Constructing Order Two Superconvergent WG Finite Elements on Rectangular Meshes

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Abstract. In this paper, we introduce a stabilizer free weak Galerkin (SFWG) finite element method for second order elliptic problems on rectangular meshes. With a special weak Gradient space, an order two superconvergence for the SFWG finite element solution is obtained, in both $L^2$ and $H^1$ norms. A local post-process lifts such a $P_k$ weak Galerkin solution to an optimal order $P_{k+2}$ solution. The numerical results confirm the theory.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, weak Galerkin method, stabilizer free, rectangular mesh.

1. Introduction

A new stabilizer free weak Galerkin method is developed to solve the following second order elliptic problem:

\begin{align}
-\Delta u &= f \quad \text{in } \Omega, \quad (1.1) \\
u &= g \quad \text{on } \partial \Omega, \quad (1.2)
\end{align}

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$, which can be subdivided into rectangular meshes.

The weak Galerkin (WG) finite element methods introduced in [24, 25] provide a general finite element technique for solving partial differential equations. The novelty of the WG method is the introduction of weak function and its weakly defined derivatives. The weak functions possess the form of \( v = \{v_0, v_b\} \) with $v = v_0$ representing the value of $v$ in the interior of each element and $v = v_b$ on the boundary of the element.

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The weak derivative $\nabla_w v$ for a weak function $v$ is defined as distributions. WG method uses polynomials $(P_k(T), P_s(e), [P_l(T)]^d)$ to approximate $(v_0, v_b, \nabla_w v)$ accordingly. The WG methods have been applied for solving various PDEs such as Sobolev equation, the Navier-Stokes equations, the Oseen equations, time-dependent Maxwell’s equations, elliptic interface problems, biharmonic equations, etc, [1, 5–17, 21–23, 26, 27, 30].

For some special combinations of the WG element $(P_k(T), P_s(e), [P_l(T)]^d)$, stabilizer is no longer needed in the corresponding weak Galerkin finite element formulations, which leads to a stabilizer free weak Galerkin method. The stabilizer free weak Galerkin method was first introduced in [28] on polygonal/polyhedral meshes and then has been applied for the second order problems, the Stokes equations and the biharmonic equation [2, 18, 29].

This paper has two purposes:

1. Developing a new SFWG method with an order two superconvergence for the problem (1.1)-(1.2).

2. Providing necessary theory for a subsequent paper, order two superconvergent conforming discontinuous Galerkin method on rectangular meshes.

A WG element $(P_k(T), P_{k+1}(e), BDM[k][T])$ on rectangular mesh is used in this stabilizer free weak Galerkin finite element method. We prove that the SFWG method converges to the true solution of (1.1)-(1.2) with a convergence rate two orders higher than the optimal order in both an energy norm and the $L^2$ norm theoretically and numerically. We further define a local post-process which lifts such a $P_k$ weak Galerkin solution to an optimal order $P_{k+2}$ solution. It is proved and numerically verified.

2. The weak Galerkin finite element scheme

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of rectangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ be the set of all interior edges. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and the mesh size by $h = \max_{T \in \mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let $V_h$ be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows:

$$V_h = \{ v = \{v_0, v_b\} : v_0 |_{T} \in P_k(T), v_b |_{e} \in P_{k+1}(e), e \subset \partial T, T \in \mathcal{T}_h \}$$

and its subspace $V_h^0$ is defined as

$$V_h^0 = \{ v : v \in V_h, v_b = 0 \text{ on } \partial \Omega \}.$$  

We would like to emphasize that any function $v \in V_h$ has a single value $v_b$ on each edge $e \in \mathcal{E}_h$.

On each rectangle $T \in \mathcal{T}_h$, the BDM finite element space is defined by [4]

$$BDM_{[k+1]}(T) = P_{k+1}(T)^2 \oplus \text{curl} x^{k+2} y \oplus \text{curl} xy^{k+2}.$$
For \( v = \{v_0, v_b\} \in V_h \), a weak gradient \( \nabla_w v \) is a piecewise vector valued polynomial such that on each \( T \in T_h \), \( \nabla_w v \in \text{BDM}_{k+1}(T) \) satisfies
\[
(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in \text{BDM}_{k+1}(T).
\] (2.3)

For simplicity, we adopt the following notations:
\[
(v, w)_{T_h} = \sum_{T \in T_h} (v, w)_T = \sum_{T \in T_h} \int_T vwdx,
\]
\[
(v, w)_{\partial T_h} = \sum_{T \in T_h} \sum_{e \subset \partial T} \langle v, w \rangle_e = \sum_{T \in T_h} \int_{\partial T} vwdx.
\]

**Algorithm 2.1 (Weak Galerkin algorithm).** A numerical approximation for (1.1)-(1.2) can be obtained by seeking \( u_h = \{u_0, u_b\} \in V_h \) satisfying \( u_b = Q_b g \) on \( \partial \Omega \) and the following equation:
\[
(\nabla_w u_h, \nabla_w v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V^0_h.
\] (2.4)

### 3. Well posedness

For any \( v \in V_h \), a semi-\( H^1 \)-like semi-norm is defined as follows:
\[
\|v\|^2 = (\nabla_w v, \nabla_w v).
\] (3.1)

We introduce a discrete semi-\( H^1 \) norm as follows:
\[
\|v\|^2_{1,h} = (\nabla v_0, \nabla v_0)_{T_h} + \langle h^{-1}_T(v_0 - v_b), v_0 - v_b \rangle_{\partial T_h}.
\] (3.2)

For any function \( \varphi \in H^1(T) \), the trace inequality holds true
\[
\|\varphi\|^2_e \leq C (h^{-1}_T \|\varphi\|^2_T + h_T \|\nabla \varphi\|^2_T).
\] (3.3)

Next we will show that \( \|\cdot\| \) also defines a norm for \( V^0_h \) by proving the equivalence of \( \|\cdot\| \) and \( \|\cdot\|_{1,h} \) in \( V_h \). For \( q \in H(\text{div}, \Omega) \), by [4], we define a BDM interpolation \( \Pi_h q \) such that \( \Pi_h q|_T \in \text{BDM}_{[k+1]}(T) \) for \( T \in T_h \) satisfies
\[
\langle (q - \Pi_h q) \cdot n, p_{k+1} \rangle_e = 0, \quad \forall p_{k+1} \in P_{k+1}(e), \quad e \subset \partial T,
\]
\[
(q - \Pi_h q, p_{k-1})_T = 0, \quad \forall p_{k-1} \in [P_{k-1}(T)]^2.
\] (3.4)

**Lemma 3.1 ([4]).** Let \( q \in H^{k+2}(\Omega)^2 \).
\[
\|q - \Pi_h q\| \leq C h^{k+2} |q|_{k+2},
\] (3.6)
\[
\|\nabla \cdot (q - \Pi_h q)\|_T \leq C h^{k+1} |\nabla \cdot q|_{k+1,T}.
\] (3.7)
Lemma 3.2. There exist two positive constants $C_1$ and $C_2$ such that
\[ C_1 \|v\|_{1,h} \leq \|v\| \leq C_2 \|v\|_{1,h}, \quad \forall v \in V_h. \tag{3.8} \]

Proof. We prove the upper bound first. By the definition of weak gradient (2.3), letting $w = \nabla_w v$, we have
\[
\|v\|^2 = \sum_{T \in T_h} - (v_0, \nabla \cdot \nabla_w v)_T + (v_b, \nabla_w \cdot n)_{\partial T} \\
= \sum_{T \in T_h} (\nabla v_0, \nabla_w v)_T + (v_b - v_0, \nabla_w \cdot n)_{\partial T} \\
\leq \sum_{T \in T_h} (\nabla v_0, \nabla_w v)_T + \|v_b - v_0\|_{\partial T} \|\nabla_w v\|_{\partial T} \\
\leq \sum_{T \in T_h} \left( \|\nabla v_0\|_T + \frac{\|v_b - v_0\|_{\partial T}}{Ch^{1/2}_T} \right) \|\nabla_w v\|_T \leq C_2 \|v\|_{1,h} \|v\|,
\]
where we applied the trace inequality (3.3) and the inverse inequality.

To prove the lower bound, we need to choose an appropriate $q$ in the definition of weak gradient (2.3) so that the above inequality can be reversed. Let $q \in BDM_{[k+1]}(T)$ be defined, similar to the BDM interpolation $\Pi_h$ in (3.4)-(3.5), by
\[
(q - \nabla v_0, p_{k-1})_{T} = 0, \quad \forall p_{k-1} \in P_{k-1}(T)^2, \tag{3.9}
\]
\[
\langle q \cdot n - h^{-1}_T (v_0 - v_b), p_{k+1} \rangle_e = 0, \quad \forall p_{k+1} \in P_{k+1}(e), \quad e \subset \partial T. \tag{3.10}
\]
By (3.4)-(3.5), (3.9)-(3.10) define a unique $q$. Further, by finite dimensional norm equivalence and scaling argument,
\[
\|q\| \leq C \|v\|_{1,h}. \tag{3.11}
\]
Using this $q$ in (2.3), we have
\[
\|v\|^2_{1,h} = (\nabla v_0, \nabla v)_T + \langle h^{-1}_T (v_0 - v_b), v_0 - v_b \rangle_{\partial T} \\
= (\nabla v_0, q)_T + \langle v_0 - v_b, q \cdot n \rangle_{\partial T} \\
= (\nabla_w v, q)_T \leq \|v\| \|q\| \leq C_1^{-1} \|v\| \|v\|_{1,h}.
\]
The lemma is proved. \hfill \square

Lemma 3.3. The weak Galerkin finite element scheme (2.4) has a unique solution.

Proof. Let $u_h^{(1)}$ and $u_h^{(2)}$ be the two solutions of (2.4), then $\varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V_h^0$ would satisfy the following equation:
\[
(\nabla_w \varepsilon_h, \nabla_w v) = 0, \quad \forall v \in V_h^0.
\]
Then by letting $v = \varepsilon_h$ in the above equation, we arrive at
\[
\|\varepsilon_h\|^2 = (\nabla_w \varepsilon_h, \nabla_w \varepsilon_h) = 0.
\]
It follows from (3.8) that $\|\varepsilon_h\|_{1,h} = 0$. Since $\|\cdot\|_{1,h}$ is a norm in $V_h^0$, one has $\varepsilon_h = 0$. This completes the proof of the lemma. \hfill \square
4. Error estimates in energy norm

We start this section with a useful lemma. First let $Q_0$ and $Q_b$ be the two elementwise defined $L^2$ projections onto $P_k(T)$ and $P_{k+1}(e)$ on each $T \in T_h$ respectively. Define $Q_h u = \{Q_0 u, Q_b u\} \in V_h$. Let $Q_h$ be the elementwise defined $L^2$ projection onto $\text{BDM}_{[k+1]}(T)$ on each $T \in T_h$.

**Lemma 4.1.** Let $\phi \in H^1(\Omega)$, then on any $T \in T_h$,

$$\nabla w(Q_h \phi) = Q_h \nabla \phi. \quad (4.1)$$

**Proof.** Using (2.3) and integration by parts, we have that for any $q \in \text{BDM}_{[k+1]}(T)$, as $\nabla \cdot q \in P_k(T)$ and $q \cdot n \in P_{k+1}(e)$,

$$(\nabla w Q_h \phi, q)_T = -(Q_0 \phi, \nabla \cdot q)_T + \langle Q_b \phi, q \cdot n \rangle_{\partial T}$$

$$= -(\phi, \nabla \cdot q)_T + \langle \phi, q \cdot n \rangle_{\partial T}$$

$$= (\nabla \phi, q)_T = (Q_h \nabla \phi, q)_T,$$

which implies the Eq. (4.1). \qed

Next we derive an equation for the error $e_h = Q_h u - u_h$.

**Lemma 4.2.** For any $v \in V_h^0$, the following error equation holds true:

$$(\nabla w e_h, \nabla w v) = \ell(u, v), \quad (4.2)$$

where

$$\ell(u, v) = \langle (\nabla u - Q_h \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T_h}.$$

**Proof.** For $v = \{v_0, v_b\} \in V_h^0$, testing (1.1) by $v_0$ and using the fact that

$$\langle \nabla u \cdot n, v_b \rangle_{\partial T_h} = 0,$$

we have

$$(\nabla u, \nabla v_0)_{\partial T_h} - \langle \nabla u \cdot n, v_0 - v_b \rangle_{\partial T_h} = (f, v_0). \quad (4.3)$$

It follows from integration by parts, (2.3) and (4.1) that

$$(\nabla u, \nabla v_0)_{\partial T_h} = (Q_h \nabla u, \nabla v_0)_{\partial T_h}$$

$$= -(v_0, \nabla \cdot (Q_h \nabla u))_{\partial T_h} + (v_0, Q_h \nabla u \cdot n)_{\partial T_h}$$

$$= (Q_h \nabla u, \nabla w v)_{\partial T_h} + \langle v_0 - v_b, Q_h \nabla u \cdot n \rangle_{\partial T_h}$$

$$= (\nabla w Q_h u, \nabla w v) + \langle v_0 - v_b, Q_h \nabla u \cdot n \rangle_{\partial T_h}. \quad (4.4)$$

Combining (4.3) and (4.4) yields

$$(\nabla w Q_h u, \nabla w v) = (f, v_0) + \ell(u, v). \quad (4.5)$$
The error equation follows from subtracting (2.4) from (4.5),
\[
(\nabla w e_h, \nabla w v) = \ell(u, v), \quad \forall v \in V^0_h.
\]
This completes the proof of the lemma.

Next we will bound \(\ell(u, v)\).

**Lemma 4.3.** For any \(w \in H^{k+3}(\Omega)\) and \(v = \{v_0, v_b\} \in V^0_h\), we have
\[
|\ell(w, v)| \leq C h^{k+2} |w|_{k+3} |v|.
\]  
(4.6)

**Proof.** Using the Cauchy-Schwarz inequality, the trace inequality (3.3), and (3.8), we have
\[
|\ell(w, v)| = \left| \sum_{T \in T_h} \left( (\nabla w - Q_h \nabla w) \cdot n, v_0 - v_b \right)_{\partial T} \right|
\leq \sum_{T \in T_h} \|\nabla w - Q_h \nabla w\|_{\partial T} \|v_0 - v_b\|_{\partial T}
\leq \left( \sum_{T \in T_h} h_T \|\nabla w - Q_h \nabla w\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{1/2}
\leq C h^{k+2} |w|_{k+3} |v|.
\]

We have proved the lemma.

**Theorem 4.1.** Let \(u_h \in V_h\) be the SFWG finite element solution of (2.4). Assume the exact solution \(u \in H^{k+3}(\Omega)\). Then, there exists a constant \(C\) such that
\[
\|Q_h u - u_h\| \leq C h^{k+2} |u|_{k+3}.
\]  
(4.7)

**Proof.** By letting \(v = e_h\) in (4.2), we have
\[
\|e_h\|^2 = (\nabla_w e_h, \nabla_w e_h) = |\ell(u, e_h)|.
\]  
(4.8)

It follows from (4.6) that
\[
\|e_h\|^2 \leq C h^{k+2} |u|_{k+3} \|e_h\|,
\]
which implies (4.7).

**5. Error estimates in \(L^2\) norm**

The duality argument is used to obtain \(L^2\) error estimate. Recall \(e_h = \{e_0, e_b\} = Q_h u - u_h\). The corresponding dual problem seeks \(\Phi \in H^1_0(\Omega)\) satisfying
\[
-\Delta \Phi = e_0 \quad \text{in } \Omega.
\]  
(5.1)

Assume that the following \(H^2\)-regularity holds:
\[
\|\Phi\|_2 \leq C \|e_0\|.
\]  
(5.2)
Theorem 5.1. Let $u_h \in V_h$ be the SFWG finite element solution of (2.4). Assume that the exact solution $u \in H^{k+3}(\Omega)$ and (5.2) holds true. Then, there exists a constant $C$ such that
\[
\|Q_0u - u_0\| \leq Ch^{k+3}|u|_{k+3}.
\] (5.3)

Proof. Testing (5.1) by $e_0$, we obtain
\[
\|e_0\|^2 = -\left(\nabla \cdot (\nabla \Phi), e_0\right) = (\nabla \Phi, \nabla e_0)_{T_h} - (\nabla \Phi \cdot n, e_0 - e_b)_{\partial T_h},
\] (5.4)
where we have used the fact $(\nabla \Phi \cdot n, e_b)_{\partial T_h} = 0$. Setting $u = \Phi$ and $v = e_h$ in (4.4) yields
\[
(\nabla \Phi, \nabla e_0)_{T_h} = (\nabla w_{Q_h} \Phi, \nabla w_{Q_h} \Phi) + (Q_h \nabla \Phi \cdot n, e_0 - e_b)_{\partial T_h}.
\] (5.5)
Substituting (5.5) into (5.4) and using (4.2) give
\[
\|e_0\|^2 = (\nabla w_{Q_h} \Phi, \nabla w_{Q_h} \Phi) + (Q_h \nabla \Phi - \nabla \Phi) \cdot n, e_0 - e_b)
\]
(\nabla \Phi, e_0)_{T_h} = (\nabla w_{Q_h} \Phi, \nabla w_{Q_h} \Phi) - \ell(\Phi, e_h) = \ell(u, Q_h \Phi) - \ell(\Phi, e_h).
\] (5.6)
Using the triangle inequality, we obtain
\[
|\ell(u, Q_h \Phi)| = \left| \sum_{T \in T_h} (\nabla u - Q_h \nabla u) \cdot n, Q_0 \Phi - Q_0 \Phi)_{\partial T} \right|
\]
\[
\leq \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{\partial T} \|Q_0 \Phi - Q_0 \Phi\|_{\partial T}
\]
\[
\leq \left( \sum_{T \in T_h} h_T^{1/2} \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|Q_0 \Phi - Q_0 \Phi\|_{\partial T}^2 \right)^{1/2}.
\] (5.7)
From the trace inequality (3.3) we have
\[
\left( \sum_{T \in T_h} h_T^{-1} \|Q_0 \Phi - \Phi\|_{\partial T}^2 \right)^{1/2} \leq Ch^{-1} \|Q_0 \Phi - \Phi\|_2 \leq Ch \|\Phi\|_2,
\]
\[
\left( \sum_{T \in T_h} h_T \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \leq C \|\nabla u - Q_h \nabla u\| \leq Ch^{k+2}|u|_{k+3}.
\]
Combining the above two estimates with (5.7) gives
\[
|\ell(u, Q_h \Phi)| \leq Ch^{k+3}|u|_{k+3} \|\Phi\|_2.
\] (5.8)
It follows from (4.6) and (4.7),
\[
|\ell(\Phi, e_h)| \leq Ch \|\Phi\|_2 \|e_h\| \leq Ch^{k+3}|u|_{k+3} \|\Phi\|_2.
\] (5.9)
Substituting (5.8) and (5.9) into (5.6) yields
\[
|e_0|^2 \leq Ch^{k+3}|u|_{k+3} \|\Phi\|_2.
\]
Using the estimate above and the regularity assumption (5.2), we obtain the error estimate (5.3) of order two superconvergence. □
6. A locally lifted $P_{k+2}$ solution

In last section, we proved that the $P_k$ weak Galerkin solution is two-order superconvergent, i.e., it converges at order $k+3$ in $L^2$ norm. We define a local post-process, which lifts the $P_k$ solution to an optimal-order $P_{k+2}$ solution.

On each element $T$, we compute a solution $\hat{u}_h \in \Pi_{T \in \mathcal{T}_h} P_{k+2}(T)$ by
\begin{align*}
(\nabla \hat{u}_h - \nabla w u_h, \nabla v)_T = 0, & \quad \forall v \in P_{k+2}(T) \setminus P_0(T), \\
(\hat{u}_h - u_0, v)_T = 0, & \quad \forall v \in P_0(T).
\end{align*}
(6.1) (6.2)

We show next the uniqueness of the above square linear system of equations (6.1)-(6.2). When $u_h = 0$, (6.1) implies $\|\nabla \hat{u}_h\|^2 = 0$ and $\hat{u}_h$ is a constant on each $T$. By (6.2), the constant is zero. As the linear system is square and finite dimensional, the uniqueness implies the existence of solution.

**Theorem 6.1.** Let $u \in H^1_0(\Omega) \cap H^{k+3}(\Omega)$ be the exact solution of (1.1)-(1.2). Let $u_h \in V_h$ in (6.1)-(6.2) be the weak Galerkin finite element solution of (2.4). Let $\hat{u}_h \in \Pi_{T \in \mathcal{T}_h} P_{k+2}(T)$ be locally lifted solution of (6.1)-(6.2). Then there exists a constant $C$ such that
\begin{equation}
\|u - \hat{u}_h\|_h \leq C h^{k+3}|u|_{k+3}.
\end{equation}
(6.3)

**Proof:** In the proof, we use $\Pi_k$ to denote the elementwise $L^2$ orthogonal projection onto either $\Pi_{T \in \mathcal{T}_h} P_k(T)$ or $\Pi_{T \in \mathcal{T}_h} [P_k(T)]^2$. Eq. (6.2) means that
\begin{equation}
\Pi_0 \hat{u}_h = \Pi_0 u_h,
\end{equation}
where $\Pi_0$ is again the $L^2$ orthogonal projection onto $P_0(T)$, on $T$. We consider the error in two parts
\begin{equation}
\|u - \hat{u}_h\|_h \leq \|\Pi_0 (u - \hat{u}_h)\|_h + \|(I - \Pi_0)(u - \hat{u}_h)\|_h.
\end{equation}

For the $\Pi_0$ part of error, by (5.3) we have
\begin{equation}
\|\Pi_0 (u - \hat{u}_h)\|_h = \|\Pi_0 (\Pi_k u - u_h)\|_h \leq C \|\Pi_k u - u_h\|_h \leq C h^{k+3}|u|_{k+3}.
\end{equation}

For the $\Pi_0$-orthogonal error, we separate it further into two
\begin{align*}
\| (I - \Pi_0)(u - \hat{u}_h)\|_h & \leq C h \|\nabla (u - \hat{u}_h)\|_h \\
& \leq C h \|\nabla (u - \Pi_{k+2} u)\|_h + C h \|\nabla (\Pi_{k+2} u - \hat{u}_h)\|_h \\
& \leq C h^{k+3}|u|_{k+3} + C h \|\nabla (\Pi_{k+2} u - \hat{u}_h)\|_h.
\end{align*}

By (4.1), i.e., $\Pi_{k+1} \nabla u = \nabla w Q_h u$, (6.1), i.e., $\nabla \hat{u}_h = \nabla w u_h$, and (4.7), letting $\mathbf{q} = \nabla (\Pi_{k+2} u - \hat{u}_h)$, we get
\begin{equation}
\|\nabla (\Pi_{k+2} u - \hat{u}_h)\|_h^2 = (\nabla (\Pi_{k+2} u - u), \mathbf{q}) + (\nabla u - \Pi_{k+1} \nabla u, \mathbf{q}) + (\nabla w Q_h u - \nabla w u_h, \mathbf{q})
\end{equation}
\[ \leq (\| \nabla (\Pi_{k+2} u - u) \|_0 + \| \nabla u - \Pi_{k+1} \nabla u \|_0 + \| Q_h u - u_h \|_0 ) \| q \|_0 + \| \nabla u - \Pi_{k+3} \nabla u \|_0 + \| Q_h u - u_h \|_0 ) \| q \|_0 \]

Combining above three inequalities yields (6.3).

\[ \square \]

7. Numerical experiments

Consider problem (1.1) with \( \Omega = (0,1)^2 \). The source term \( f \) and the boundary value \( g \) are chosen so that the exact solution is

\[ u(x,y) = \sin \pi x \sin \pi y. \]  \hfill (7.1)

Function \( f \) and \( g \) in (1.1)-(1.2) cannot be valid to all functions for nonlinear PDEs. The conditions for valid \( f \) and \( g \) are discussed in [3, 19, 20].

We use the uniform square meshes shown as in Fig. 1. The results of \( P_1 \), \( P_2 \), \( P_3 \) and \( P_4 \) WG methods are listed in Table 1. Two orders of superconvergence are obtained for new element, in both \( L^2 \) and \( H^1 \)-like norms.

As we have order two superconvergence, we lift each \( P_k \) weak Galerkin finite element solution \( u_h \) to a \( P_{k+2} \) solution \( \hat{u}_h \) elementwise. From Table 2, the lifted \( P_{k+2} \) solution converges at order \( k + 3 \) in \( L^2 \) norm, two orders above that of the original \( P_k \) solution (which is from solving a linear system of equations.)

<table>
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<tr>
<th>Grid</th>
<th>( Q_h u - u_h |_0 )</th>
<th>Rate</th>
<th>( Q_h u - u_h |_0 )</th>
<th>Rate</th>
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Table 2: The errors of $P_k$ WG solution $u_h$ and lifted $P_{k+2}$ solution $\hat{u}_h$, and the convergence rate for problem (7.1).

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<th>Rate</th>
<th>$|u - \hat{u}_h|$</th>
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Figure 1: The first three levels of square grids used in the computation.

Acknowledgments

Xiu Ye was supported in part by National Science Foundation Grant DMS-1620016.

References


