omega) = \Big\{ v \mid v \in W^{m,\infty}(\Omega), \partial_x^l v(a) = \partial_x^l v(a),\ l = 0,1,\ldots,m-1 \Big\}, \quad m \geq 1$$

with $0 < T < T^*$ (T^* is the maximal existence time of the solution to the KGD system (2.1). We denote

$$N_\phi = \|\phi\|_{L_\infty}, \quad N_\Psi = \|\Psi\|_{L_\infty} \tag{2.37}$$

with

$$\begin{aligned}
L^\infty &= L^\infty\big([0,T];L^\infty(\Omega)\big) \quad \text{for} \quad \phi,\\
L^\infty &= L^\infty\big([0,T];[L^\infty(\Omega)]^2\big) \quad \text{for} \quad \Psi,
\end{aligned} \tag{2.38}$$

and the grid error functions $\eta^n \in \widetilde{X}_M$ and $e^n \in X_M$

$$\eta_j^n = \phi(t_n,x_j) - \phi_j^n, \quad e_j^n = \Psi(t_n,x_j) - \Psi_j^n, \quad j = 0,1,\ldots,M, \quad n \geq 0 \tag{2.39}$$

with ϕ_j^n and Ψ_j^n being the numerical approximations obtained from the proposed CNFD scheme, then the error estimates can be established as follows:

Theorem 2.2. *Under Assumption (A), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, such that, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimates for the nonlinear CNFD scheme* (2.4)-(2.6)*:*

$$\|\eta^n\| + \|\delta_x^+\eta^n\| \lesssim h^2 + \tau^2, \quad \|e^n\| + \|\delta_x^+ e^n\| \lesssim h^2 + \tau^2, \tag{2.40}$$

$$\|\eta^n\|_\infty \lesssim h^2 + \tau^2, \quad \|e^n\|_\infty \lesssim h^2 + \tau^2, \tag{2.41}$$

$$\|\phi^n\|_\infty \leq 1 + N_\phi, \quad \|\Psi^n\|_\infty \leq 1 + N_\Psi, \quad 0 < n \leq T/\tau. \tag{2.42}$$

Theorem 2.3. *Under Assumption (B), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, such that, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimates for the nonlinear CNFD scheme* (2.4)-(2.6)*:*

$$\|\eta^n\| + \|\delta_x^+\eta^n\| \lesssim h^2 + \tau^2, \quad \|e^n\| \lesssim h^2 + \tau^2, \tag{2.43}$$

$$\|\eta^n\|_\infty \lesssim h^2 + \tau^2, \quad \|e^n\|_\infty \lesssim h^2 + \tau^2, \tag{2.44}$$

$$\|\phi^n\|_\infty \leq 1 + N_\phi, \quad \|\Psi^n\|_\infty \leq 1 + N_\Psi, \quad 0 < n \leq T/\tau. \tag{2.45}$$

Remark 2.1. Obviously, the assumption on the exact solution ϕ and Ψ given in Assumption (B) is weaker than that given in Assumption (A). Hence, the error estimates given in Theorem 3.2 is weaker than those given in Theorem 2.3, and one can see in Section 3 that the proofs of the two theorems are quite different.

3. Error estimates

In this section, we propose the rigorous error estimates of the CNFD scheme (2.4)-(2.7) for solving the KGD equations (2.1).

3.1. The proof of Theorems 3.1 and 3.2

Similar to the nonlinear Schrödinger equations [3, 27, 38], we truncate the nonlinearity of the KGD equation to a global Lipschitz function with compact support in 1D. There is a good news, in [6, 18, 37], the same main difficulties were tackled by this method. This idea ensures that the numerical solution is close to the continuous solution if the continuous solution is bounded. Here, we apply the same cut-off technique to choose a function $\rho(s) \in C_0^\infty(\mathbb{R})$ as

$$\rho(s) = \begin{cases} 1, & |s| \le 1, \\ \in [0,1], & 1 \le |s| \le 2, \\ 0, & |s| \ge 2. \end{cases}$$

Denote $N_1 = (1+N_\Psi)^2$, $N_2 = (1+N_\phi)^2$ and define

$$f_{N_1}(\Psi) = \rho\big(|\Psi|^2/N_1\big)\Psi, \quad f_{N_2}(\phi) = \rho\big(|\phi|^2/N_1\big)\phi, \quad \Psi \in \mathbb{C}^2, \quad \phi \in \mathbb{R}. \tag{3.1}$$

Then $f_{N_1}(\Psi)$ and $f_{N_2}(\phi)$ have compact supports, and are smooth and global Lipschitz, i.e., there exist $C_{N_1} > 0$ and $C_{N_2} > 0$ such that $\forall \Psi_1, \Psi_2 \in \mathbb{C}^2$, $\forall \phi_1, \phi_2 \in \mathbb{R}$, and

$$\begin{aligned} &\big|f_{N_1}(\Psi_1) - f_{N_1}(\Psi_2)\big| \le C_{N_1}|\Psi_1 - \Psi_2|, \\ &\big|f_{N_2}(\phi_1) - f_{N_2}(\phi_2)\big| \le C_{N_2}|\phi_1 - \phi_2|. \end{aligned} \tag{3.2}$$

Set $\widehat{\phi}^0 = \phi_0$, $\widehat{\Psi}^0 = \Psi_0$, $\widehat{u}^0 = \phi_1$ and determine $\widehat{\phi}^n, \widehat{u}^n \in \widetilde{X}_M$ and $\widehat{\Psi}^n \in X_M$ as follows for $j = 0, 1, \ldots, M$

$$\begin{aligned} &\delta_t^+ \widehat{u}_j^n - \delta_x^2 \widehat{\phi}_j^{n+\frac{1}{2}} + \widehat{\phi}_j^{n+\frac{1}{2}} \\ &= \frac{g}{2}\left[\big(\widehat{\Psi}_j^{n+1}\big)^* \sigma_3 f_{N_1}\big(\widehat{\Psi}_j^{n+1}\big) + \big(\widehat{\Psi}_j^{n}\big)^* \sigma_3 f_{N_1}\big(\Psi_j^{n}\big)\right], \qquad n \geqslant 0, \end{aligned} \tag{3.3}$$

$$\delta_t^+ \widehat{\phi}_j^n = \widehat{u}_j^{n+\frac{1}{2}}, \qquad n \geqslant 0, \tag{3.4}$$

$$i\delta_t^+ \widehat{\Psi}_j^n + i\sigma_1 \delta_x \widehat{\Psi}_j^{n+\frac{1}{2}} - \omega\sigma_3 \widehat{\Psi}_j^{n+\frac{1}{2}} = g f_{N_2,j}^{n+\frac{1}{2}} \sigma_3 f_{N_1,j}^{n+\frac{1}{2}}, \quad n \geqslant 0, \tag{3.5}$$

where

$$\begin{aligned} &\widehat{\phi}_j^{n+\frac{1}{2}} = \frac{1}{2}\big(\widehat{\phi}_j^{n+1} + \widehat{\phi}_j^n\big), \quad \widehat{\Psi}_j^{n+\frac{1}{2}} = \frac{1}{2}\big(\widehat{\Psi}_j^{n+1} + \widehat{\Psi}_j^n\big), \quad \widehat{u}_j^{n+\frac{1}{2}} = \frac{1}{2}\big(\widehat{u}_j^{n+1} + \widehat{u}_j^n\big), \\ &f_{N_1,j}^{n+\frac{1}{2}} = \frac{1}{2}\left(f_{N_1}\big(\widehat{\Psi}_j^{n+1}\big) + f_{N_1}\big(\widehat{\Psi}_j^n\big)\right), \quad f_{N_2,j}^{n+\frac{1}{2}} = \frac{1}{2}\left(f_{N_2}\big(\widehat{\phi}_j^{n+1}\big) + f_{N_2}\big(\widehat{\phi}_j^n\big)\right). \end{aligned} \tag{3.6}$$

In fact, $\widehat{\phi}_j^n$ and $\widehat{\Psi}_j^n$ can be viewed as the approximations to $\phi(t_n, x_j)$ and $\Psi(t_n, x_j)$ respectively. Using the properties of ρ and standard techniques [3], it is easy to check that the above scheme (3.3)-(3.5) is uniquely solvable for sufficiently small time step τ. Define the error functions as

$$\widehat{\eta}_j^n = \phi(t_n, x_j) - \widehat{\phi}_j^n, \quad \widehat{e}_j^n = \Psi(t_n, x_j) - \widehat{\Psi}_j^n, \quad j = 0, 1, \ldots, M, \quad n \geq 0. \tag{3.7}$$

Concerning the errors bounds on $\widehat{\eta}^n$ and $\widehat{e}^n$, we have the following estimates.

Theorem 3.1. *Under Assumption (A), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, such that, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimates for the nonlinear CNFD scheme* (3.3)-(3.5)*:*

$$\|\widehat{\eta}^n\| + \|\delta_x^+ \widehat{\eta}^n\| \lesssim h^2 + \tau^2, \quad \|\widehat{e}^n\| + \|\delta_x^+ \widehat{e}^n\| \lesssim h^2 + \tau^2, \tag{3.8}$$

$$\|\widehat{\eta}^n\|_\infty \lesssim h^2 + \tau^2, \quad \|\widehat{e}^n\|_\infty \lesssim h^2 + \tau^2, \tag{3.9}$$

$$\|\widehat{\phi}^n\|_\infty \leq 1 + N_\phi, \quad \|\widehat{\Psi}^n\|_\infty \leq 1 + N_\Psi, \quad 0 < n \leq T/\tau. \tag{3.10}$$

Theorem 3.2. *Under Assumption (B), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, such that, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimates for the nonlinear CNFD scheme* (3.3)-(3.5)*:*

$$\|\widehat{\eta}^n\| + \|\delta_x^+ \widehat{\eta}^n\| \lesssim h^2 + \tau^2, \quad \|\widehat{e}^n\| \lesssim h^2 + \tau^2, \tag{3.11}$$

$$\|\widehat{\eta}^n\|_\infty \lesssim h^2 + \tau^2, \quad \|\widehat{e}^n\|_\infty \lesssim h^2 + \tau^2, \tag{3.12}$$

$$\|\widehat{\phi}^n\|_\infty \leq 1 + N_\phi, \quad \|\widehat{\Psi}^n\|_\infty \leq 1 + N_\Psi, \quad 0 < n \leq T/\tau. \tag{3.13}$$

We start with the local truncation errors of (3.3)-(3.5) $\widehat{\zeta}^n, \widehat{\xi}^n \in \widetilde{X}_M$ and $\widehat{\theta}^n \in X_M$ as follows:

$$\begin{aligned}
\widehat{\zeta}_j^n &:= \delta_t^+ u(t_n, x_j) - \frac{1}{2}\delta_x^2\big(\phi(t_{n+1}, x_j) + \phi(t_n, x_j)\big) + \frac{1}{2}\big(\phi(t_{n+1}, x_j) + \phi(t_n, x_j)\big) \\
&\quad - \frac{g}{2}\big[\Psi^*(t_{n+1}, x_j)\sigma_3\Psi(t_{n+1}, x_j) + \Psi^*(t_n, x_j)\sigma_3\Psi(t_n, x_j)\big], \quad n \geqslant 0,
\end{aligned} \tag{3.14}$$

$$\widehat{\xi}_j^n := \delta_t^+ \phi(t_n, x_j) - \frac{1}{2}\big(u(t_{n+1}, x_j) + u(t_n, x_j)\big), \quad n \geqslant 0, \tag{3.15}$$

$$\begin{aligned}
\widehat{\theta}_j^n &:= i\delta_t^+ \Psi(t_n, x_j) + \frac{i}{2}\sigma_1\delta_x\big(\Psi(t_{n+1}, x_j) + \Psi(t_n, x_j)\big) \\
&\quad - \frac{\omega}{2}\sigma_3\big(\Psi(t_{n+1}, x_j) + \Psi(t_n, x_j)\big) \\
&\quad - \frac{g}{4}\big(\phi(t_{n+1}, x_j) + \phi(t_n, x_j)\big)\sigma_3\big(\Psi(t_{n+1}, x_j) + \Psi(t_n, x_j)\big), \quad n \geqslant 0.
\end{aligned} \tag{3.16}$$

And we have the following estimates held for $\widehat{\zeta}^n, \widehat{\xi}^n \in \widetilde{X}_M$ and $\widehat{\theta}^n \in X_M$.

Lemma 3.1. *Under Assumption (A) or Assumption (B), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small such that when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, the local truncation errors* (3.14)-(3.16) *satisfy*

(i) Under Assumption (A), we have

$$\begin{aligned}&\|\widehat{\zeta}^n\| \lesssim h^2+\tau^2, \quad \|\widehat{\xi}^n\| + \|\delta_t^+\widehat{\xi}^n\| \lesssim h^2+\tau^2,\\ &\|\widehat{\theta}^n\| + \|\delta_x^+\widehat{\theta}^n\| \lesssim h^2+\tau^2, \quad 0\le n\le T/\tau-1.\end{aligned} \tag{3.17}$$

(ii) Under Assumption (B), we have

$$\begin{aligned}&\|\widehat{\zeta}^n\| \lesssim h^2+\tau^2, \quad \|\widehat{\xi}^n\| + \|\delta_t^+\widehat{\xi}^n\| \lesssim h^2+\tau^2,\\ &\|\widehat{\theta}^n\| \lesssim h^2+\tau^2, \quad 0\le n\le T/\tau-1.\end{aligned} \tag{3.18}$$

Proof. Noticing (2.1), we apply Taylor expansion to the local truncation errors (3.14)-(3.16) under Assumptions (A), then we obtain the following inequalities with the help of triangle inequality and the Cauchy-Schwartz inequality:

$$\begin{aligned}
|\widehat{\zeta}_j^n| &\le \frac{\tau^2}{24}\|\partial_{ttt}u\|_\infty + \frac{\tau^2}{2}\|\partial_{xxtt}\phi\|_\infty + \frac{h^2}{12}\|\partial_{xxxx}\phi\|_\infty\\
&\quad + \frac{\tau^2}{8}\|\partial_{tt}\phi\|_\infty + \frac{g\tau^2}{4}\|\partial_t\Psi^*\|_\infty\|\partial_t\Psi\|_\infty,\\
|\widehat{\xi}_j^n| &\le \frac{\tau^2}{24}\|\partial_{ttt}\phi\|_\infty + \frac{\tau^2}{8}\|\partial_{tt}u\|_\infty,\\
|\delta_t^+\widehat{\xi}_j^n| &\le \frac{\tau^2}{24}\|\partial_{tttt}\phi\|_\infty + \frac{\tau^2}{8}\|\partial_{ttt}u\|_\infty,\\
|\widehat{\theta}_j^n| &\le \frac{\tau^2}{24}\|\partial_{ttt}\Psi\|_\infty + \frac{h^2}{6}\|\partial_{xxx}\Psi\|_\infty + \frac{\tau^2}{8}\|\partial_{xtt}\Psi\|_\infty\\
&\quad + \frac{\tau^2}{8}\|\partial_{tt}\Psi\|_\infty + \frac{g\tau^2}{4}\|\partial_t\phi\|_\infty\|\partial_t\Psi\|_\infty,\\
|\delta_x^+\widehat{\theta}_j^n| &\le \frac{\tau^2}{24}\|\partial_{tttx}\Psi\|_\infty + \frac{h^2}{6}\|\partial_{xxxx}\Psi\|_\infty + \frac{\tau^2}{8}\|\partial_{xxtt}\Psi\|_\infty + \frac{\tau^2}{8}\|\partial_{ttx}\Psi\|_\infty\\
&\quad + \frac{g\tau^2}{4}\big(\|\partial_{tx}\phi\|_\infty\|\partial_t\Psi\|_\infty + \|\partial_t\phi\|_\infty\|\partial_{tx}\Psi\|_\infty\big).
\end{aligned}$$

These immediately imply

$$\|\widehat{\zeta}^n\|_\infty + \|\widehat{\xi}^n\|_\infty \lesssim h^2+\tau^2, \quad \|\widehat{\theta}^n\|_\infty + \|\delta_x^+\widehat{\theta}^n\|_\infty \lesssim h^2+\tau^2, \quad 0\le n\le T/\tau-1. \tag{3.19}$$

Then we can obtain (3.17). Similarly, we can also obtain (3.18). Thus, the conclusions for the local truncation errors are derived. □

Next, we study the growth of the errors. Subtracting (3.3)-(3.5) from (3.14)-(3.16), respectively. Then we denote $\widehat{\mu}_j^n := u(t_n,x_j) - \widehat{u}_j^n$, the error equations can be obtained as for $j=0,1,\dots,M$

$$\delta_t^+\widehat{\mu}_j^n - \frac{1}{2}\delta_x^2\big(\widehat{\eta}_j^{n+1}+\widehat{\eta}_j^n\big) + \frac{1}{2}\big(\widehat{\eta}_j^{n+1}+\widehat{\eta}_j^n\big) = \widehat{\zeta}_j^n + \widehat{\lambda}_j^n, \qquad 0\le n\le T/\tau-1, \tag{3.20}$$

$$\delta_t^+\widehat{\eta}_j^n - \frac{1}{2}\big(\widehat{\mu}_j^{n+1} + \widehat{\mu}_j^n\big) = \widehat{\xi}_j^n, \qquad 0 \le n \le T/\tau - 1, \quad (3.21)$$

$$i\delta_t^+\widehat{e}_j^n + \frac{i}{2}\sigma_1\delta_x\big(\widehat{e}_j^{n+1} + \widehat{e}_j^n\big) - \frac{\omega}{2}\sigma_3\big(\widehat{e}_j^{n+1} + \widehat{e}_j^n\big) = \widehat{\theta}_j^n + \widehat{\chi}_j^n, \quad 0 \le n \le T/\tau - 1, \quad (3.22)$$

$$\widehat{\eta}_0^n = \widehat{\eta}_M^n, \quad \widehat{\eta}_{-1}^n = \widehat{\eta}_{M-1}^n, \quad \widehat{\eta}_j^0 = 0, \qquad 0 \le n \le T/\tau - 1, \quad (3.23)$$

$$\widehat{e}_0^n = \widehat{e}_M^n, \quad \widehat{e}_{-1}^n = \widehat{e}_{M-1}^n, \quad \widehat{e}_j^0 = 0, \qquad 0 \le n \le T/\tau - 1, \quad (3.24)$$

$$\widehat{\mu}_0^n = \widehat{\mu}_M^n, \quad \widehat{\mu}_{-1}^n = \widehat{\mu}_{M-1}^n, \quad \widehat{\mu}_j^0 = 0, \qquad 0 \le n \le T/\tau - 1, \quad (3.25)$$

where $\widehat{\lambda}^n = (\widehat{\lambda}_0^n, \widehat{\lambda}_1^n, \cdots, \widehat{\lambda}_M^n)^T \in \widetilde{X}_M$ and $\widehat{\chi}^n = (\widehat{\chi}_0^n, \widehat{\chi}_1^n, \cdots, \widehat{\chi}_M^n)^T \in X_M$ are the errors of the nonlinear terms as

$$\begin{aligned}\widehat{\lambda}_j^n := \frac{g}{2}\Big[&\big(\Psi^*(t_{n+1}, x_j)\sigma_3\Psi(t_{n+1}, x_j) + \Psi^*(t_n, x_j)\sigma_3\Psi(t_n, x_j)\big)\\ &- \big((\widehat{\Psi}_j^{n+1})^*\sigma_3 f_{N_1}(\widehat{\Psi}_j^{n+1}) + (\widehat{\Psi}_j^n)^*\sigma_3 f_{N_1}(\Psi_j^n)\big)\Big], \end{aligned} \quad (3.26)$$

$$\begin{aligned}\widehat{\chi}_j^n := \frac{g}{4}\Big[&\big(\phi(t_{n+1}, x_j) + \phi(t_n, x_j)\big)\sigma_3\big(\Psi(t_{n+1}, x_j) + \Psi(t_n, x_j)\big)\\ &- \big(f_{N_2}(\widehat{\phi}_j^{n+1}) + f_{N_2}(\widehat{\phi}_j^n)\big)\sigma_3\big(f_{N_1}(\widehat{\Psi}_j^{n+1}) + f_{N_1}(\widehat{\Psi}_j^n)\big)\Big]. \end{aligned} \quad (3.27)$$

In order to prove Theorems 3.1 and 3.2, we propose the following lemmas with regard to the nonlinear terms $\widehat{\lambda}_j^n$ and $\widehat{\chi}_j^n$.

Lemma 3.2. *Under Assumption (A) or Assumption (B), the nonlinear terms $\widehat{\lambda}^n$ and $\widehat{\chi}^n$ $(0 \le n \le T/\tau - 1)$ satisfy*

(i) Under Assumption (A), we have

$$\begin{aligned}&\|\widehat{\lambda}^n\| \lesssim \|\widehat{e}^{n+\frac{1}{2}}\|, \quad \|\widehat{\chi}^n\| \lesssim \|\widehat{e}^{n+\frac{1}{2}}\| + \|\widehat{\eta}^{n+\frac{1}{2}}\|,\\ &\|\delta_x^+\widehat{\chi}^n\| \lesssim \|\widehat{e}^{n+\frac{1}{2}}\| + \|\delta_x^+\widehat{e}^{n+\frac{1}{2}}\| + \|\widehat{\eta}^{n+\frac{1}{2}}\| + \|\delta_x^+\widehat{\eta}^{n+\frac{1}{2}}\|. \end{aligned} \quad (3.28)$$

(ii) Under Assumption (B), we have

$$\|\widehat{\lambda}^n\| \lesssim \|\widehat{e}^{n+\frac{1}{2}}\|, \quad \|\widehat{\chi}^n\| \lesssim \|\widehat{e}^{n+\frac{1}{2}}\| + \|\widehat{\eta}^{n+\frac{1}{2}}\|. \quad (3.29)$$

Proof. Under Assumption (A), rewriting the nonlinear term $\widehat{\lambda}^n$, we get

$$\begin{aligned}\widehat{\lambda}_j^n = \frac{g}{2}\Big[&\Psi^*(t_{n+1}, x_j)\sigma_3\Psi(t_{n+1}, x_j) - \Psi^*(t_{n+1}, x_j)\sigma_3 f_{N_1}(\widehat{\Psi}_j^{n+1})\\ &+ \Psi^*(t_{n+1}, x_j)\sigma_3 f_{N_1}(\widehat{\Psi}_j^{n+1}) - (\widehat{\Psi}_j^{n+1})^*\sigma_3 f_{N_1}(\widehat{\Psi}_j^{n+1})\\ &+ \Psi^*(t_n, x_j)\sigma_3\Psi(t_n, x_j) - \Psi^*(t_n, x_j)\sigma_3 f_{N_1}(\widehat{\Psi}_j^n)\\ &+ \Psi^*(t_n, x_j)\sigma_3 f_{N_1}(\widehat{\Psi}_j^n) - (\widehat{\Psi}_j^n)^*\sigma_3 f_{N_1}(\widehat{\Psi}_j^n)\Big], \end{aligned}$$

and in view of the properties of f_{N_1} and f_{N_2}, we get

$$\left|\widehat{\lambda}_j^n\right| \le \frac{g}{2}C\left[|\widehat{e}^{n+1}|\big(N_\Psi + |f_{N_1}(\widehat{\Psi}_j^{n+1})|\big) + |\widehat{e}^n|\big(N_\Psi + |f_{N_1}(\widehat{\Psi}_j^n)|\big)\right].$$

Similarly, rewrite the nonlinear term $\widehat{\chi}^n$ as follows:

$$\begin{aligned}\widehat{\chi}_j^n = \frac{g}{4}\Big[&\big(\phi(t_{n+1},x_j)+\phi(t_n,x_j)\big)\sigma_3\big(\Psi(t_{n+1},x_j)+\Psi(t_n,x_j)\big)\\ &-\big(\phi(t_{n+1},x_j)+\phi(t_n,x_j)\big)\sigma_3\big(f_{N_1}(\widehat{\Psi}_j^{n+1})+f_{N_1}(\widehat{\Psi}_j^n)\big)\\ &+\big(\phi(t_{n+1},x_j)+\phi(t_n,x_j)\big)\sigma_3\big(f_{N_1}(\widehat{\Psi}_j^{n+1})+f_{N_1}(\widehat{\Psi}_j^n)\big)\\ &-\big(f_{N_2}(\widehat{\phi}_j^{n+1})+f_{N_2}(\widehat{\phi}_j^n)\big)\sigma_3\big(f_{N_1}(\widehat{\Psi}_j^{n+1})+f_{N_1}(\widehat{\Psi}_j^n)\big)\Big],\end{aligned}$$

which combined with (3.2) to get

$$\left|\widehat{\chi}_j^n\right| \le \frac{g}{4}C\Big[N_\phi\left|\widehat{e}_j^{n+1}+\widehat{e}_j^n\right| + \left|f_{N_1,j}^{n+\frac{1}{2}}\right|\left|\widehat{\eta}_j^{n+1}+\widehat{\eta}_j^n\right|\Big],$$

which also implies that

$$\begin{aligned}\left|\delta_x^+\widehat{\chi}_j^n\right| \le \frac{g}{4}C\Big[&\|\partial_x\Psi\|_\infty\left|\widehat{e}_j^{n+1}+\widehat{e}_j^n\right| + N_\phi\delta_x^+\left|\widehat{e}_j^{n+1}+\widehat{e}_j^n\right|\\ &+\delta_x^+\left|f_{N_1,j}^{n+\frac{1}{2}}\right|\left|\widehat{\eta}_j^{n+1}+\widehat{\eta}_j^n\right| + \left|f_{N_1,j}^{n+\frac{1}{2}}\right|\delta_x^+\left|\widehat{\eta}_j^{n+1}+\widehat{\eta}_j^n\right|\Big],\end{aligned}$$

where the constant C is independent of h and τ. Under Assumption (A), applying the properties of f_{N_1} and f_{N_2}, then combining the above inequalities, we obtain (3.28), similarly for Assumption (B), we can obtain (3.29). Then, we complete the proof. □

Based on Lemmas 3.1 and 3.2, we are ready to prove the error estimates of scheme (3.3)-(3.5) under Assumptions (A) and (B).

Proof of Theorem 3.1. When $n=0$, the estimates are obvious and for $1\le n\le T/\tau$, the proof is divided into two parts.

Part 1. (Estimates of $\|\widehat{\eta}^n\|+\|\delta_x^+\widehat{\eta}^n\|+\|\widehat{e}^n\|+\|\delta_x^+\widehat{e}^n\|$ for $1\le n\le T/\tau$). Computing the inner product of (3.20) and (3.21) with $(\widehat{\eta}^{n+1}-\widehat{\eta}^n)$ and $(\widehat{\mu}^{n+1}-\widehat{\mu}^n)$, respectively, then subtracting the two obtained equations, we get

$$\begin{aligned}&\frac{1}{2}\big(\|\widehat{\mu}^{n+1}\|^2-\|\widehat{\mu}^n\|^2\big)+\frac{1}{2}\big(\|\delta_x^+\widehat{\eta}^{n+1}\|^2-\|\delta_x^+\widehat{\eta}^n\|^2\big)+\frac{1}{2}\big(\|\widehat{\eta}^{n+1}\|^2-\|\widehat{\eta}^n\|^2\big)\\ &=\tau h\sum_{j=0}^{M-1}\delta_t^+\widehat{\eta}_j^n\big(\widehat{\zeta}_j^n+\widehat{\lambda}_j^n\big)-\tau h\sum_{j=0}^{M-1}\delta_t^+\widehat{\mu}_j^n\widehat{\xi}_j^n.\end{aligned}\tag{3.30}$$

In (3.30), replacing n by l, then summing it up for $l=0,1,\dots,n-1\le T/\tau$, we get

$$\frac{1}{2}\big(\|\widehat{\mu}^n\|^2-\|\widehat{\mu}^0\|^2\big)+\frac{1}{2}\big(\|\delta_x^+\widehat{\eta}^n\|^2-\|\delta_x^+\widehat{\eta}^0\|^2\big)+\frac{1}{2}\big(\|\widehat{\eta}^n\|^2-\|\widehat{\eta}^0\|^2\big)$$

$$
\begin{aligned}
&= \tau h \sum_{l=0}^{n-1}\sum_{j=0}^{M-1} \delta_t^+ \widehat{\eta}_j^l \big(\widehat{\zeta}_j^l + \widehat{\lambda}_j^l\big) + \tau h \sum_{l=0}^{n-2}\sum_{j=0}^{M-1} \widehat{\mu}_j^{l+1} \delta_t^+ \widehat{\xi}_j^l - \big(\widehat{\mu}^n, \widehat{\xi}^{n-1}\big) \\
&= \tau h \sum_{l=0}^{n-1}\sum_{j=0}^{M-1} \big(\widehat{\mu}_j^{l+\frac{1}{2}} + \widehat{\xi}_j^l\big)\big(\widehat{\zeta}_j^l + \widehat{\lambda}_j^l\big) + \tau h \sum_{l=0}^{n-2}\sum_{j=0}^{M-1} \widehat{\mu}_j^{l+1} \delta_t^+ \widehat{\xi}_j^l - \big(\widehat{\mu}^n, \widehat{\xi}^{n-1}\big), \qquad (3.31)
\end{aligned}
$$

and in (3.31), making use of Cauchy inequality, triangle inequality and Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
&\left(\frac{1}{4}\|\widehat{\mu}^n\|^2 - \frac{1}{2}\|\widehat{\mu}^0\|^2\right) + \frac{1}{2}\left(\|\delta_x^+\widehat{\eta}^n\|^2 - \|\delta_x^+\widehat{\eta}^0\|^2\right) + \frac{1}{2}\left(\|\widehat{\eta}^n\|^2 - \|\widehat{\eta}^0\|^2\right) \\
&\lesssim \tau \sum_{l=0}^{n-1}\left(\|\widehat{\mu}^{l+1}\|^2 + \|\widehat{\mu}^l\|^2 + \|\widehat{\zeta}^l\|^2 + \|\widehat{\lambda}^l\|^2 + \|\delta_t^+\widehat{\xi}^l\|^2\right) + \|\widehat{\xi}^{n+1}\|^2 \\
&\lesssim \tau \sum_{l=0}^{n-1}\left(\|\widehat{\mu}^{l+1}\|^2 + \|\widehat{\mu}^l\|^2 + \|\widehat{e}^{l+1}\|^2 + \|\widehat{e}^l\|^2 + (h^2+\tau^2)^2\right). \qquad (3.32)
\end{aligned}
$$

Next, computing the inner product of $\tau(\widehat{e}^{n+1} + \widehat{e}^n)$ and $\tau\delta_x^2(\widehat{e}^{n+1} + \widehat{e}^n)$ with (3.22), respectively, then subtracting the two obtained equations and taking the imaginary parts, we get

$$
\begin{aligned}
&\|\widehat{e}^{n+1}\|^2 - \|\widehat{e}^n\|^2 + \|\delta_x^+\widehat{e}^{n+1}\|^2 - \|\delta_x^+\widehat{e}^n\|^2 \\
&= \mathrm{Im}\left\{2\tau h \sum_{j=0}^{M-1} \big(\widehat{e}_j^{n+1} + \widehat{e}_j^n\big)^* \big(\widehat{\theta}_j^n + \widehat{\chi}_j^n\big) + \delta_x^+\big(\widehat{e}_j^{n+1} + \widehat{e}_j^n\big)^* \delta_x^+\big(\widehat{\theta}_j^n + \widehat{\chi}_j^n\big)\right\}. \qquad (3.33)
\end{aligned}
$$

In (3.33), replacing n by l, then summing it up for $l = 0, 1, \ldots, n-1 \le T/\tau$ and applying Cauchy inequality, triangle inequality and Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
&\|\widehat{e}^n\|^2 - \|\widehat{e}^0\|^2 + \|\delta_x^+\widehat{e}^n\|^2 - \|\delta_x^+\widehat{e}^0\|^2 \\
&= \mathrm{Im}\left\{2\tau h \sum_{l=0}^{n-1}\sum_{j=0}^{M-1} \big(\widehat{e}_j^{l+1} + \widehat{e}_j^l\big)^* \big(\widehat{\theta}_j^l + \widehat{\chi}_j^l\big) + \delta_x^+\big(\widehat{e}_j^{l+1} + \widehat{e}_j^l\big)^* \delta_x^+\big(\widehat{\theta}_j^l + \widehat{\chi}_j^l\big)\right\} \\
&\lesssim \tau \sum_{l=0}^{n-1}\left(\|\widehat{e}^{l+1}\|^2 + \|\widehat{e}^l\|^2 + \|\delta_x^+\widehat{e}^{l+1}\|^2 + \|\delta_x^+\widehat{e}^l\|^2 + \|\widehat{\theta}^l\|^2 + \|\widehat{\chi}^l\|^2 + \|\delta_x^+\widehat{\theta}^l\|^2 + \|\delta_x^+\widehat{\chi}^l\|^2\right) \\
&\lesssim \tau \sum_{l=0}^{n-1}\Big(\|\widehat{e}^{l+1}\|^2 + \|\widehat{e}^l\|^2 + \|\delta_x^+\widehat{e}^{l+1}\|^2 + \|\delta_x^+\widehat{e}^l\|^2 + \|\widehat{\eta}^{l+1}\|^2 + \|\widehat{\eta}^l\|^2 \\
&\qquad + \|\delta_x^+\widehat{\eta}^{l+1}\|^2 + \|\delta_x^+\widehat{\eta}^l\|^2 + (h^2+\tau^2)^2\Big). \qquad (3.34)
\end{aligned}
$$

Denote

$$
S^n := \frac{1}{4}\|\widehat{\mu}^n\|^2 + \frac{1}{2}\|\delta_x^+\widehat{\eta}^n\|^2 + \frac{1}{2}\|\widehat{\eta}^n\|^2 + \|\widehat{e}^n\|^2 + \|\delta_x^+\widehat{e}^n\|^2, \qquad (3.35)
$$

then the inequalities (3.32) and (3.34) imply

$$S^n \lesssim \tau \sum_{l=0}^{n-1} S^l + (h^2+\tau^2)^2, \quad 1 \le n \le T/\tau, \tag{3.36}$$

therefore, the inequality implies that there exists a constant τ_1 sufficiently small, such that when $0 < \tau \le \tau_1$, by making using of the discrete Gronwall's inequality, the following holds

$$S^n \lesssim (h^2+\tau^2)^2, \quad 1 \le n \le T/\tau, \tag{3.37}$$

this together with (3.35) gives

$$\|\widehat{\eta}^n\| + \|\delta_x^+\widehat{\eta}^n\| + \|\widehat{e}^n\| + \|\delta_x^+\widehat{e}^n\| \lesssim h^2+\tau^2, \quad 1 \le n \le T/\tau. \tag{3.38}$$

Part 2. (Estimates of $\|\widehat{\phi}^n\|_\infty$ and $\|\widehat{\psi}^n\|_\infty$ for $1 \le n \le T/\tau$). Applying the discrete Sobolev inequality, we get from (3.38) that

$$\|\widehat{\eta}^n\|_\infty \le C\big(\|\widehat{\eta}^n\| + \|\delta_x^+\widehat{\eta}^n\|\big) \le C(h^2+\tau^2), \quad \|\widehat{e}^n\|_\infty \le C(h^2+\tau^2), \tag{3.39}$$

thus, there exist $h_1 > 0$ and $\tau_2 > 0$ sufficiently small, such that, when $0 < h \le h_1$ and $0 < \tau \le \tau_2$, there are

$$\begin{aligned} &\|\widehat{\phi}^n\|_\infty \le \|\phi(t_n,x_j)\|_\infty + \|\widehat{\eta}^n\|_\infty \le N_\phi + 1, \\ &\|\widehat{\psi}^n\|_\infty \le \|\Psi(t_n,x_j)\|_\infty + \|\widehat{e}^n\|_\infty \le N_\Psi + 1. \end{aligned} \tag{3.40}$$

Choosing $\tau_0 = \min\{\tau_1,\tau_2\}$ and $h_0 = h_1$, the proof of the Theorem 3.1 is completed. □

Based on the proof of Theorem 3.1, and recalling the definition of ρ, Theorem 3.1 implies that (3.3)-(3.5) collapse to (2.4)-(2.6). Thus Theorem 2.2 is a direct consequence of Theorem 3.1.

Proof of Theorem 3.2. Under Assumption (B), we give a brief proof process for the error estimates of scheme (3.8)-(3.10). Similarly, when $n = 0$, the estimates are obvious and for $1 \le n \le T/\tau$, the proof is divided into two parts.

Part 1. (Estimates of $\|\widehat{\eta}^n\| + \|\delta_x^+\widehat{\eta}^n\| + \|\widehat{e}^n\|$ for $1 \le n \le T/\tau$). Recalling the proof of Theorem 3.1, we can get the following results directly. Denote

$$S^n := \frac{1}{4}\|\widehat{\mu}^n\|^2 + \frac{1}{2}\|\delta_x^+\widehat{\eta}^n\|^2 + \frac{1}{2}\|\widehat{\eta}^n\|^2 + \|\widehat{e}^n\|^2, \tag{3.41}$$

we can get

$$S^n \lesssim \tau \sum_{l=0}^{n-1} S^l + T(h^2+\tau^2)^2, \quad 1 \le n \le T/\tau, \tag{3.42}$$

therefore, the inequality implies that there exists a constant τ_3 sufficiently small, such that, when $0 < \tau \le \tau_3$ by using the discrete Gronwall's inequality, the following holds

$$S^n \lesssim (h^2+\tau^2)^2, \quad 1 \le n \le T/\tau, \tag{3.43}$$

due to (3.41), we get

$$\|\widehat{\eta}^n\| + \|\delta_x^+\widehat{\eta}^n\| + \|\widehat{e}^n\| \lesssim h^2 + \tau^2, \quad 1 \le n \le T/\tau. \tag{3.44}$$

Part 2. (Estimates of $\|\widehat{\phi}^n\|_\infty$ and $\|\widehat{\psi}^n\|_\infty$ for $1 \le n \le T/\tau$). Applying the discrete Sobolev inequality, we get

$$\|\widehat{\eta}^n\|_\infty \le C\big(\|\widehat{\eta}^n\| + \|\delta_x^+\widehat{\eta}^n\|\big) \le C(h^2 + \tau^2). \tag{3.45}$$

By inverse inequality we can get for $1 \le n \le T/\tau$,

$$\|\widehat{e}^n\|_\infty \le \frac{1}{\sqrt{h}}\|\widehat{e}^n\| \le \frac{h^2 + \tau^2}{\sqrt{h}}. \tag{3.46}$$

On the other hand, due to the triangle inequality, the error equation (3.22) implies

$$\begin{aligned}
\|\delta_x(\widehat{e}^{n+1} + \widehat{e}^n)\| &\le C\left(\|\delta_t^+\widehat{e}^n\| + \|\widehat{e}^n\| + \|\widehat{e}^{n+1}\| + \|\widehat{\eta}^n\| + \|\widehat{\eta}^{n+1}\| + \|\widehat{\theta}^n|\right) \\
&\le C\left(\frac{\|\widehat{e}^n\| + \|\widehat{e}^{n+1}\|}{\tau} + (h^2 + \tau^2)\right) \\
&\le C\frac{h^2 + \tau^2}{\tau}, \quad 0 \le n \le T/\tau - 1
\end{aligned} \tag{3.47}$$

for sufficiently small τ and the constant C is independent of h and τ. The Sobolev inequality shows that

$$\begin{aligned}
\|\widehat{e}^{n+1}\|_\infty - \|\widehat{e}^n\|_\infty &\le \|\widehat{e}^{n+1} + \widehat{e}^n\|_\infty \\
&\le C\|\widehat{e}^{n+1} + \widehat{e}^n\|^{\frac{1}{2}}\big(\|\widehat{e}^{n+1} + \widehat{e}^n\| + \|\delta_x^+(\widehat{e}^{n+1} + \widehat{e}^n)\|\big)^{\frac{1}{2}} \\
&\le C\frac{h^2 + \tau^2}{\tau^{\frac{1}{2}}}, \quad 0 \le n \le T/\tau - 1.
\end{aligned} \tag{3.48}$$

Summing the (3.48) together for time step $0, 1, \ldots, n-1$, we have

$$\|\widehat{e}^n\|_\infty \le Cn\frac{h^2 + \tau^2}{\tau^{\frac{3}{2}}} \le CT\frac{h^2 + \tau^2}{\tau^{\frac{3}{2}}}, \quad 1 \le n \le T/\tau. \tag{3.49}$$

Therefore, for $1 \le n \le T/\tau$, in view of (3.46) and (3.49), we have

$$\|\widehat{e}^n\|_\infty \le C\min\left\{\frac{h^2 + \tau^2}{\sqrt{h}}, \frac{h^2 + \tau^2}{\tau^{\frac{3}{2}}}\right\} \le C\big(2\tau^{\frac{1}{4}}h^{\frac{3}{4}} + \tau^{\frac{1}{2}} + h^{\frac{3}{2}}\big), \tag{3.50}$$

thus, there exist $h_2 > 0$ and $\tau_4 > 0$ sufficiently small, such that, when $0 < h \le h_2$ and $0 < \tau \le \tau_4$, we obtain

$$\begin{aligned}
\|\widehat{\phi}^n\|_\infty &\le \|\phi(t_n, x_j)\|_\infty + \|\widehat{\eta}^n\|_\infty \le N_\phi + 1, \\
\|\widehat{\psi}^n\|_\infty &\le \|\Psi(t_n, x_j)\|_\infty + \|\widehat{e}^n\|_\infty \le N_\Psi + 1.
\end{aligned} \tag{3.51}$$

Choosing $\tau_0' = \min\{\tau_3, \tau_4\}$ and $h_0' = h_2$, the proof of Theorem 3.2 is complete. □

Based on the proof of Theorem 3.2, and recalling the definition of ρ, Theorem 3.2 implies that (3.3)-(3.5) collapse to (2.4)-(2.7). Thus Theorem 2.3 is a direct consequence of Theorem 3.2.

Remark 3.1. The above theorems can be directly generalized to the two-dimensional case. That is to say, they are still valid under the technical conditions $0 < \tau \lesssim 1/\sqrt{C_d(h)}$ and $0 < h \lesssim 1/\sqrt{C_d(h)}$. The key observation is that $\|\eta^n\|_\infty$ and $\|e^n\|_\infty$ can be controlled by the discrete Sobolev inequality [3] in 2D as

$$\|U\|_\infty \lesssim C_d(h)\big[\|U\| + \|\delta_x^+ U\|\big], \quad C_d(h) = \begin{cases} 1, & d = 1, \\ |\ln h|, & d = 2, \end{cases} \tag{3.52}$$

where U is a periodic 2D mesh function.

4. Numerical examples

In this section, we report numerical results to verify our theoretical error analysis on the difference solutions. The computational interval Ω is chosen as $\Omega = [-128, 128]$, i.e. $a = -128$ and $b = 128$. In the computation, the problem is solved numerically with the coefficients $g = 1$ and $\omega = 1$. Let $\phi(t, x)$ and $\Psi(t, x)$ be the 'exact' solution which is obtained numerically by using the scheme (2.4)-(2.7) with very fine mesh size and small time step, i.e. $h_e = 1/2048$ and $\tau_e = 5e^{-6}$. In order to quantify the convergence, we define the error functions (l^2-error, discrete H^1-error and l^∞-error) as

$$\begin{aligned}
&e_{H^1}^{\phi}(t_n) = \sqrt{\|\phi(t_n,\cdot) - \phi^n\|_{l^2}^2 + \|\delta_x^+(\phi(t_n,\cdot) - \phi^n)\|_{l^2}^2}, \\
&e_{H^1}^{\Psi}(t_n) = \sqrt{\|\Psi(t_n,\cdot) - \Psi^n\|_{l^2}^2 + \|\delta_x^+(\Psi(t_n,\cdot) - \Psi^n)\|_{l^2}^2}, \\
&e_{l^2}^{\phi}(t_n) = \|\phi(t_n,\cdot) - \phi^n\|_{l^2}, \quad e_{l^2}^{\Psi}(t_n) = \|\Psi(t_n,\cdot) - \Psi^n\|_{l^2}, \\
&e_{\infty}^{\phi}(t_n) = \|\phi(t_n,\cdot) - \phi^n\|_{\infty}, \quad e_{\infty}^{\Psi}(t_n) = \|\Psi(t_n,\cdot) - \Psi^n\|_{\infty}.
\end{aligned}$$

For the initial conditions, here we take

$$\phi_0(x) = e^{-\frac{x^2}{2}}, \quad \phi_1(x) = \frac{3}{2}e^{-\frac{x^2}{2}}, \quad \Psi_0(x) = \big(e^{-\frac{x^2}{2}}, e^{-\frac{(x-1)^2}{2}}\big)^T. \tag{4.1}$$

We test and study the temporal and spatial error separately. Firstly, we measure the spatial discretization errors. In order to do this, we fix the time step size $\tau = 5e^{-4}$ sufficiently small, such that the errors from time discretization are negligible, and solve KGD with CNFD method under different mesh sizes h. Tables 1 and 2 list the numerical errors e_ϕ, Ψ at $t = 1$ with different mesh sizes h for scheme (2.4)-(2.7).

Next we test the temporal errors at $t = 1$, listed in Tables 3 and 4 under different τ with a very small mesh size $h = h_e$ such that the discretization errors in space are negligible.

Finally, Figs. 1 and 2 display the values of total mass and energy in the discrete level at different times with $h = 1/128$ and $\tau = 0.001$ which confirms the mass and energy conservation of the proposed CNFD scheme.

Table 1: Spatial errors of CNFD for ϕ at $t = 1$.

Spatial error	$h_0 = 1/8$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$e^{\phi}_{l^2}$	7.71E-4	1.93E-4	4.82E-5	1.21E-6	3.04E-6
Order	—	2.00	2.00	2.00	1.99
e^{ϕ}_{∞}	5.81E-4	1.45E-4	3.63E-5	9.11E-6	2.30E-6
Order	—	2.00	2.00	2.00	1.99
$e^{\phi}_{H^1}$	1.80E-3	4.52E-4	1.13E-4	2.83E-5	7.12E-6
Order	—	2.00	2.00	2.00	1.99

Table 2: Spatial errors of CNFD for Ψ at $t = 1$.

Spatial error	$h_0 = 1/8$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$e^{\Psi}_{l^2}$	8.20E-3	2.10E-3	5.18E-4	1.30E-4	3.26E-5
Order	—	1.99	2.00	2.00	1.99
e^{Ψ}_{∞}	4.10E-3	1.00E-3	2.58E-4	6.46E-5	1.61E-5
Order	—	1.98	2.00	2.00	2.00
$e^{\Psi}_{H^1}$	2.00E-2	5.00E-3	1.30E-3	3.16E-4	7.94E-5
Order	—	1.98	2.00	2.00	1.99

Table 3: Temporal errors of CNFD for ϕ at $t = 1$.

Temporal error	$\tau_0 = 1/20$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$e^{\phi}_{l^2}$	1.00E-3	2.58E-4	6.46E-5	1.62E-5	4.04E-6
Order	—	2.00	2.00	2.00	2.00
e^{ϕ}_{∞}	8.27E-4	2.07E-4	5.16E-5	1.29E-5	3.23E-6
Order	—	2.00	2.00	2.00	2.00
$e^{\phi}_{H^1}$	1.50E-3	3.83E-4	9.56E-5	2.39E-5	5.98E-6
Order	—	2.00	2.00	2.00	2.00

Table 4: Temporal errors of CNFD for Ψ at $t = 1$.

Temporal error	$\tau_0 = 1/20$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$e^{\Psi}_{l^2}$	6.70E-3	1.70E-3	4.19E-4	1.05E-4	2.62E-5
Order	—	2.00	2.00	2.00	2.00
e^{Ψ}_{∞}	4.10E-3	1.00E-3	2.56E-4	6.41E-5	1.60E-5
Order	—	2.00	2.00	2.00	2.00
$e^{\Psi}_{H^1}$	9.10E-3	2.30E-3	5.68E-4	1.42E-4	3.55E-5
Order	—	2.00	2.00	2.00	2.00

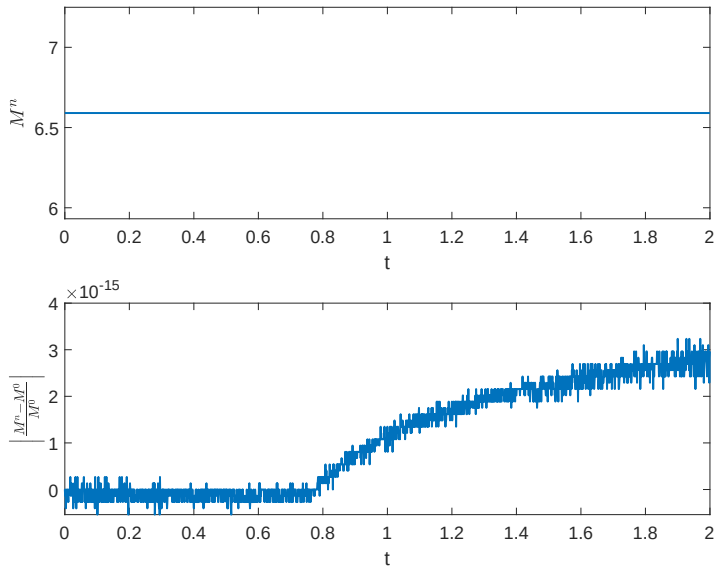

Figure 1: The discrete mass M^n with $\tau = 0.001$ and $h = 1/128$.

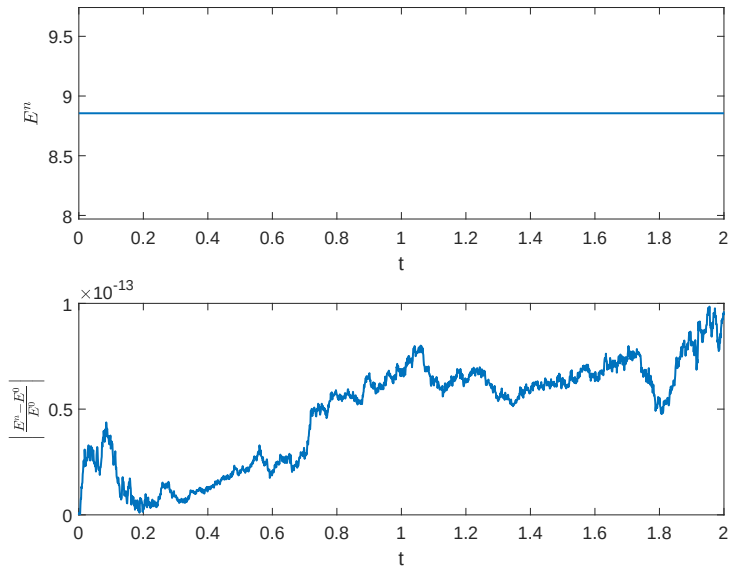

Figure 2: The discrete energy E^n with $\tau = 0.001$ and $h = 1/128$.

5. Conclusion and further question

In conclusion, a new conservative CNFD method was proposed and analyzed for solving the KGD system. After converting the auxiliary function, we prove that the new scheme perfectly conserved the total mass and energy in the discrete level. By utiliz-

ing 'cut-off' function technique and energy method, two type of error estimates were rigorously established under two different hypotheses, which showed that the CNFD method was second-order accurate in both space and time. Numerical experiments were conducted to confirm our estimates of the proposed scheme. In future, our further exploration is to give an unconditional error estimates of the proposed scheme for solving the two-dimensional KGD equations, and we will discuss it in detail in our next research work.

References

[1] A. ALVAREZ, *Linearized Crank-Nicholson scheme for nonlinear Dirac equations*, J. Comput. Phys. 99 (1992), 348–350.

[2] A. BACHELOT, *Global Existence of Large Amplitude Solutions for Dirac-Klein-Gordon Systems in Minkowski Space*, in: *Nonlinear Hyperbolic Problems, C. Carasso, P. Charrier, B. Hanouzet, JL. Joly (Eds)*, Lecture Notes in Mathematics, Vol. 1402, Springer, (1989).

[3] W. BAO AND Y. CAI, *Optimal error estimates of finite difference methods for the Gross-Pitaevskii equation with angular momentum rotation*, Math. Comp. 82 (2013), 99–128.

[4] W. BAO, Y. CAI, X. JIA, AND Q. TANG, *A uniformly accurate multiscale time integrator pseudospectral method for the Dirac equation in the nonrelativistic limit regime*, SIAM J. Numer. Anal. 54 (2016), 1785–1812.

[5] W. BAO, Y. CAI, X. JIA, AND Q. TANG, *Numerical methods and comparison for the Dirac equation in the nonrelativistic limit regime*, J. Comput. Phys. 71 (2017), 1094–1134.

[6] W. BAO, Y. CAI, X. JIA, AND J. YIN, *Error estimates of numerical methods for the nonlinear Dirac equation in the nonrelativistic limit regime*, Sci. China Math. 59 (2016), 1461–1494.

[7] W. BAO, Y. CAI, AND X.ZHAO, *A uniformly accurate multiscale time integrator pseudospectral method for the Klein-Gordon equation in the nonrelativistic limit regime*, SIAM. J. Numer. Anal. 52 (2014), 2488–2511.

[8] W. BAO AND X. DONG, *Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime*, Numer. Math. 120 (2012), 189–229.

[9] W. BAO AND X. LI, *An efficient and stable numerical method for the Maxwell-Dirac system*, J. Comput. Phys. 199 (2004), 663–687.

[10] W. BAO AND Y. LI, *Efficient and accurate numerical methods for the Klein-Gordon-Schrödinger equations*, J. Comput. Phys. 225 (2007), 1863–1893.

[11] W. BAO AND X. ZHAO, *A uniformly accurate multiscale time integrator spectral method for the Klein-Gordon-Zakharov system in the high-plasma-frequency limit regime*, J. Comput. Phys. 327 (2016), 270–293.

[12] S. BAUMSTARK, E. FAOU, AND K. SCHRATZ, *Uniformly accurate exponential-type integrators for Klein-Gordon equations with asymptotic convergence to the classical NLS splitting*, Math. Comp. 87 (2018), 1227–1254.

[13] J. D. BJORKEN AND S. D. DRELL, *Relativistic Quantum Fields*, McGraw-Hill. Inc, (1965).

[14] N. BOURNAVEAS, *Local existence of energy class solutions for the Dirac-Klein-Gordon equations*, Commun. Partial. Differ. Equ. 24 (1999), 1167–1193.

[15] N. BOURNAVEAS, *Low regularity solutions of the Dirac-Klein-Gordon equations in two space dimensions*, Commun. Partial. Differ. Equ. 26 (2001), 1345–1366.

[16] J. CAI AND H. LIANG, *Explicit multisymplectic Fourier pseudospectral scheme for the Klein-Gordon-Zakharov equations*, Chinese Phys. Lett. 29 (2012), 1028–1032.

[17] Y. CAI AND W. YI, *Error estimates of finite difference time domain methods for the Klein-Gordon-Dirac system in the nonrelativistic limit regime*, Commun. Math. Sci. 16 (2018), 1325–1346.

[18] Y. CAI AND Y. YUAN, *Uniform error estimates of the conservative finite difference method for the Zakharov system in the subsonic limit regime*, Math. Comput. 87 (2018), 1191–1225.

[19] J. M. CHADAM AND R. T. GLASSEY, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions*, Arch. Ration. Mech. An. 54 (1974), 223–237.

[20] G. CHEN AND Y. ZHENG, *Solitary waves for the Klein-Gordon-Dirac model*, J. Differential Equations 253 (2012), 2263–2284.

[21] M. DEHGHAN AND A. TALEEI, *Numerical solution of the Yukawa-coupled Klein-Gordon-Schrödinger equations via a Chebyshev pseudospectral multidomain method*, Appl. Math. Model. 36 (2012), 2340–2349.

[22] X. DONG, Z.XU, AND X.ZHAO, *On time-splitting pseudospectral discretization for nonlinear Klein-Gordon equation in nonrelativistic limit regime*, Commun. Comput. Phys. 16 (2014), 440–466.

[23] M. J. ESTEBAN, V. GEORGIEV, AND E. SÉRÉ, *Bound-state solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac systems*, Lett. Math. Phys. 38 (1996), 217–220.

[24] Y. FANG, *A direct proof of global existence for the Dirac-Klein-Gordon equations in one space dimension*, Taiwanese J. Math. 8 (2004), 33–41.

[25] W. GREINER, *Relativistic Quantum Mechanics. Wave Equations*, Springer, (1994).

[26] R. HAMMER, W. PÖTZ, AND A. ARNOLD, *A dispersion and norm preserving finite difference scheme with transparent boundary conditions for the Dirac equation in (1+1)D*, J. Comput. Phys. 256 (2014), 728–747.

[27] J. W. V. HOLTEN, *On the electrodynamics of spinning particles*, Nucl. Phys. B. 356 (1991), 3–26.

[28] Z. HUANG, S. JIN, P. MARKOWICH, C. SPARBER, AND C. ZHENG, *A time-splitting spectral scheme for the Maxwell-Dirac system*, J. Sci. Comput. 208 (2005), 761–789.

[29] S. JIMNEZ AND L. VZQUEZ, *Analysis of four numerical schemes for a nonlinear Klein-Gordon equation*, Appl. Math. Comput. 35 (1990), 61–94.

[30] M. LEMOU, F. MHATS, AND X. ZHAO, *Uniformly accurate numerical schemes for the nonlinear Dirac equation in the nonrelativistic limit regime*, Commun. Math. Sci. 15 (2017), 1107–1128.

[31] E. LORIN AND A. BANDRAUK, *A simple and accurate mixed p0-q1 solver for the Maxwell-Dirac equations*, Nonlinear Anal. Real World Appl. 12 (2011), 190–202.

[32] S. MACHIHARA AND T. OMOSO, *The explicit solutions to the nonlinear Dirac equation and Dirac-Klein-Gordon equation*, Ric. Mat. 56 (2007), 19–30.

[33] T. OHLSSON, *Relativistic Quantum Physics: From Advanced Quantum Mechanics to Introductory Quantum Field Theory*, Cambridge University Press, (2011).

[34] P. J. PASCUAL, S. JIMNEZZ, AND L. VZQUEZ, *Numerical Simulations of a Nonlinear Klein-Gordon Model and Applications*, Springer, (1995).

[35] S. SELBERG AND A. TESFAHUN, *Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension*, Commun. Contemp. Math. 10 (2008), 181–194.

[36] J. J. SLAWIANOWSKI AND V. KOVALCHUK, *Klein-Gordon-Dirac equation: Physical justification and quantization attempts*, Rep. Math. Phys. 49 (2002), 249–257.

[37] C. SU AND W. YI, *Error estimates of a finite difference method for the Klein-Gordon-Zakharov system in the subsonic limit regime*, IMA J. Numer. Anal. 38 (2017), 2055–2073.

[38] V. THOMÉE, *Galerkin Finite Element Methods for Parabolic Problems*, in: *Springer Series in*

Computational Mathematics, Springer-Verlag, (1986).

[39] J. Xu, S. Shao, and H. Tang, *Numerical methods for nonlinear Dirac equation*, J. Comput. Phys. 245 (2013), 131–149.

[40] J. Yan, H. Zhang, X. Qian, and S. Song, *Regularised finite difference methods for the logarithmic Klein-Gordon equation*, East Asian J. Appl. Math. 11 (2021), 119–142.

[41] W. Yi and Y. Cai, *Optimal error estimates of finite difference time domain methods for the Klein-Gordon-Dirac system*, IMA J. Numer. Anal. 40 (2020), 1266–1293.

[42] W. Yi, X. Ruan, and C. Su, *Optimal resolution methods for the Klein-Gordon-Dirac system in the nonrelativistic limit regime*, J. Sci. Comput. 79 (2019), 1907–1935.

[43] C. Zhang and H. Sun, *A linearised three-point combined compact difference method with weighted approximation for nonlinear time fractional Klein-Gordon equations*, East Asian J. Appl. Math. 11 (2021), 604–617.