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Constructing Order Two Superconvergent WG Finite Elements on Rectangular Meshes

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Abstract. In this paper, we introduce a stabilizer free weak Galerkin (SFWG) finite element method for second order elliptic problems on rectangular meshes. With a special weak Gradient space, an order two superconvergence for the SFWG finite element solution is obtained, in both L^2 and H^1 norms. A local post-process lifts such a P_k weak Galerkin solution to an optimal order P_{k+2} solution. The numerical results confirm the theory.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, weak Galerkin method, stabilizer free, rectangular mesh.

1. Introduction

A new stabilizer free weak Galerkin method is developed to solve the following second order elliptic problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, which can be subdivided into rectangular meshes.

The weak Galerkin (WG) finite element methods introduced in [24, 25] provide a general finite element technique for solving partial differential equations. The novelty of the WG method is the introduction of weak function and its weakly defined derivatives. The weak functions possess the form of $v = \{v_0, v_b\}$ with $v = v_0$ representing the value of v in the interior of each element and $v = v_b$ on the boundary of the element.

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The weak derivative $\nabla_w v$ for a weak function v is defined as distributions. WG method uses polynomials $(P_k(T), P_s(e), [P_\ell(T)]^d)$ to approximate $(v_0, v_b, \nabla_w v)$ accordingly. The WG methods have been applied for solving various PDEs such as Sobolev equation, the Navier-Stokes equations, the Oseen equations, time-dependent Maxwell's equations, elliptic interface problems, biharmonic equations, etc, [1, 5–17, 21–23, 26, 27, 30].

For some special combinations of the WG element $(P_k(T), P_s(e), [P_\ell(T)]^d)$, stabilizer is no longer needed in the corresponding weak Galerkin finite element formulations, which leads to a stabilizer free weak Galerkin method. The stabilizer free weak Galerkin method was first introduced in [28] on polygonal/polyhedral meshes and then has been applied for the second order problems, the Stokes equations and the biharmonic equation [2, 18, 29].

This paper has two purposes:

1. Developing a new SFWG method with an order two superconvergence for the problem (1.1)-(1.2).

2. Providing necessary theory for a subsequent paper, order two superconvergent conforming discontinuous Galerkin method on rectangular meshes.

A WG element $(P_k(T),\ P_{k+1}(e), \mathrm{BDM}_{[k]}[T])$ on rectangular mesh is used in this stabilizer free weak Galerkin finite element method. We prove that the SFWG method converges to the true solution of (1.1)-(1.2) with a convergence rate two orders higher than the optimal order in both an energy norm and the L^2 norm theoretically and numerically. We further define a local post-process which lifts such a P_k weak Galerkin solution to an optimal order P_{k+2} solution. It is proved and numerically verified.

2. The weak Galerkin finite element scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of rectangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and the mesh size by $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows:

$$V_h = \big\{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k+1}(e), e \subset \partial T, T \in \mathcal{T}_h\big\} \tag{2.1}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \{v : v \in V_h, v_b = 0 \ \text{ on } \ \partial\Omega\}. \tag{2.2}$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

On each rectangle $T \in \mathcal{T}_h$, the BDM finite element space is defined by [4]

$$\mathrm{BDM}_{[k+1]}(T) = P_{k+1}(T)^2 \oplus \mathbf{curl} x^{k+2} y \oplus \mathbf{curl} x y^{k+2}.$$

For $v = \{v_0, v_b\} \in V_h$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w v \in \mathrm{BDM}_{[k+1]}(T)$ satisfies

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{q} \in \mathrm{BDM}_{[k+1]}(T). \tag{2.3}$$

For simplicity, we adopt the following notations:

$$\begin{aligned} (v, w)_{\mathcal{T}_h} &= \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw d\mathbf{x}, \\ \langle v, w \rangle_{\partial \mathcal{T}_h} &= \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T} \langle v, w \rangle_e = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw ds. \end{aligned}$$

Algorithm 2.1 (Weak Galerkin algorithm)**.** A numerical approximation for (1.1)-(1.2) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h$ satisfying $u_b = Q_b g$ on $\partial\Omega$ and the following equation:

$$(\nabla_w u_h, \nabla_w v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_h^0. \tag{2.4}$$

3. Well posedness

For any $v \in V_h$, a semi-H^1-like semi-norm is defined as follows:

$$|||v|||^2 = (\nabla_w v, \nabla_w v). \tag{3.1}$$

We introduce a discrete semi-H^1 norm as follows:

$$\|v\|_{1,h}^2 = (\nabla v_0, \nabla v_0)_{\mathcal{T}_h} + \left\langle h_T^{-1}(v_0 - v_b), v_0 - v_b \right\rangle_{\partial \mathcal{T}_h}. \tag{3.2}$$

For any function $\varphi \in H^1(T)$, the trace inequality holds true

$$\|\varphi\|_e^2 \le C \left(h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2 \right). \tag{3.3}$$

Next we will show that $|||\cdot|||$ also defines a norm for V_h^0 by proving the equivalence of $|||\cdot|||$ and $\|\cdot\|_{1,h}$ in V_h. For $\mathbf{q} \in H(\mathrm{div}, \Omega)$, by [4], we define a BDM interpolation Π_h such that $\Pi_h \mathbf{q}|_T \in \mathrm{BDM}_{[k+1]}(T)$ for $T \in \mathcal{T}_h$ satisfies

$$\begin{aligned} &\left\langle (\mathbf{q} - \Pi_h \mathbf{q}) \cdot \mathbf{n}, p_{k+1} \right\rangle_e = 0, && \forall p_{k+1} \in P_{k+1}(e), \quad e \subset \partial T, && (3.4) \\ &(\mathbf{q} - \Pi_h \mathbf{q}, \mathbf{p}_{k-1})_T = 0, && \forall \mathbf{p}_{k-1} \in [P_{k-1}(T)]^2. && (3.5) \end{aligned}$$

Lemma 3.1 ([4])**.** *Let* $\mathbf{q} \in H^{k+2}(\Omega)^2$.

$$\begin{aligned} &\|\mathbf{q} - \Pi_h \mathbf{q}\| \le C h^{k+2} |\mathbf{q}|_{k+2}, && (3.6) \\ &\|\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q})\|_T \le C h_T^{k+1} |\nabla \cdot \mathbf{q}|_{k+1,T}. && (3.7) \end{aligned}$$

Lemma 3.2. *There exist two positive constants C_1 and C_2 such that*

$$C_1\|v\|_{1,h} \le |||v||| \le C_2\|v\|_{1,h}, \quad \forall v \in V_h. \tag{3.8}$$

Proof. We prove the upper bound first. By the definition of weak gradient (2.3), letting $w = \nabla_w v$, we have

$$\begin{aligned}
|||v|||^2 &= \sum_{T\in\mathcal{T}_h} -(v_0, \nabla\cdot\nabla_w v)_T + \langle v_b, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (\nabla v_0, \nabla_w v)_T + \langle v_b - v_0, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T} \\
&\le \sum_{T\in\mathcal{T}_h} (\nabla v_0, \nabla_w v)_T + \|v_b - v_0\|_{\partial T}\|\nabla_w v\|_{\partial T} \\
&\le \sum_{T\in\mathcal{T}_h} \left(\|\nabla v_0\|_T + \frac{\|v_b - v_0\|_{\partial T}}{C h_T^{1/2}}\right)\|\nabla_w v\|_T \le C_2\|v\|_{1,h}|||v|||,
\end{aligned}$$

where we applied the trace inequality (3.3) and the inverse inequality.

To prove the lower bound, we need to choose an appropriate $\mathbf{q}$ in the definition of weak gradient (2.3) so that the above inequality can be reversed. Let $\mathbf{q} \in \mathrm{BDM}_{[k+1]}(T)$ be defined, similar to the BDM interpolation Π_h in (3.4)-(3.5), by

$$(\mathbf{q} - \nabla v_0, \mathbf{p}_{k-1})_T = 0, \qquad \forall \mathbf{p}_{k-1} \in P_{k-1}(T)^2, \tag{3.9}$$

$$\left\langle \mathbf{q}\cdot\mathbf{n} - h_T^{-1}(v_0 - v_b), p_{k+1}\right\rangle_e = 0, \quad \forall p_{k+1} \in P_{k+1}(e), \quad e \subset \partial T. \tag{3.10}$$

By (3.4)-(3.5), (3.9)-(3.10) define a unique $\mathbf{q}$. Further, by finite dimensional norm equivalence and scaling argument,

$$\|\mathbf{q}\| \le C\|v\|_{1,h}. \tag{3.11}$$

Using this $\mathbf{q}$ in (2.3), we have

$$\begin{aligned}
\|v\|_{1,h}^2 &= (\nabla v_0, \nabla v_0)_{\mathcal{T}_h} + \left\langle h_T^{-1}(v_0 - v_b), v_0 - v_b\right\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla v_0, \mathbf{q})_{\mathcal{T}_h} + \langle v_0 - v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla_w v, \mathbf{q})_{\mathcal{T}_h} \le |||v|||\|\mathbf{q}\| \le C_1^{-1}|||v|||\|v\|_{1,h}.
\end{aligned}$$

The lemma is proved. □

Lemma 3.3. *The weak Galerkin finite element scheme* (2.4) *has a unique solution.*

Proof. Let $u_h^{(1)}$ and $u_h^{(2)}$ be the two solutions of (2.4), then $\varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V_h^0$ would satisfy the following equation:

$$(\nabla_w\varepsilon_h, \nabla_w v) = 0, \quad \forall v \in V_h^0.$$

Then by letting $v = \varepsilon_h$ in the above equation, we arrive at

$$|||\varepsilon_h|||^2 = (\nabla_w\varepsilon_h, \nabla_w\varepsilon_h) = 0.$$

It follows from (3.8) that $\|\varepsilon_h\|_{1,h} = 0$. Since $\|\cdot\|_{1,h}$ is a norm in V_h^0, one has $\varepsilon_h = 0$. This completes the proof of the lemma. □

4. Error estimates in energy norm

We start this section with a useful lemma. First let Q_0 and Q_b be the two elementwise defined L^2 projections onto $P_k(T)$ and $P_{k+1}(e)$ on each $T \in \mathcal{T}_h$ respectively. Define $Q_h u = \{Q_0 u, Q_b u\} \in V_h$. Let $\mathbb{Q}_h$ be the elementwise defined L^2 projection onto $\mathrm{BDM}_{[k+1]}(T)$ on each $T \in \mathcal{T}_h$.

Lemma 4.1. *Let $\phi \in H^1(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$\nabla_w(Q_h\phi) = \mathbb{Q}_h \nabla\phi. \tag{4.1}$$

Proof. Using (2.3) and integration by parts, we have that for any $\mathbf{q} \in \mathrm{BDM}_{[k+1]}(T)$, as $\nabla \cdot \mathbf{q} \in P_k(T)$ and $\mathbf{q} \cdot \mathbf{n} \in P_{k+1}(e)$,

$$\begin{aligned}(\nabla_w Q_h\phi, \mathbf{q})_T &= -(Q_0\phi, \nabla \cdot \mathbf{q})_T + \langle Q_b\phi, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial T}\\ &= -(\phi, \nabla \cdot \mathbf{q})_T + \langle \phi, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial T}\\ &= (\nabla\phi, \mathbf{q})_T = (\mathbb{Q}_h\nabla\phi, \mathbf{q})_T,\end{aligned}$$

which implies the Eq. (4.1). $\square$

Next we derive an equation for the error $e_h = Q_h u - u_h$.

Lemma 4.2. *For any $v \in V_h^0$, the following error equation holds true:*

$$(\nabla_w e_h, \nabla_w v) = \ell(u, v), \tag{4.2}$$

where

$$\ell(u, v) = \big\langle (\nabla u - \mathbb{Q}_h \nabla u) \cdot \mathbf{n},\ v_0 - v_b\big\rangle_{\partial\mathcal{T}_h}.$$

Proof. For $v = \{v_0, v_b\} \in V_h^0$, testing (1.1) by v_0 and using the fact that

$$\langle \nabla u \cdot \mathbf{n}, v_b\rangle_{\partial\mathcal{T}_h} = 0,$$

we have

$$(\nabla u, \nabla v_0)_{\mathcal{T}_h} - \langle \nabla u \cdot \mathbf{n}, v_0 - v_b\rangle_{\partial\mathcal{T}_h} = (f, v_0). \tag{4.3}$$

It follows from integration by parts, (2.3) and (4.1) that

$$\begin{aligned}(\nabla u, \nabla v_0)_{\mathcal{T}_h} &= (\mathbb{Q}_h\nabla u, \nabla v_0)_{\mathcal{T}_h}\\ &= -\big(v_0, \nabla \cdot (\mathbb{Q}_h\nabla u)\big)_{\mathcal{T}_h} + \langle v_0, \mathbb{Q}_h\nabla u \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\mathbb{Q}_h\nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, \mathbb{Q}_h\nabla u \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w Q_h u, \nabla_w v) + \langle v_0 - v_b, \mathbb{Q}_h\nabla u \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) yields

$$(\nabla_w Q_h u, \nabla_w v) = (f, v_0) + \ell(u, v). \tag{4.5}$$

The error equation follows from subtracting (2.4) from (4.5),

$$(\nabla_w e_h, \nabla_w v) = \ell(u, v), \quad \forall v \in V_h^0.$$

This completes the proof of the lemma. □

Next we will bound $\ell(u, v)$.

Lemma 4.3. *For any $w \in H^{k+3}(\Omega)$ and $v = \{v_0, v_b\} \in V_h^0$, we have*

$$|\ell(w, v)| \le Ch^{k+2}|w|_{k+3}|||v|||. \tag{4.6}$$

Proof. Using the Cauchy-Schwarz inequality, the trace inequality (3.3), and (3.8), we have

$$\begin{aligned}
|\ell(w, v)| &= \left| \sum_{T\in\mathcal{T}_h} \big\langle (\nabla w - \mathbb{Q}_h \nabla w)\cdot \mathbf{n}, v_0 - v_b \big\rangle_{\partial T} \right| \\
&\le \sum_{T\in\mathcal{T}_h} \|\nabla w - \mathbb{Q}_h\nabla w\|_{\partial T}\|v_0 - v_b\|_{\partial T} \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T\|(\nabla w - \mathbb{Q}_h\nabla w)\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch^{k+2}|w|_{k+3}|||v|||.
\end{aligned}$$

We have proved the lemma. □

Theorem 4.1. *Let $u_h \in V_h$ be the SFWG finite element solution of* (2.4). *Assume the exact solution $u \in H^{k+3}(\Omega)$. Then, there exists a constant C such that*

$$|||Q_h u - u_h||| \le Ch^{k+2}|u|_{k+3}. \tag{4.7}$$

Proof. By letting $v = e_h$ in (4.2), we have

$$|||e_h|||^2 = (\nabla_w e_h, \nabla_w e_h) = |\ell(u, e_h)|. \tag{4.8}$$

It follows from (4.6) that

$$|||e_h|||^2 \le Ch^{k+2}|u|_{k+3}|||e_h|||,$$

which implies (4.7). □

5. Error estimates in L^2 norm

The duality argument is used to obtain L^2 error estimate. Recall $e_h = \{e_0, e_b\} = Q_h u - u_h$. The corresponding dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\Delta\Phi = e_0 \quad \text{in } \Omega. \tag{5.1}$$

Assume that the following H^2-regularity holds:

$$\|\Phi\|_2 \le C\|e_0\|. \tag{5.2}$$

Theorem 5.1. *Let $u_h \in V_h$ be the SFWG finite element solution of* (2.4)*. Assume that the exact solution $u \in H^{k+3}(\Omega)$ and* (5.2) *holds true. Then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \le Ch^{k+3}|u|_{k+3}. \tag{5.3}$$

Proof. Testing (5.1) by e_0, we obtain

$$\begin{aligned}\|e_0\|^2 &= -\big(\nabla\cdot(\nabla\Phi), e_0\big)\\ &= (\nabla\Phi, \nabla e_0)_{\mathcal{T}_h} - \langle \nabla\Phi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h},\end{aligned} \tag{5.4}$$

where we have used the fact $\langle \nabla\Phi\cdot\mathbf{n}, e_b\rangle_{\partial\mathcal{T}_h} = 0$. Setting $u = \Phi$ and $v = e_h$ in (4.4) yields

$$(\nabla\Phi, \nabla e_0)_{\mathcal{T}_h} = (\nabla_w Q_h\Phi, \nabla_w e_h) + \langle \mathbb{Q}_h\nabla\Phi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}. \tag{5.5}$$

Substituting (5.5) into (5.4) and using (4.2) give

$$\begin{aligned}\|e_0\|^2 &= (\nabla_w e_h, \nabla_w Q_h\Phi) + \big\langle(\mathbb{Q}_h\nabla\Phi - \nabla\Phi)\cdot\mathbf{n}, e_0 - e_b\big\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w e_h, \nabla_w Q_h\Phi) - \ell(\Phi, e_h) = \ell(u, Q_h\Phi) - \ell(\Phi, e_h).\end{aligned} \tag{5.6}$$

Using the triangle inequality, we obtain

$$\begin{aligned}|\ell(u, Q_h\Phi)| &= \Big|\sum_{T\in\mathcal{T}_h}\big\langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ Q_0\Phi - Q_b\Phi\big\rangle_{\partial T}\Big|\\ &\le \sum_{T\in\mathcal{T}_h}\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}\|Q_0\Phi - Q_b\Phi\|_{\partial T}\\ &\le \Big(\sum_{T\in\mathcal{T}_h} h_T\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\Phi - \Phi\|_{\partial T}^2\Big)^{\frac12}.\end{aligned} \tag{5.7}$$

From the trace inequality (3.3) we have

$$\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\Phi - \Phi\|_{\partial T}^2\Big)^{\frac12} \le Ch^{-1}\|Q_0\Phi - \Phi\| \le Ch\|\Phi\|_2,$$

$$\Big(\sum_{T\in\mathcal{T}_h} h_T\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\Big)^{\frac12} \le C\|\nabla u - \mathbb{Q}_h\nabla u\| \le Ch^{k+2}|u|_{k+3}.$$

Combining the above two estimates with (5.7) gives

$$|\ell(u, Q_h\Phi)| \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2. \tag{5.8}$$

It follows from (4.6) and (4.7),

$$|\ell(\Phi, e_h)| \le Ch\|\Phi\|_2 |||e_h||| \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.6) yields

$$\|e_0\|^2 \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2.$$

Using the estimate above and the regularity assumption (5.2), we obtain the error estimate (5.3) of order two superconvergence. $\square$

6. A locally lifted P_{k+2} solution

In last section, we proved that the P_k weak Galerkin solution is two-order superconvergent, i.e., it converges at order $k+3$ in L^2 norm. We define a local post-process, which lifts the P_k solution to an optimal-order P_{k+2} solution.

On each element T, we compute a solution $\hat{u}_h \in \Pi_{T\in\mathcal{T}_h} P_{k+2}(T)$ by

$$(\nabla \hat{u}_h - \nabla_w u_h, \nabla v)_T = 0, \quad \forall v \in P_{k+2}(T) \setminus P_0(T), \tag{6.1}$$

$$(\hat{u}_h - u_0, v)_T = 0, \quad \forall v \in P_0(T). \tag{6.2}$$

We show next the uniqueness of the above square linear system of equations (6.1)-(6.2). When $u_h = 0$, (6.1) implies $\|\nabla \hat{u}_h\|^2 = 0$ and $\hat{u}_h$ is a constant on each T. By (6.2), the constant is zero. As the linear system is square and finite dimensional, the uniqueness implies the existence of solution.

Theorem 6.1. *Let $u \in H_0^1(\Omega) \cap H^{k+3}(\Omega)$ be the exact solution of* (1.1)-(1.2). *Let $u_h \in V_h$ in* (6.1)-(6.2) *be the weak Galerkin finite element solution of* (2.4). *Let $\hat{u}_h \in \Pi_{T\in\mathcal{T}_h} P_{k+2}(T)$ be locally lifted solution of* (6.1)-(6.2). *Then there exists a constant C such that*

$$\|u - \hat{u}_h\|_0 \le C h^{k+3} |u|_{k+3}. \tag{6.3}$$

Proof. In the proof, we use Π_k to denote the elementwise L^2 orthogonal projection onto either $\Pi_{T\in\mathcal{T}_h} P_k(T)$ or $\Pi_{T\in\mathcal{T}_h} [P_k(T)]^2$. Eq. (6.2) means that

$$\Pi_0 \hat{u}_h = \Pi_0 u_h,$$

where Π_0 is again the L^2 orthogonal projection onto $P_0(T)$, on T. We consider the error in two parts

$$\|u - \hat{u}_h\|_0 \le \|\Pi_0(u - \hat{u}_h)\|_0 + \|(I - \Pi_0)(u - \hat{u}_h)\|_0.$$

For the P_0 part of error, by (5.3) we have

$$\|\Pi_0(u - \hat{u}_h)\|_0 = \|\Pi_0(\Pi_k u - u_h)\|_0 \le C\|\Pi_k u - u_h\|_0 \le C h^{k+3} |u|_{k+3}.$$

For the P_0-orthogonal error, we separate it further into two

$$\begin{aligned}\|(I - \Pi_0)(u - \hat{u}_h)\|_0 &\le Ch\|\nabla(u - \hat{u}_h)\|_0 \\ &\le Ch\|\nabla(u - \Pi_{k+2}u)\|_0 + Ch\|\nabla(\Pi_{k+2}u - \hat{u}_h)\|_0 \\ &\le Ch^{k+3}|u|_{k+3} + Ch\|\nabla(\Pi_{k+2}u - \hat{u}_h)\|_0.\end{aligned}$$

By (4.1), i.e., $\Pi_{k+1}\nabla u = \nabla_w Q_h u$, (6.1), i.e., $\nabla \hat{u}_h = \nabla_w u_h$, and (4.7), letting

$$\mathbf{q} = \nabla(\Pi_{k+2}u - \hat{u}_h),$$

we get

$$\|\nabla(\Pi_{k+2}u - \hat{u}_h)\|_0^2 = \big(\nabla(\Pi_{k+2}u - u), \mathbf{q}\big) + (\nabla u - \Pi_{k+1}\nabla u, \mathbf{q}) + (\nabla_w Q_h u - \nabla_w u_h, \mathbf{q})$$

$$\leq \big(\|\nabla(\Pi_{k+2}u - u)\|_0 + \|\nabla u - \Pi_{k+1}\nabla u\|_0 + \|\!|Q_h u - u_h|\!\|\big)\|\mathbf{q}\|_0$$
$$\leq Ch^{k+2}|u|_{k+3}\|\nabla(\Pi_{k+2}u - \hat{u}_h)\|_0.$$

Combining above three inequalities yields (6.3). □

7. Numerical experiments

Consider problem (1.1) with $\Omega = (0,1)^2$. The source term f and the boundary value g are chosen so that the exact solution is

$$u(x,y) = \sin \pi x \sin \pi y. \tag{7.1}$$

Function f and g in (1.1)-(1.2) cannot be valid to all functions for nonlinear PDEs. The conditions for valid f and g are discussed in [3, 19, 20].

We use the uniform square meshes shown as in Fig. 1. The results of P_1, P_2, P_3 and P_4 WG methods are listed in Table 1. Two orders of superconvergence are obtained for new element, in both L^2 and H^1-like norms.

As we have order two superconvergence, we lift each P_k weak Galerkin finite element solution u_h to a P_{k+2} solution $\hat{u}_h$ elementwise. From Table 2, the lifted P_{k+2} solution converges at order $k+3$ in L^2 norm, two orders above that of the original P_k solution (which is from solving a linear system of equations.)

Table 1: The error and the convergence rate for problem (7.1).

Grid	$\|Q_h u - u_h\|$	Rate	$\|\!|Q_h u - u_h|\!\|$	Rate
	The P_1 weak Galerkin element			
6	0.770E-06	4.00	0.170E-03	3.00
7	0.482E-07	4.00	0.213E-04	3.00
8	0.301E-08	4.00	0.266E-05	3.00
	The P_2 weak Galerkin element			
5	0.600E-06	4.99	0.112E-03	3.99
6	0.188E-07	5.00	0.703E-05	4.00
7	0.586E-09	5.00	0.440E-06	4.00
	The P_3 weak Galerkin element			
4	0.170E-05	5.98	0.221E-03	4.98
5	0.267E-07	5.99	0.693E-05	5.00
6	0.419E-09	5.99	0.217E-06	5.00
	The P_4 weak Galerkin element			
3	0.160E-04	6.93	0.138E-02	5.94
4	0.127E-06	6.98	0.218E-04	5.99
5	0.995E-09	6.99	0.341E-06	6.00

Table 2: The errors of P_k WG solution u_h and lifted P_{k+2} solution $\hat{u}_h$, and the convergence rate for problem (7.1).

Grid	$\|u - u_h\|$	Rate	$\|u - \hat{u}_h\|$	Rate
	P_1 WG solution		Lifted P_3 solution	
6	0.175E-02	2.00	0.102E-05	4.00
7	0.438E-03	2.00	0.636E-07	4.00
8	0.109E-03	2.00	0.397E-08	4.00
	P_2 WG solution		Lifted P_4 solution	
5	0.271E-03	3.00	0.674E-06	4.99
6	0.339E-04	3.00	0.211E-07	5.00
7	0.424E-05	3.00	0.659E-09	5.00
	P_3 WG solution		Lifted P_5 solution	
4	0.237E-03	3.98	0.176E-05	5.98
5	0.149E-04	4.00	0.275E-07	6.00
6	0.930E-06	4.00	0.432E-09	5.99
	P_4 WG solution		Lifted P_6 solution	
3	0.700E-03	4.95	0.161E-04	6.93
4	0.221E-04	4.99	0.128E-06	6.98
5	0.691E-06	5.00	0.100E-08	6.99

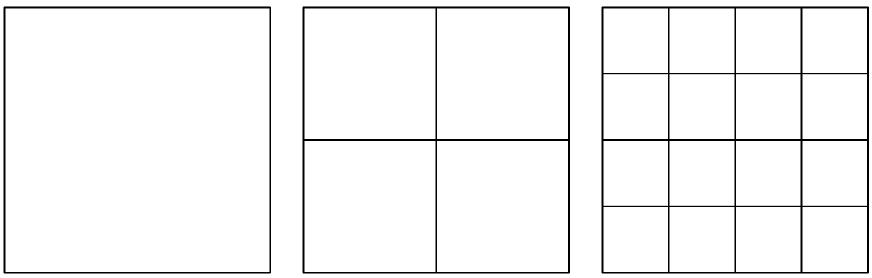

Figure 1: The first three levels of square grids used in the computation.

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