Numer. Math. Theor. Meth. Appl. Vol. 16, No. 1, pp. 79-110
doi: 10.4208/nmtma.OA-2022-0024 February 2023

Semi-Discrete and Fully Discrete Weak Galerkin Finite Element Methods for a Quasistatic Maxwell Viscoelastic Model

Jihong Xiao[1,2], Zimo Zhu[1] and Xiaoping Xie[1,*]

[1] *School of Mathematics, Sichuan University, Chengdu 610064, China*
[2] *Mathematics Department of Jinjiang College, Sichuan University, Pengshan 620860, China*

Received 10 February 2022; Accepted (in revised version) 27 May 2022

Abstract. This paper considers weak Galerkin finite element approximations on polygonal/polyhedral meshes for a quasistatic Maxwell viscoelastic model. The spatial discretization uses piecewise polynomials of degree k ($k \geq 1$) for the stress approximation, degree $k+1$ for the velocity approximation, and degree k for the numerical trace of velocity on the inter-element boundaries. The temporal discretization in the fully discrete method adopts a backward Euler difference scheme. We show the existence and uniqueness of the semi-discrete and fully discrete solutions, and derive optimal a priori error estimates. Numerical examples are provided to support the theoretical analysis.

AMS subject classifications: 35Q74, 65M12, 65M60

Key words: Quasistatic Maxwell viscoelastic model, weak Galerkin method, semi-discrete scheme, fully discrete scheme, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a convex polyhedral domain with boundary $\partial\Omega$, and T be a positive constant. We consider the following quasistatic Maxwell viscoelastic model:

$$-\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f}, \qquad (x,t) \in \Omega \times [0,T], \tag{1.1a}$$

$$\boldsymbol{\sigma} + \boldsymbol{\sigma}_t = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u_t}), \qquad (x,t) \in \Omega \times [0,T], \tag{1.1b}$$

$$\boldsymbol{u} = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.1c}$$

$$\boldsymbol{u}(x,0) = \phi_0(x), \qquad x \in \Omega, \tag{1.1d}$$

$$\boldsymbol{\sigma}(x,0) = \psi_0(x), \qquad x \in \Omega. \tag{1.1e}$$

*Corresponding author. *Email addresses:* xiaojh2752@163.com (J. Xiao), zzm@stu.scu.edu.cn (Z. Zhu), xpxie@scu.edu.cn (X. Xie)

Here $\boldsymbol{u} \in \mathbb{R}^d$ is the displacement field, $\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}$ the symmetric stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)}{2}$ the strain tensor, f the body force, $\phi_0(x)$ and $\psi_0(x)$ are initial data, $g_t := \frac{\partial g}{\partial t}$ for any function $g(\boldsymbol{x}, t)$, and $\mathbb{C}$ denotes an elastic module tensor such that for any symmetric tensor $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$ a.e. $x \in \Omega$ one has

$$0 < M_0 \boldsymbol{\tau} : \boldsymbol{\tau} \leq \mathbb{C}^{-1} \boldsymbol{\tau} : \boldsymbol{\tau} \leq M_1 \boldsymbol{\tau} : \boldsymbol{\tau}, \tag{1.2}$$

where M_0 and M_1 are two positive constants, and

$$\boldsymbol{\iota} : \boldsymbol{\tau} := \sum_{i=1}^{d} \sum_{j=1}^{d} \iota_{ij} \tau_{ij} \quad \text{for} \quad \boldsymbol{\iota}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}.$$

Note that for an isotropic elastic medium we have

$$\mathbb{C} \boldsymbol{\varepsilon}(\boldsymbol{u_t}) = 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u_t}) + \lambda (\nabla \cdot \boldsymbol{u_t}) I,$$

where μ and λ are Lamé constants, and I the identity matrix.

In material science and continuum mechanics, viscoelasticity is the property of materials that exhibit both viscous and elastic characteristic when undergoing deformation. The Maxwell model, characterized by the governing constitutive relation (1.1b), is one of classical models of viscoelasticity (see, e.g. [2, 12–14, 16, 18, 19, 33, 40, 41] for some related works on the development and applications of viscoelasticity theory). These models, including the Kelvin-Voigt model and the Zener model, are represented by different combinations of purely elastic springs, which obey Hooke's law, and purely viscous dashpots, which obey Newton law. The Maxwell model consists of a spring and a dashpot connected in series. We note that the general constitutive law of viscoelasticity can be described in a unified framework by using convolution integrals in time with some kernels [12, 16, 40].

In [5, 6] Carcione *et al.* gave the first numerical simulation of wave propagation in viscoelastic materials, and introduced memory variables to avoid the computation of convolution integrals in the constitutive relation. Janovsky *et al.* [25] applied continuous/discontinuous Galerkin finite element methods to discretize a linear viscoelasticity model involving the hereditary constitutive relations for compressible solids. Ha *et al.* [20] proposed a nonconforming finite element method for a viscoelastic complex model in the space frequency domain. Bécache *et al.* [1] presented a family of mass lumped mixed finite element methods, together with a leap-frog scheme in the time discretization, for the Zener model. In [36–38] Rivière *et al.* analyzed discontinuous Galerkin finite element discretizations of the quasistatic linear viscoelasticity and linear/nonlinear diffusion viscoelastic models, where a Crank-Nicolson temporal scheme is used in the full discretization. Rognes and Winther [39] considered mixed finite element approximations with weak symmetric stresses for the quasistatic Maxwell and Kelvin-Voigt models, where the temporal discretization uses a second backward difference scheme. In [42] Shi and Zhang applied the standard p-order rectangular finite elements to solve a kind of nonlinear viscoelastic wave equations with nonlinear boundary conditions. Lee [26] studied mixed finite element methods with weak symmetry for the Zener, Kelvin-Voigt and Maxwell models, and employed the Crank-Nicolson scheme in the temporal discretization. In [33] Marques and Creuso gave an overview of numerical methods of viscoelasticity problems including finite element, boundary element and finite volume formulations. Li *et al.* [31] proposed a space-time continuous finite element method for a 2D viscoelastic wave equation. In [50], Wang and Xie analyzed a hybrid stress finite element method for the Maxwell model, where a second order implicit difference was used in the fully discrete scheme. Recently, Yuan and Xie [54] showed that the mixed finite element framework for Maxwell-model-based problems of wave propagation in linear viscoelastic solid allows the use of a large class of existing mixed conforming finite elements for elasticity in the spatial discretization.

This paper is to consider a class of weak Galerkin finite element discretizations of the quasistatic Maxwell viscoelastic model (1.1). The weak Galerkin (WG) method was firstly proposed and analyzed by Wang and Ye for second order elliptic problems [46, 47]. Due to adopting weakly defined gradient/divergence operators over functions with discontinuity, the WG method allows the use of totally discontinuous functions on finite element partitions with arbitrary shape of polygons/polyhedra, and allows the local elimination of unknowns defined in the interior of elements. Later on, this method was extended to some other models of partial differential equations, such as convection-diffusion equations [4, 8, 17, 30, 32, 56], linear elasticity problems [9, 23, 45, 49], Stokes equations [7, 48, 55, 57, 58], Maxwell equations [35, 44], natural convection problems [21, 22], Biot models [11, 24], biharmonic equations [3, 34, 51] and p-Laplacian problem [53]. We also refer the reader to [10, 27–29] for some WG fast solvers and to [52] for a low regularity error analysis of a WG discretization.

In this contribution, we develop semi-discrete and fully discrete WG methods for a velocity-stress system of the quasistatic Maxwell viscoelastic model (1.1) on polygonal/polyhedral meshes, where the velocity variable $\boldsymbol{v} = \boldsymbol{u}_t$ is introduced (cf. (2.1)). In the spatial discretization, the stress variable is approximated by piecewise polynomials of degree k $(k \geq 1)$, the velocity variable is approximated by piecewise polynomials of degree $k+1$, and the velocity trace on the inter-element boundaries is approximated by piecewise polynomials of degree k. In the fully-discrete method, the backward Euler difference scheme is adopted for the temporal discretization.

The rest of this paper is organized as follows. Section 2 introduces some notations and the weak variational problem. Sections 3 and 4 are devoted to the stability and error estimation for the semi-discrete and fully discrete weak Galerkin schemes, respectively. Finally, we report some numerical results to demonstrate the performance of the proposed WG methods.

2. Weak formulation

We first introduce some notations. For any bounded domain $D \subset \mathbb{R}^s$ $(s = d, d-1)$ and nonnegative integer m, we denote by $H^m(D)$ and $H_0^m(D)$ the usual m-th order

Sobolev spaces with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. $H^0(D) = L^2(D)$ is the space of square integrable functions defined on D. We use $(\cdot,\cdot)_{m,D}$ to denote the inner product of $H^m(D)$, with $(\cdot,\cdot)_D = (\cdot,\cdot)_{0,D}$. When $D = \Omega$, we set $\|\cdot\|_m := \|\cdot\|_{m,\Omega}$, $|\cdot|_m := |\cdot|_{m,\Omega}$ and $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$. In particular, for $D \subset \mathbb{R}^{d-1}$, we use $\langle\cdot,\cdot\rangle_D$ to replace $(\cdot,\cdot)_D$. For any integer $j \geq 0, P_j(D)$ denotes the set of all polynomials defined on D with degree no greater than j.

For any vector-valued (or tensor-valued) space X, defined on D, with norm $\|\cdot\|_X$, we set

$$L^p\big([0,T];X\big) := \big\{\boldsymbol{v} : [0,T] \to X;\ \|\boldsymbol{v}\|_{L^p(X)} < \infty\big\},$$

where

$$\|\boldsymbol{v}\|_{L^p(X)} := \begin{cases} \left(\displaystyle\int_0^T \|\boldsymbol{v}(t)\|_X^p\right)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{0\leq t\leq T} \|\boldsymbol{v}(t)\|_X, & \text{if} \quad p = \infty, \end{cases}$$

and $\boldsymbol{v}(t)$ abbreviates $\boldsymbol{v}(\boldsymbol{x},t)$. For simplicity, we set $L^p(X) := L^p(0,T;X)$. For any integer $r \geq 0$, the spaces $H^r(X) := H^r(0,T;X)$ and $C^r(X) := C^r([0,T];X)$ can be defined similarly.

For convenience, throughout this paper we use $a \lesssim b$ to represent $a \leq Cb$, where C is a generic positive constant C independent of the spatial mesh size h and the temporal mesh size Δt.

Introducing the velocity variable $\boldsymbol{v} = \boldsymbol{u}_t$, we reformulate the quasistatic Maxwell viscoelastic model (1.1) in the velocity-stress form

$$\begin{cases} -\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f}(t), & (x,t) \in \Omega \times [0,T], \\ \boldsymbol{\sigma} + \boldsymbol{\sigma}_t = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}), & (x,t) \in \Omega \times [0,T], \\ \boldsymbol{v} = 0, & (x,t) \in \partial\Omega \times [0,T], \\ \boldsymbol{\sigma}(0) = \psi_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

It is easy to see that

$$\boldsymbol{u}(x,t) = \phi_0(x) + \int_0^t \boldsymbol{v}(x,s)ds.$$

Define

$$\boldsymbol{L^2_S}(\Omega) := \left\{\boldsymbol{\tau} = (\tau_{ij})_{d\times d} \in \left[L^2(\Omega)\right]^{d\times d} \,|\, \tau_{ij} = \tau_{ji},\ i,j = 1,\ldots,d\right\}.$$

Then, based on the system (2.1), we can get the following weak problem: Find $(\boldsymbol{\sigma},\boldsymbol{v}) \in H^1(\boldsymbol{L^2_S}(\Omega)) \times L^2([H_0^1(\Omega)]^d)$ such that for any $t \in (0,T]$,

$$\begin{cases} a(\boldsymbol{\sigma_t},\boldsymbol{\tau}) + a(\boldsymbol{\sigma},\boldsymbol{\tau}) - b(\boldsymbol{\tau},\boldsymbol{v}) = 0, & \forall\boldsymbol{\tau} \in \boldsymbol{L^2_S}(\Omega), \quad (2.2a) \\ b(\boldsymbol{\sigma},\boldsymbol{w}) = (\boldsymbol{f},\boldsymbol{w}), & \forall\boldsymbol{w} \in \left[H_0^1(\Omega)\right]^d, \quad (2.2b) \\ \boldsymbol{\sigma}(0) = \psi_0(x), & x \in \Omega, \quad (2.2c) \end{cases}$$

where $a(\boldsymbol{\sigma},\boldsymbol{\tau}) := (\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau})$ and $b(\boldsymbol{\tau},\boldsymbol{w}) := (\boldsymbol{\tau},\boldsymbol{\varepsilon}(\boldsymbol{w}))$.

Introduce a norm $\|\cdot\|_a$ on $\boldsymbol{L}^2_{\boldsymbol{S}}(\Omega)$, with $\|\cdot\|_a^2 := a(\cdot,\cdot)$. Then from (1.2) it follows:

$$M_0\|\boldsymbol{\tau}\|_0^2 \leq \|\boldsymbol{\tau}\|_a^2 \leq M_1\|\boldsymbol{\tau}\|_0^2, \quad \forall \boldsymbol{\tau} \in \boldsymbol{L}^2_{\boldsymbol{S}}(\Omega). \tag{2.3}$$

Thus, we have

$$M_0\|\boldsymbol{\tau}\|_0^2 \leq a(\boldsymbol{\tau},\boldsymbol{\tau}), \quad a(\boldsymbol{\sigma},\boldsymbol{\tau}) \leq M_1\|\boldsymbol{\sigma}\|_0\|\boldsymbol{\tau}\|_0, \quad \forall \boldsymbol{\tau},\boldsymbol{\sigma} \in \boldsymbol{L}^2_{\boldsymbol{S}}(\Omega). \tag{2.4}$$

For the bilinear form $b(\cdot,\cdot)$, the Korn inequality indicates the inf-sup condition

$$\|\boldsymbol{w}\|_1 \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_0 \leq \sup_{0\neq\boldsymbol{\tau}\in\boldsymbol{L}^2_S(\Omega)} \frac{b(\boldsymbol{\tau},\boldsymbol{w})}{\|\boldsymbol{\tau}\|_0}, \quad \forall \boldsymbol{w} \in \left[H_0^1(\Omega)\right]^d. \tag{2.5}$$

We need the following continuous Grönwall's inequality.

Lemma 2.1. *Let $\phi(t)$ be such that*

$$\frac{d\phi(t)}{dt} + \rho(t)\phi(t) \leq \psi(t), \quad 0 \leq t \leq T,$$

where $\rho(t),\psi(t) \in L^1([0,T])$. Then it holds

$$\phi(t) \leq e^{-\int_0^t \rho(s)ds}\left(\phi(0) + \int_0^t \psi(s)e^{\int_0^s \rho(\tau)d\tau}ds\right), \quad \forall t \in [0,T]. \tag{2.6}$$

In particular, if $\rho \leq 0$ is a constant and $\psi(t) \geq 0$, then

$$\phi(t) \leq e^{-\rho T}\left(\phi(0) + \int_0^T \psi(s)ds\right), \quad \forall t \in [0,T]. \tag{2.7}$$

By following a similar routine to that in [39] for a weak formulation of the Maxwell model with weak symmetry, we can derive existence, uniqueness and stability results for the system (2.2).

Lemma 2.2. *Assume that*

$$\boldsymbol{f} \in C^1\big([L^2(\Omega)]^d\big), \quad \psi_0 \in L^2(\Omega). \tag{2.8}$$

The weak problem (2.2) *has a unique solution $(\boldsymbol{\sigma},\boldsymbol{v}) \in C^1(\boldsymbol{L}^2_{\boldsymbol{S}}(\Omega)) \times C^0([H_0^1(\Omega)]^d)$, and the following stability results hold:*

$$\|\boldsymbol{\sigma}(t)\|_0^2 \lesssim e^{-\frac{M_0}{M_1}t}\|\psi_0\|_0^2 + \int_0^t e^{-\frac{M_0}{M_1}(t-s)}\left(\|\boldsymbol{f}(s)\|_0^2 + \|\boldsymbol{f}_t(s)\|_0^2\right)ds, \tag{2.9}$$

$$\|\boldsymbol{v}(t)\|_1^2 + \|\boldsymbol{\sigma_t}(t)\|_0^2 \lesssim \|\boldsymbol{\sigma}(t)\|_0^2 + \|\boldsymbol{f}_t(t)\|_0^2 \tag{2.10}$$

for any $t \in (0,T]$, where M_0 and M_1 are positive constants given in (1.2).

Proof. On one hand, the conditions (2.4),(2.5),(2.8) and the Babuska-Brezzi's theory for saddle-point formulations imply that there exist $\boldsymbol{\sigma}_e(t) \in C^1(\boldsymbol{L}^{\mathbf{2}}_{\boldsymbol{S}}(\Omega))$ and $\boldsymbol{v}_e(t) \in C^1([H_0^1(\Omega)]^d)$ solving the elasticity problem

$$\begin{cases} a(\boldsymbol{\sigma}_e, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \boldsymbol{v}_e) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{L}^{\mathbf{2}}_{\boldsymbol{S}}(\Omega), \\ b(\boldsymbol{\sigma}_e, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}), & \forall \boldsymbol{w} \in \left[H_0^1(\Omega)\right]^d \end{cases} \tag{2.11}$$

for any $t \in [0, T]$. Introduce

$$\Sigma_0 := \left\{ \boldsymbol{\tau} \in \boldsymbol{L}^{\mathbf{2}}_{\boldsymbol{S}}(\Omega) \mid b(\boldsymbol{\tau}, \boldsymbol{w}) = 0,\ \forall \boldsymbol{w} \in \left[H_0^1(\Omega)\right]^d \right\}.$$

From (2.4) and the standard theory of ordinary differential equations we know that there exists $\boldsymbol{\sigma}_0 \in C^1(\Sigma_0)$ satisfying the ordinary differential equation

$$\begin{cases} a(\boldsymbol{\sigma}_{0,t}, \tau) + a(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) = -a(\boldsymbol{\sigma}_{e,t}, \tau), & \forall \tau \in \Sigma_0, \\ \boldsymbol{\sigma}_{\mathbf{0}}(0) = \psi_0 - \boldsymbol{\sigma}_e(0). \end{cases} \tag{2.12}$$

On the other hand, the inf-sup condition (2.5) yields the existence of $\boldsymbol{v}_0 \in C^0([H_0^1(\Omega)]^d)$ such that for $t \in [0, T]$,

$$a\big((\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_e)_t, \boldsymbol{\tau}\big) + a(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \boldsymbol{v}_{\mathbf{0}}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{L}^{\mathbf{2}}_{\boldsymbol{S}}(\Omega). \tag{2.13}$$

As the result, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_e$ and $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_e$ solve the weak problem (2.2).

To prove the uniqueness of the solution, it suffices to establish the stability results (2.9) and (2.10). We first prove (2.10). Take $\boldsymbol{\tau} = \boldsymbol{\sigma}_t$ in Eq. (2.2a) and differentiate Eq. (2.2b) with respect to t to obtain

$$a(\boldsymbol{\sigma}, \boldsymbol{\sigma}_{\boldsymbol{t}}) + \|\boldsymbol{\sigma}_{\boldsymbol{t}}\|_a^2 = (\boldsymbol{f}_t, \boldsymbol{v}). \tag{2.14}$$

In light of Eq. (2.2a), the inf-sup condition (2.5), the Cauchy-Schwarz inequality and (2.3), we have

$$\|\boldsymbol{v}(t)\|_1 \leq \beta M_1^{\frac{1}{2}} \big(\|\boldsymbol{\sigma}(t)\|_a + \|\boldsymbol{\sigma}_t(t)\|_a\big),$$

which, together with (2.14), implies

$$\|\boldsymbol{v}(t)\|_1^2 + \|\boldsymbol{\sigma}_{\boldsymbol{t}}(t)\|_a^2 \leq C\|\boldsymbol{\sigma}(t)\|_a^2 + \|\boldsymbol{f}_t(t)\|_0^2. \tag{2.15}$$

Here C is a positive constant depending only on β, M_0, M_1. Thus, from (2.3) the desired estimate (2.10) follows.

The thing left is to show (2.9). Take $\boldsymbol{\tau} = \boldsymbol{\sigma}$ and $\boldsymbol{w} = \boldsymbol{v}$ in (2.2) and employ the Young's inequality and (2.15) to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\sigma}(t)\|_a^2 + \|\boldsymbol{\sigma}(t)\|_a^2 = (\boldsymbol{f}, \boldsymbol{v}) &\leq \frac{q}{2}\|\boldsymbol{f}(t)\|_0^2 + \frac{1}{2q}\|\boldsymbol{v}(t)\|_0^2 \\ &\leq \frac{q}{2}\|\boldsymbol{f}(t)\|_0^2 + \frac{C}{2q}\left(\|\boldsymbol{\sigma}(t)\|_a^2 + \|\boldsymbol{f}_t(t)\|_0^2\right). \end{aligned}$$

Then, taking $q = \frac{M_1 C}{2M_1 - M_0} > 0$ in this inequality implies

$$\frac{d}{dt}\|\boldsymbol{\sigma}(t)\|_a^2 + \frac{M_0}{M_1}\|\boldsymbol{\sigma}(t)\|_a^2 \leq c\left(\|\boldsymbol{f}(t)\|^2 + \|\boldsymbol{f}_t(t)\|^2\right).$$

Here c is a positive constant depending only on β, M_0, M_1. Hence, using the Grönwall's inequality (2.6), we obtain

$$\|\boldsymbol{\sigma}(t)\|_a^2 \leq e^{-\frac{M_0}{M_1}t}\|\psi_0\|_a^2 + c\int_0^t e^{-\frac{M_0}{M_1}(t-s)}\left(\|\boldsymbol{f}(s)\|_0^2 + \|\boldsymbol{f}_t(s)\|_0^2\right) ds,$$

which, together with (2.3), yields (2.9). This completes the proof of the lemma. □

3. Semi-discrete weak Galerkin method

3.1. Semi-discrete WG scheme

Let $\mathscr{T}_h = \bigcup\{K\}$ be a shape-regular decomposition of the domain $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$ consisting of polygons/polyhedrons, in the sense that the following two assumptions hold (cf. [9]):

(A1) There exists a positive constant θ_* such that for each element $K \in \mathscr{T}_h$, there is a point $M_K \in K$ with K being star-shaped with respect to every point in the ball of center M_K and radius $\theta_* h_K$.

(A2) There exists a positive constant l_* such that for every element $K \in \mathscr{T}_h$, the distance between any two vertexes is no less than $l_* h_K$.

Let $\mathcal{E}_h$ be the set of all edges/faces of all elements in $\mathscr{T}_h$. For any $K \in \mathscr{T}_h$ and $E \in \mathcal{E}_h$, we denote by h_K and h_E the diameters of K and E, respectively, and set $h := \max_{K\in\mathscr{T}_h} h_K$. Let ∇_h be the piecewise-defined gradient with respect to $\mathscr{T}_h$. Moreover, let

$$\begin{aligned}
\mathcal{V}(K) &:= \left\{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K)\right\},\\
\mathcal{W}(K) &:= \left\{\boldsymbol{v} = \{\boldsymbol{v_0}, \boldsymbol{v_b}\} : \boldsymbol{v_0} \in [L^2(K)]^d, \boldsymbol{v_b}\cdot \boldsymbol{n}_K \in H^{-\frac{1}{2}}(\partial K)\right\}.
\end{aligned}$$

We follow [46] to introduce the definitions of discrete weak gradient/divergence operators.

Definition 3.1. *For any $K \in \mathscr{T}_h, v \in \mathcal{V}(K)$ and integer $j \geq 0$, the discrete weak gradient $\boldsymbol{\nabla_{w,j,K}} v \in [P_j(K)]^d$ of v is defined by*

$$(\boldsymbol{\nabla_{w,j,K}} v, \boldsymbol{q})_K := -(v_0, \nabla\cdot\boldsymbol{q})_K + \langle v_b, \boldsymbol{q}\cdot\mathbf{n}_K\rangle_{\partial K}, \ \ \forall \boldsymbol{q} \in [P_j(K)]^d, \tag{3.1}$$

where $\mathbf{n}_K$ is the unit outward normal vector along ∂K. The global discrete weak gradient operator $\boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j}}$ on $\mathcal{V}(\mathscr{T}_h) := \{v:\ v|_K \in \mathcal{V}(K),\ \forall K \in \mathscr{T}_h\}$ is defined by

$$\boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j}}|_K = \boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j},\boldsymbol{K}}, \forall K \in \mathscr{T}_h.$$

For a vector $\boldsymbol{v} = (v_1, \cdots, v_d)^T \in [\mathcal{V}(\mathscr{T}_h)]^d$, its discrete weak gradient $\nabla_{w,j}\boldsymbol{v}$ is defined as

$$\boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j}}\boldsymbol{v} := (\boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j}}v_1, \cdots, \boldsymbol{\nabla}_{\boldsymbol{w},\boldsymbol{j}}v_d)^T.$$

Definition 3.2. *For any $K \in \mathscr{T}_h, \boldsymbol{v} \in \mathcal{W}(K)$ and integer $j \geq 0$, the discrete weak divergence $\nabla_{w,j,K} \cdot \boldsymbol{v} \in P_j(K)$ of $\boldsymbol{v}$ is defined by*

$$(\nabla_{w,j,K} \cdot \boldsymbol{v}, q)_K = -(\boldsymbol{v_0}, \nabla q)_K + \langle \boldsymbol{v_b} \cdot \boldsymbol{n}_K, q\rangle_{\partial K}, \quad \forall q \in P_j(K). \tag{3.2}$$

The global discrete weak divergence operator $\nabla_{w,j}\cdot$ is defined by

$$\nabla_{w,j} \cdot |_K = \nabla_{w,j,K}\cdot, \quad \forall K \in \mathscr{T}_h.$$

For any $K \in \mathscr{T}_h$, $E \in \mathcal{E}_h$ and any integer $j \geq 0$, let

$$Q_j^0 : L^2(K) \to P_j(K), \quad Q_j^b : L^2(E) \to P_j(E)$$

be the usual L^2 projection operators. For convenience, vector and tensor analogues of Q_j^0 and Q_j^b are still denoted by Q_j^0 and Q_j^b, respectively.

For any integer $k \geq 1$, we introduce the following finite dimensional spaces:

$$\Sigma_h := \left\{\boldsymbol{\tau_h} \in \boldsymbol{L^2_S}(\Omega) : \boldsymbol{\tau_h}|_K \in [P_k(K)]^{d\times d},\ \forall K \in \mathscr{T}_h\right\}, \tag{3.3}$$

$$V_h := \Big\{\boldsymbol{v_h} = \{\boldsymbol{v_{h0}}, \boldsymbol{v_{hb}}\} : \boldsymbol{v_{h0}}|_K \in [P_{k+1}(K)]^d, \boldsymbol{v_{hb}}|_E \in [P_k(E)]^d, \forall K \in \mathscr{T}_h, E \in \mathcal{E}_h\Big\}, \tag{3.4}$$

$$V_h^0 := \big\{\boldsymbol{v_h} \in V_h : \boldsymbol{v_{hb}}|_{\partial\Omega} = 0\big\}. \tag{3.5}$$

The semi-discrete WG scheme reads as follows: For any $t \in [0, T]$, find $\boldsymbol{\sigma_h}(\cdot, t) \in \Sigma_h$, $\boldsymbol{v_h}(\cdot, t) = \{\boldsymbol{v_{h0}}(\cdot, t), \boldsymbol{v_{hb}}(\cdot, t)\} \in V_h^0$ such that

$$a_h(\boldsymbol{\sigma_{h,t}}, \boldsymbol{\tau_h}) + a_h(\boldsymbol{\sigma_h}, \boldsymbol{\tau_h}) - b_h(\boldsymbol{\tau_h}, \boldsymbol{v_h}) = 0, \quad \forall \boldsymbol{\tau_h} \in \Sigma_h, \tag{3.6a}$$

$$b_h(\boldsymbol{\sigma_h}, \boldsymbol{w_h}) + s_h(\boldsymbol{v_h}, \boldsymbol{w_h}) = (\boldsymbol{f}, \boldsymbol{w_{h0}}), \quad \forall \boldsymbol{w_h} \in V_h^0, \tag{3.6b}$$

$$\boldsymbol{\sigma_h}(0) = Q_k^0\psi_0, \tag{3.6c}$$

where

$$a_h(\boldsymbol{\sigma_h}, \boldsymbol{\tau_h}) = \big(\mathbb{C}^{-1}\boldsymbol{\sigma_h}, \boldsymbol{\tau_h}\big), \quad b_h(\boldsymbol{\tau_h}, \boldsymbol{w_h}) = \big(\boldsymbol{\varepsilon_{w,k}}(\boldsymbol{w_h}), \boldsymbol{\tau_h}\big),$$
$$s_h(\boldsymbol{v_h}, \boldsymbol{w_h}) = \big\langle \alpha\big(Q_k^b\boldsymbol{v_{h0}} - \boldsymbol{v_{hb}}\big), Q_k^b\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big\rangle_{\partial\mathscr{T}_h}$$

with

$$\boldsymbol{\varepsilon_{w,k}}(\boldsymbol{w_h}) := \frac{1}{2}\left(\boldsymbol{\nabla_{w,k}}\boldsymbol{w_h} + (\boldsymbol{\nabla_{w,k}}\boldsymbol{w_h})^{\boldsymbol{T}}\right), \quad \langle\cdot,\cdot\rangle_{\partial\mathscr{T}_h} := \sum_{K\in\mathscr{T}_h}\langle\cdot,\cdot\rangle_{\partial K}$$

and the stabilization parameter $\alpha|_E = 2\mu h_E^{-1}$ for any $E \in \mathcal{E}_h$.

Remark 3.1. Notice that by the definition of the discrete weak gradient we have

$$b_h(\boldsymbol{\tau_h},\boldsymbol{w_h}) = (\boldsymbol{\nabla_{w,k}}\boldsymbol{w_h},\boldsymbol{\tau_h}) = -(\boldsymbol{w_{h0}},\boldsymbol{\nabla}\cdot\boldsymbol{\tau_h})_{\mathscr{T}_h} + \langle \boldsymbol{w_{hb}},\boldsymbol{\tau_h n}\rangle_{\partial\mathscr{T}_h}. \tag{3.7}$$

Then the Eqs. (3.6a) and (3.6b) lead to the relations

$$\begin{aligned} &a_h(\boldsymbol{\sigma_{h,t}},\boldsymbol{\tau_h}) + a_h(\boldsymbol{\sigma_h},\boldsymbol{\tau_h}) + (\boldsymbol{v}_{h0},\nabla\cdot\boldsymbol{\tau}_h) - \langle \boldsymbol{v}_{hb},\boldsymbol{\tau}_h\boldsymbol{n}\rangle_{\partial\mathscr{T}_h} = 0,\\ -&(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma_h},\boldsymbol{w}_{h0}) + \left\langle \alpha\left(Q_k^b\boldsymbol{v_{h0}} - \boldsymbol{v_{hb}}\right),\boldsymbol{w_{h0}}\right\rangle_{\partial\mathscr{T}_h} = (\boldsymbol{f},\boldsymbol{w_{h0}}),\\ &\left\langle \boldsymbol{\sigma_h n} - \alpha\left(Q_k^b\boldsymbol{v_{h0}} - \boldsymbol{v_{hb}}\right),\boldsymbol{w_{hb}}\right\rangle_{\partial\mathscr{T}_h} = 0 \end{aligned}$$

for all $(\boldsymbol{\tau_h},\{\boldsymbol{w_{h0}},\boldsymbol{w_{hb}}\}) \in \Sigma_h \times V_h^0$.

By using standard techniques, we can show the existence and uniqueness of the semi-discrete solution.

Theorem 3.1. *The semi-discrete scheme* (3.6) *has a unique solution* $(\boldsymbol{\sigma_h},\boldsymbol{v_h}) \in \Sigma_h \times V_h^0$.

Proof. Let $\{\Phi_i\}_{i=1}^{r_1}$ and $\{\{\phi_{0i}\}_{i=1}^{r_2},\{\phi_{bi}\}_{i=1}^{r_3}\}$ be the basis functions of Σ_h and V_h^0, respectively. We write

$$\begin{aligned} &\boldsymbol{\sigma_h}(t) = \sum_{i=1}^{r_1}\eta_i(t)\Phi_i, \quad \boldsymbol{v_{h0}}(t) = \sum_{i=1}^{r_2}\beta_i(t)\phi_{0i},\\ &\boldsymbol{v_{hb}} = \sum_{i=1}^{r_3}\gamma_i(t)\phi_{bi}, \qquad \mathcal{F}_i = (\boldsymbol{f},\phi_{0j}), \end{aligned}$$

and denote by $\eta(t),\beta(t),\gamma(t)$ the corresponding vectors of $\eta_i(t),\beta_i(t),\gamma_i(t)$, respectively. Let $\mathcal{M}_{s,ij}$ the (i,j)-th components of matrix $\mathcal{M}_s$ $(s=0,1,\dots,6)$ be given by

$$\begin{aligned} &\mathcal{M}_{0,ij} = \left(\mathbb{C}^{-1}\Phi_j,\Phi_i\right), && \mathcal{M}_{1,ij} = (\phi_{0j},\nabla\cdot\Phi_i),\\ &\mathcal{M}_{2,ij} = -\langle\phi_{bj},\Phi_i\boldsymbol{n}\rangle_{\partial\mathscr{T}_h}, && \mathcal{M}_{3,ij} = \left\langle\alpha Q_k^b\phi_{0j},\phi_{0i}\right\rangle_{\partial\mathscr{T}_h},\\ &\mathcal{M}_{4,ij} = -\langle\alpha\phi_{bj},\phi_{0i}\rangle_{\partial\mathscr{T}_h}, && \mathcal{M}_{5,ij} = -\langle\alpha Q_k^b\phi_{0j},\phi_{bi}\rangle_{\partial\mathscr{T}_h},\\ &\mathcal{M}_{6,ij} = \langle\alpha\phi_{bj},\phi_{bi}\rangle_{\partial\mathscr{T}_h}. \end{aligned}$$

Then the system (3.6) can be written as the following matrix forms:

$$\mathcal{M}_0\frac{d\eta(t)}{dt} + \mathcal{M}_0\eta(t) + \mathcal{M}_1\beta(t) + \mathcal{M}_2\gamma(t) = 0, \tag{3.8}$$

$$-\mathcal{M}_1^T\eta(t) + \mathcal{M}_3\beta(t) + \mathcal{M}_4\gamma(t) = \mathcal{F}(t), \tag{3.9}$$

$$-\mathcal{M}_2^T\eta(t) + \mathcal{M}_5\beta(t) + \mathcal{M}_6\gamma(t) = 0. \tag{3.10}$$

Here we have used the relation (3.7) for the terms $b_h(\cdot,\cdot)$ in the scheme. Since $\mathcal{M}_0$ and $\mathcal{M}_6$ are symmetric positive definite, we can eliminate $\beta(t)$ and $\gamma(t)$ from (3.8)-(3.10) to get

$$\mathcal{M}_0\frac{d\eta(t)}{dt} + \mathcal{P}\eta(t) = \mathcal{Q}(t), \tag{3.11}$$

where

$$\begin{aligned}\mathcal{P} &:= \mathcal{M}_0 + \mathcal{M}_2\mathcal{M}_6^{-1}\mathcal{M}_2^T + \big(\mathcal{M}_1 - \mathcal{M}_2\mathcal{M}_6^{-1}\mathcal{M}_5\big)\\ &\quad\times \big(\mathcal{M}_3 - \mathcal{M}_4\mathcal{M}_6^{-1}\mathcal{M}_5\big)^{-1}\big(\mathcal{M}_1^T - \mathcal{M}_4\mathcal{M}_6^{-1}\mathcal{M}_2^T\big),\\ \mathcal{Q} &:= \big(\mathcal{M}_2\mathcal{M}_6^{-1}\mathcal{M}_5 - \mathcal{M}_1\big)\big(\mathcal{M}_3 - \mathcal{M}_4\mathcal{M}_6^{-1}\mathcal{M}_5\big)^{-1}\mathcal{F}(t).\end{aligned}$$

By the standard theory of ordinary differential equations (cf. [15]), the above system has a unique solution $\eta(t)$. And the existence and uniqueness of $\beta(t)$ and $\gamma(t)$ follow from (3.9) and (3.10). This completes the proof. □

3.2. A priori error estimation

To establish error estimates for the proposed WG scheme, we need the following properties of the L_2-projections Q_j^0, Q_j^b with nonnegative integer j.

Lemma 3.1 ([9]). *It holds the commutative property*

$$\nabla_{w,j}\big\{Q_{j+1}^0\boldsymbol{v}, Q_j^b\boldsymbol{v}\big\} = Q_j^0\nabla\boldsymbol{v} \quad \textit{for all} \quad \boldsymbol{v} \in \big[H^1(K)\big]^d. \tag{3.12}$$

Lemma 3.2 ([9,43]). *Let m be an integer with $1 \le m \le j+1$. For any $K \in \mathscr{T}_h, E \in \mathcal{E}_h$, it holds*

$$\begin{aligned}&\big\|v - Q_j^0 v\big\|_{0,K} + h_K\big|v - Q_j^0 v\big|_{1,K} \lesssim h_K^m |v|_{m,K}, && \forall v \in H^m(K),\\ &\big\|v - Q_j^b v\big\|_{0,\partial K} \lesssim h_k^{m-\frac{1}{2}}|v|_{m,K}, && \forall v \in H^m(K),\\ &\big|v - Q_j^0 v\big|_{s,K} \lesssim h_K^{m-s}|v|_{m,K}, && \forall v \in H^m(K), \quad 0 \le s \le m,\\ &\big\|Q_j^0 v\big\|_{0,K} \le \|v\|_{0,K}, && \forall v \in L^2(K),\\ &\big\|Q_j^b v\big\|_{0,E} \le \|v\|_{0,E}, && \forall v \in L^2(E).\end{aligned}$$

For the bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, we easily get the following continuity and coercivity results.

Lemma 3.3. *For all $\boldsymbol{\sigma_h}, \boldsymbol{\tau_h} \in \Sigma_h, \boldsymbol{v_h} = \{\boldsymbol{v_{h0}}, \boldsymbol{v_{hb}}\} \in V_h$, it holds*

$$\begin{aligned}&a_h(\boldsymbol{\sigma_{h0}}, \boldsymbol{\tau_{h0}}) \le M_1\|\boldsymbol{\sigma_{h0}}\|_0\|\boldsymbol{\tau_{h0}}\|_0,\\ &b_h(\boldsymbol{\tau_h}, \boldsymbol{w_h}) \le \|\boldsymbol{\tau_h}\|_0\|\boldsymbol{\varepsilon_{w,k}}(\boldsymbol{w_h})\|_0,\\ &a_h(\boldsymbol{\tau_h}, \boldsymbol{\tau_h}) \ge M_0\|\boldsymbol{\tau_h}\|_0^2.\end{aligned}$$

We also need the following inf-sup stability condition for the bilinear form $b_h(\cdot,\cdot)$.

Lemma 3.4 ([9]). *For any $\boldsymbol{w_h} = \{\boldsymbol{w_{h0}}, \boldsymbol{w_{hb}}\} \in V_h$, it holds*

$$\|\boldsymbol{\varepsilon_{w,k}}(\boldsymbol{w_h})\|_0 \lesssim \sup_{0\ne\boldsymbol{\tau_h}\in\Sigma_h} \frac{b_h(\boldsymbol{\tau_h}, \boldsymbol{w_h})}{\|\boldsymbol{\tau_h}\|_0}. \tag{3.13}$$

Lemma 3.5 ([9]). *For any $\boldsymbol{w_h} = \{\boldsymbol{w_{h0}}, \boldsymbol{w_{hb}}\} \in V_h^0$ and sufficiently small h, it holds*

$$\|\boldsymbol{\nabla_h w_{h0}}\|_0^2 \lesssim \|\boldsymbol{\varepsilon_h(w_{h0})}\|_0^2 + \left\|\alpha^{\frac{1}{2}}(\boldsymbol{Q_k^b w_{h0}} - \boldsymbol{w_{hb}})\right\|_{\partial\mathscr{T}_h}^2, \tag{3.14}$$

$$\|\boldsymbol{\varepsilon_h(w_{h0})}\|_0^2 \lesssim \|\boldsymbol{\varepsilon_{w,k}(w_h)}\|_0^2 + \left\|\alpha^{\frac{1}{2}}(\boldsymbol{Q_k^b w_{h0}} - \boldsymbol{w_{hb}})\right\|_{\partial\mathscr{T}_h}^2, \tag{3.15}$$

where

$$\boldsymbol{\varepsilon_h(w)} := \frac{1}{2}\left(\boldsymbol{\nabla_h w} + (\boldsymbol{\nabla_h w})^{\boldsymbol{T}}\right), \quad \|\cdot\|_{\partial\mathscr{T}_h} := \langle\cdot,\cdot\rangle_{\partial\mathscr{T}_h}^{\frac{1}{2}}.$$

The following lemma shows the error equations of the weak solution $(\boldsymbol{\sigma}, \boldsymbol{v})$ and its projection $(Q_k^0\boldsymbol{\sigma}, \{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\})$.

Lemma 3.6. *Let $(\boldsymbol{\sigma}, \boldsymbol{v}) \in C^1(\boldsymbol{L_S^2}(\Omega) \bigcap \boldsymbol{H}(\mathbf{div}, \Omega)) \times C^0([H_0^1(\Omega)]^d)$ be the weak solution of system* (2.2)*, then, for all $\boldsymbol{\tau_h} \in \Sigma_h$ and $\boldsymbol{w_h} = \{\boldsymbol{w_{h0}}, \boldsymbol{w_{hb}}\} \in V_h^0$ it holds*

$$\begin{aligned} &a_h(Q_k^0\boldsymbol{\sigma}_t, \boldsymbol{\tau_h}) + a_h(Q_k^0\boldsymbol{\sigma}, \boldsymbol{\tau_h}) - b_h\left(\boldsymbol{\tau_h}, \{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\}\right) \\ =\ & a_h\left((Q_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma})_t, \boldsymbol{\tau_h}\right), \end{aligned} \tag{3.16a}$$

$$\begin{aligned} &b_h(Q_k^0\boldsymbol{\sigma}, \boldsymbol{w_h}) + s_h\left(\{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\}, \boldsymbol{w_h}\right) \\ =\ & (\boldsymbol{f}, \boldsymbol{w_{h0}}) + l_1(\boldsymbol{\sigma}, \boldsymbol{w_h}) + l_2(\boldsymbol{v}, \boldsymbol{w_h}), \end{aligned} \tag{3.16b}$$

where

$$\begin{aligned} l_1(\boldsymbol{\sigma}, \boldsymbol{w_h}) &:= \left\langle \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}, \boldsymbol{\sigma n} - Q_k^0\boldsymbol{\sigma n}\right\rangle_{\partial\mathscr{T}_h}, \\ l_2(\boldsymbol{v}, \boldsymbol{w_h}) &:= \left\langle \alpha(Q_k^bQ_{k+1}^0\boldsymbol{v} - Q_k^b\boldsymbol{v}), Q_k^b\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\right\rangle_{\partial\mathscr{T}_h}. \end{aligned}$$

Proof. By the commutative property (3.1) and the definitions of $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, we obtain

$$\begin{aligned} &a_h(Q_k^0\boldsymbol{\sigma}_t, \boldsymbol{\tau_h}) + a_h(Q_k^0\boldsymbol{\sigma}, \boldsymbol{\tau_h}) - b_h(\boldsymbol{\tau_h}, \{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\}) \\ =\ & (\mathbb{C}^{-1}Q_k^0\boldsymbol{\sigma}_t, \boldsymbol{\tau_h}) + (\mathbb{C}^{-1}Q_k^0\boldsymbol{\sigma}, \boldsymbol{\tau_h}) - (\boldsymbol{\nabla_{w,k}}\{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\}, \boldsymbol{\tau_h}) \\ =\ & (\mathbb{C}^{-1}\boldsymbol{\sigma}_t, \boldsymbol{\tau_h}) + (\mathbb{C}^{-1}Q_k^0\boldsymbol{\sigma}, \boldsymbol{\tau_h}) - (Q_k^0\nabla\boldsymbol{v}, \boldsymbol{\tau_h}) + (\mathbb{C}^{-1}(Q_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma})_t, \boldsymbol{\tau_h}) \\ =\ & (\mathbb{C}^{-1}\boldsymbol{\sigma}_t, \boldsymbol{\tau_h}) + (\mathbb{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau_h}) - (\nabla\boldsymbol{v}, \boldsymbol{\tau_h}) + (\mathbb{C}^{-1}(Q_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma})_t, \boldsymbol{\tau_h}) \\ =\ & (\mathbb{C}^{-1}(Q_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma})_t, \boldsymbol{\tau_h}). \end{aligned} \tag{3.17}$$

From the definition of weak gradient, the projection property and the Green's formula, it follows:

$$\begin{aligned} &b_h(Q_k^0\boldsymbol{\sigma}, \boldsymbol{w_h}) + s_h(\{Q_{k+1}^0\boldsymbol{v}, Q_k^b\boldsymbol{v}\}, \boldsymbol{w_h}) \\ =\ & (\boldsymbol{\nabla_{w,k}w_h}, Q_k^0\boldsymbol{\sigma}) + \left\langle \alpha(Q_k^bQ_{k+1}^0\boldsymbol{v} - Q_k^b\boldsymbol{v}), Q_k^b\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\right\rangle_{\partial\mathscr{T}_h} \\ =\ & -(\boldsymbol{w_{h0}}, \nabla_h\cdot Q_k^0\boldsymbol{\sigma}) + \left\langle \boldsymbol{w_{hb}}, Q_k^0\boldsymbol{\sigma n}\right\rangle_{\partial\mathscr{T}_h} \\ &+ \left\langle \alpha(Q_k^bQ_{k+1}^0\boldsymbol{v} - Q_k^b\boldsymbol{v}), Q_k^b\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\right\rangle_{\partial\mathscr{T}_h} \end{aligned}$$

$$
\begin{aligned}
&= \big(\nabla_h \boldsymbol{w_{h0}}, Q_k^0 \boldsymbol{\sigma}\big) - \big\langle \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}, Q_k^0 \boldsymbol{\sigma n}\big\rangle_{\partial\mathscr{T}_h} \\
&\quad + \big\langle \alpha\big(Q_k^b Q_{k+1}^0 \boldsymbol{v} - Q_k^b \boldsymbol{v}\big), Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big\rangle_{\partial\mathscr{T}_h} \\
&= (-\nabla\cdot\boldsymbol{\sigma}, \boldsymbol{w_h}) + \big\langle \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}, (\boldsymbol{\sigma} - Q_k^0 \boldsymbol{\sigma})\boldsymbol{n}\big\rangle_{\partial\mathscr{T}_h} \\
&\quad + \big\langle \alpha\big(Q_k^b Q_{k+1}^0 \boldsymbol{v} - Q_k^b \boldsymbol{v}\big), Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big\rangle_{\partial\mathscr{T}_h} \\
&= (\boldsymbol{f}, \boldsymbol{w_h}) + l_1(\boldsymbol{\sigma}, \boldsymbol{w_h}) + l_2(\boldsymbol{v}, \boldsymbol{w_h}). \qquad (3.18)
\end{aligned}
$$

This finishes the proof. □

Lemma 3.7. *Let* $(\boldsymbol{\sigma}, \boldsymbol{v}) \in C^1(\boldsymbol{L^2_S}(\Omega)\bigcap[H^{k+1}(\Omega)]^{d\times d}) \times C^0([H_0^1(\Omega)\bigcap H^{k+2}(\Omega)]^d)$ *be the weak solution of system* (2.2) *and* $\boldsymbol{w_h} = \{\boldsymbol{w_{h0}}, \boldsymbol{w_{hb}}\} \in V_h$*, it holds*

$$
\begin{aligned}
&|l_1(\boldsymbol{\sigma}, \boldsymbol{w_h})| \lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1}\|\nabla_h \boldsymbol{w_{h0}}\|_0 + h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h}, \\
&|l_2(\boldsymbol{v}, \boldsymbol{w_h})| \lesssim h^{k+1}|\boldsymbol{v}|_{k+2}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h}.
\end{aligned}
$$

Proof. Using the Cauchy-Schwarz inequality, the projection properties, the trace inequality and the triangle inequality, we obtain

$$
\begin{aligned}
|l_1(\boldsymbol{\sigma}, \boldsymbol{w_h})| &\le \|\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\|_{\partial\mathscr{T}_h}\big\|\boldsymbol{\sigma n} - Q_k^0 \boldsymbol{\sigma n}\big\|_{\partial\mathscr{T}_h} \\
&= \big\|\alpha^{\frac{1}{2}}(\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}})\big\|_{\partial\mathscr{T}_h}\big\|\alpha^{-\frac{1}{2}}\big(\boldsymbol{\sigma n} - Q_k^0 \boldsymbol{\sigma n}\big)\big\|_{\partial\mathscr{T}_h} \\
&\lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}(\boldsymbol{w_{h0}} - \boldsymbol{w_{hb}})\big\|_{\partial\mathscr{T}_h} \\
&\le h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(\boldsymbol{w_{h0}} - Q_k^b \boldsymbol{w_{h0}}\big)\big\|_{\partial\mathscr{T}_h} \\
&\quad + h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h} \\
&\lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1} h^{-\frac{1}{2}}\big\|\boldsymbol{w_{h0}} - Q_k^b \boldsymbol{w_{h0}}\big\|_{\partial\mathscr{T}_h} \\
&\quad + h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h} \\
&\lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1}\|\nabla_h \boldsymbol{w_{h0}}\|_0 + h^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h}. \qquad (3.19)
\end{aligned}
$$

Similarly, by the Cauchy-Schwarz inequality and the projection properties we get

$$
\begin{aligned}
|l_2(\boldsymbol{v}, \boldsymbol{w_h})| &\le \big\|\alpha^{\frac{1}{2}}\big(Q_k^b Q_{k+1}^0 \boldsymbol{v} - Q_k^b \boldsymbol{v}\big)\big\|_{\partial\mathscr{T}_h}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h} \\
&\lesssim \big\|\alpha^{\frac{1}{2}}\big(Q_{k+1}^0 \boldsymbol{v} - \boldsymbol{v}\big)\big\|_{\partial\mathscr{T}_h}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h} \\
&\lesssim h^{k+1}|\boldsymbol{v}|_{k+2}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b \boldsymbol{w_{h0}} - \boldsymbol{w_{hb}}\big)\big\|_{\partial\mathscr{T}_h}. \qquad (3.20)
\end{aligned}
$$

This completes the proof. □

The following lemma gives an estimate of the error between the semi-discrete solution $(\boldsymbol{\sigma}_h, \boldsymbol{v}_h = \{\boldsymbol{v}_{h0}, \boldsymbol{v}_{hb}\})$ and the projection $(Q_k^0 \boldsymbol{\sigma}, \{Q_{k+1}^0 \boldsymbol{v}, Q_k^b \boldsymbol{v}\})$ of the weak solution.

Lemma 3.8. *Let* $(\boldsymbol{\sigma},\boldsymbol{v}) \in C^1(\boldsymbol{L}^2_{\boldsymbol{S}}(\Omega)\bigcap[H^{k+1}(\Omega)]^{d\times d}) \times C^1([H^1_0(\Omega)\bigcap H^{k+2}(\Omega)]^d)$ *be the weak solution of* (2.2) *and* $(\boldsymbol{\sigma}_h,\{\boldsymbol{v}_{h0},\boldsymbol{v}_{hb}\}) \in C^1(\Sigma_h)\times C^1(V_h^0)$ *be the semi-discrete solution of the WG scheme* (3.6). *Then it holds*

$$\|\zeta_h\|_0^2 + s_h(\xi_h,\xi_h) \lesssim h^{2k+2}\big(\tilde{M}_0(\boldsymbol{\sigma},\boldsymbol{v}) + \tilde{M}_2(\boldsymbol{\sigma},\boldsymbol{v})\big), \tag{3.21}$$

$$\|\boldsymbol{\varepsilon_h}(\xi_{h0})\|_0^2 \lesssim h^{2k+2}\big(\tilde{M}_0(\boldsymbol{\sigma},\boldsymbol{v}) + \tilde{M}_1(\boldsymbol{\sigma},\boldsymbol{v}) + \tilde{M}_2(\boldsymbol{\sigma},\boldsymbol{v})\big), \tag{3.22}$$

where

$$\zeta_h := Q_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma_h}, \quad \xi_h := \{\xi_{h0},\xi_{hb}\}$$

with

$$\xi_{h0} = Q_{k+1}^0\boldsymbol{v} - \boldsymbol{v}_{h0}, \quad \xi_{hb} = Q_k^b\boldsymbol{v} - \boldsymbol{v}_{hb},$$

and

$$\begin{aligned}
&\tilde{M}_0(\boldsymbol{\sigma},\boldsymbol{v}) := |\boldsymbol{\sigma}(0)|_{k+1}^2 + |\boldsymbol{v}(0)|_{k+2}^2 + |\boldsymbol{\sigma}_t(0)|_{k+1}^2,\\
&\tilde{M}_1(\boldsymbol{\sigma},\boldsymbol{v}) := |\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2,\\
&\tilde{M}_2(\boldsymbol{\sigma},v) := \int_0^t \big(|\boldsymbol{\sigma}|_{k+1}^2 + |\boldsymbol{v}|_{k+2}^2 + |\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+1}^2\big)\,ds.
\end{aligned}$$

Proof. Subtracting (3.6a) and (3.6b) from (3.16a) and (3.16b), respectively, we obtain

$$a_h(\zeta_{h,t},\tau_h) + a_h(\zeta_h,\tau_h) - b_h(\tau_h,\xi_h) = a_h\big(\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t,\tau_h\big), \tag{3.23}$$

$$b_h(\zeta_h,w_h) + s_h(\xi_h,w_h) = l_1(\boldsymbol{\sigma},w_h) + l_2(\boldsymbol{v},w_h). \tag{3.24}$$

Taking

$$(\tau_h,w_h) = (\tau_h,\{w_{h0},w_{hb}\}) = (\zeta_h,\{\xi_{h0},\xi_{hb}\}) = (\zeta_h,\xi_h)$$

in the above equations yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}a_h(\zeta_h,\zeta_h) + a_h(\zeta_h,\zeta_h) + s_h(\xi_h,\xi_h)\\
&= a_h\big(\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t,\zeta_h\big) + l_1(\boldsymbol{\sigma},\xi_h) + l_2(\boldsymbol{v},\xi_h).
\end{aligned} \tag{3.25}$$

From Lemmas 3.7, 3.5 and the Young's inequality with any $\kappa > 1$ it follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\zeta_h\|_a^2 + \|\zeta_h\|_a^2 + s_h(\xi_h,\xi_h)\\
\le\ &\frac{1}{2}\big\|\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t\big\|_a^2 + \frac{1}{2}\|\zeta_h\|_a^2 + Ch^{k+1}|\boldsymbol{\sigma}|_{k+1}\|\nabla_h w_{h0}\|_0\\
&\quad + Ch^{k+1}|\boldsymbol{\sigma}|_{k+1}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}-\xi_{hb}\big)\big\|_{\partial\mathcal{T}_h}\\
&\quad + h^{k+1}|\boldsymbol{v}|_{k+2}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}-\xi_{hb}\big)\big\|_{\partial\mathcal{T}_h}\\
\le\ &\frac{1}{2M_0}\big\|\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t\big\|_0^2 + \frac{1}{2}\|\zeta_h\|_a^2 + \kappa Ch^{2k+2}\big(|\boldsymbol{\sigma}|_{k+1}^2 + |\boldsymbol{v}|_{k+2}^2\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{C}{2\kappa}\|\nabla_h \xi_{h0}\|_0^2 + \frac{C}{\kappa}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0} - \xi_{hb}\big)\big\|_{\partial\mathcal{T}_h}^2 \\
&\le \frac{C}{2M_0}h^{2k+2}|\boldsymbol{\sigma}_t|_{k+1}^2 + \frac{1}{2}\|\zeta_h\|_a^2 + Ch^{2k+2}\big(|\boldsymbol{\sigma}|_{k+1}^2 + |\boldsymbol{v}|_{k+2}^2\big) \\
&\quad + \frac{C}{\kappa}\big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0} - \xi_{hb}\big)\big\|_{\partial\mathcal{T}_h}^2 + \frac{C}{2\kappa}\|\boldsymbol{\varepsilon_h}(\xi_h)\|_0^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{d}{dt}\|\zeta_h\|_a^2 + \|\zeta_h\|_a^2 + s_h(\xi_h,\xi_h) \\
&\lesssim h^{2k+2}\big(|\boldsymbol{\sigma}|_{k+1}^2 + |\boldsymbol{v}|_{k+2}^2 + |\boldsymbol{\sigma}_t|_{k+1}^2\big) + \frac{C}{\kappa}\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0^2.
\end{aligned} \tag{3.26}
$$

By Lemmas 3.4, 3.2 and Eq. (3.23), we have

$$
\begin{aligned}
\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0 &\lesssim \sup_{0\neq\tau_h\in\Sigma_h}\frac{b_h(\tau_h,\xi_h)}{\|\tau_h\|_0} \\
&= \sup_{0\neq\tau_h\in\Sigma_h}\frac{a_h(\zeta_{h,t},\tau_h) + a_h(\zeta_h,\tau_h) - a_h((Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma})_t,\tau_h)}{\|\tau_h\|_0} \\
&\le c\big(\|\zeta_{h,t}\|_0 + \|\zeta_h\|_0 + h^{k+1}|\boldsymbol{\sigma}_t|_{k+1}\big).
\end{aligned} \tag{3.27}
$$

Here c is a positive constant independent of h. To bound the term $\|\zeta_{h,t}\|_0$, substitute $\tau_h = \zeta_{h,t}$ into (3.23) and take $w_h = \xi_h$ in (3.24) after differentiating in time, then we get

$$
\begin{aligned}
&a_h(\zeta_{h,t},\zeta_{h,t}) + a_h(\zeta_h,\zeta_{h,t}) - b_h(\zeta_{h,t},\xi_h) = a_h\big(\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t,\zeta_{h,t}\big), \\
&b_h(\zeta_{h,t},\xi_h) + \frac{1}{2}\frac{d}{dt}s_h(\xi_h,\xi_h) = l_1(\boldsymbol{\sigma}_t,\xi_h) + l_2(\boldsymbol{v}_t,\xi_h).
\end{aligned}
$$

Summing up the above two equalities and using Lemmas 3.7, 3.5, the Cauchy-Schwarz and the Young's inequality, for any $\kappa > 1$ we have

$$
\begin{aligned}
&\|\zeta_{h,t}\|_a^2 + \frac{1}{2}\frac{d}{dt}\|\zeta_h\|_a^2 + \frac{1}{2}\frac{d}{dt}s_h(\xi_h,\xi_h) \\
&= a_h\big(\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t,\zeta_{h,t}\big) + l_1(\boldsymbol{\sigma}_t,\xi_h) + l_2(\boldsymbol{v}_t,\xi_h) \\
&\le \frac{1}{2}\big\|\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t\big\|_a^2 + \frac{1}{2}\|\zeta_{h,t}\|_a^2 + Ch^{k+1}|\boldsymbol{\sigma}_t|_{k+1}\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0 \\
&\quad + Ch^{k+1}\big(|\boldsymbol{\sigma}_t|_{k+1} + |\boldsymbol{v}_t|_{k+2}\big)s_h(\xi_h,\xi_h)^{\frac{1}{2}} \\
&\le \frac{1}{2}\big\|\big(Q_k^0\boldsymbol{\sigma}-\boldsymbol{\sigma}\big)_t\big\|^2 + \frac{1}{2}\|\zeta_{h,t}\|_a^2 + \frac{C}{2\kappa}\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0^2 \\
&\quad + \frac{\kappa C}{2}h^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\big) + \frac{C}{2\kappa}s_h(\xi_h,\xi_h) \\
&\le Ch^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\big) + \frac{1}{2}\|\zeta_{h,t}\|_a^2 + \frac{C}{2\kappa}\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0^2 + \frac{C}{2\kappa}s_h(\xi_h,\xi_h),
\end{aligned}
$$

which implies

$$\begin{aligned}&\|\zeta_{h,t}\|_a^2+\frac{d}{dt}\left(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\right)\\&\le Ch^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2+|v_t|_{k+2}^2\big)+\frac{C}{\kappa}\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h)\|_0^2+\frac{C}{\kappa}s_h(\xi_h,\xi_h).\end{aligned}\tag{3.28}$$

From (3.27) and the norm equivalence (2.3), we have

$$\begin{aligned}\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h)\|_0^2&\le 3c^2\big(\|\zeta_{h,t}\|_0^2+\|\zeta_h\|_0^2+h^{2k+2}|\boldsymbol{\sigma}_{\boldsymbol{t}}|_{k+1}^2\big)\\&\le\frac{3c^2}{M_0}\big(\|\zeta_{h,t}\|_a^2+\|\zeta_h\|_a^2\big)+3c^2h^{2k+2}|\boldsymbol{\sigma}_t|_{k+1}^2,\end{aligned}$$

which, together with (3.28), yields

$$\begin{aligned}&\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h)\|_0^2+\frac{3c^2}{M_0}\frac{d}{dt}\left(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\right)\\&\le\frac{3c^2}{M_0}\left[\|\zeta_{h,t}\|_a^2+\frac{d}{dt}\big(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\big)\right]+\frac{3c^2}{M_0}\|\zeta_h\|_a^2+3c^2h^{2k+2}|\boldsymbol{\sigma}_t|_{k+1}^2\\&\le\frac{3c^2}{M_0}\left[Ch^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big)+\frac{C}{\kappa}\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h)\|_0^2+\frac{C}{\kappa}s_h(\xi_h,\xi_h)\right]\\&\quad+\frac{3c^2}{M_0}\|\zeta_h\|_a^2+3c^2h^{2k+2}|\boldsymbol{\sigma}_t|_{k+1}^2.\end{aligned}$$

Then we get

$$\begin{aligned}&\frac{M_0}{3c^2}\left(1-\frac{C}{\kappa}\right)\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h)\|_0^2+\frac{d}{dt}\left(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\right)\\&\le Ch^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big)+\|\zeta_h\|_a^2+\frac{C}{\kappa}s_h(\xi_h,\xi_h).\end{aligned}\tag{3.29}$$

By taking a sufficiently large positive constant κ in this inequality and using the norm equivalence (2.3), from (3.26) and (3.27) it follows:

$$\begin{aligned}&\frac{d}{dt}\left(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\right)+\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\\&\lesssim h^{2k+2}\big(|\boldsymbol{\sigma}|_{k+1}^2+|\boldsymbol{v}|_{k+2}^2+|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big).\end{aligned}\tag{3.30}$$

By the continuous Grönwall's inequality (2.6), we can get

$$\begin{aligned}\|\zeta_h(t)\|_a^2+s_h(\xi_h,\xi_h)&\lesssim\|\zeta_h(0)\|_a^2+s_h\big(\xi_h(0),\xi_h(0)\big)\\&\quad+h^{2k+2}\int_0^t\big(|\boldsymbol{\sigma}|_{k+1}^2+|\boldsymbol{v}|_{k+2}^2+|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big)ds.\end{aligned}\tag{3.31}$$

In view of (3.6c), it holds

$$\zeta_h(0)=Q_k^0\boldsymbol{\sigma}(0)-\boldsymbol{\sigma}_{\boldsymbol{h}}(\mathbf{0})=0.$$

The thing left is to estimate the term $s_h(\xi_h(0),\xi_h(0))$. To this end, we take $w_h=\xi_h$ in (3.24) and use Lemma 3.7 to get

$$\begin{aligned} s_h\big(\xi_h(0),\xi_h(0)\big) &= l_1\big(\boldsymbol{\sigma}(0),\xi_h(0)\big)+l_2\big(\boldsymbol{v}(0),\xi_h(0)\big)-b_h\big(\zeta_h(0),\xi_h(0)\big)\\ &= l_1\big(\boldsymbol{\sigma}(0),\xi_h(0)\big)+l_2\big(\boldsymbol{v}(0),\xi_h(0)\big)\\ &\lesssim h^{k+1}|\boldsymbol{\sigma}(0)|_{k+1}\cdot\|\nabla_h\xi_h(0)\|_0\\ &\quad+h^{k+1}|\boldsymbol{\sigma}(\mathbf{0})|_{k+1}\big\|\alpha^{\frac12}\big(Q_k^b\xi_{h0}(0)-\xi_{hb}(0)\big)\big\|_{\partial\mathscr{T}_h}\\ &\quad+h^{k+1}|\boldsymbol{v}(0)|_{k+1}\big\|\alpha^{\frac12}\big(Q_k^b\xi_{h0}(0)-\xi_{hb}(0)\big)\big\|_{\partial\mathscr{T}_h}, \end{aligned}$$

which, together with (3.14) and (3.15), leads to

$$s_h\big(\xi_h(0),\xi_h(0)\big)\lesssim h^{2k+2}\big(|\boldsymbol{\sigma}(0)|_{k+1}^2+|\boldsymbol{v}(0)|_{k+2}^2+|\boldsymbol{\sigma}_t(0)|_{k+1}^2\big). \tag{3.32}$$

Combining this estimate with (3.31) indicates the desired result (3.21).

Now let us prove the estimate (3.22). From (3.15) and (3.29) with a sufficiently large κ, we get

$$\begin{aligned} &\|\boldsymbol{\varepsilon_h}(\xi_{h0})\|_0^2+\frac{d}{dt}\big(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\big)\\ \lesssim\;&\|\boldsymbol{\varepsilon_{w,k}}(\xi_h)\|_0^2+\big\|\alpha^{\frac12}\big(Q_k^b\xi_{h0}-\xi_{hb}\big)\big\|_{\partial\mathscr{T}_h}^2+\frac{d}{dt}\big(\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)\big)\\ \lesssim\;&\|\zeta_h\|_a^2+s_h(\xi_h,\xi_h)+h^{2k+2}\big(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big), \end{aligned}$$

which, together with (3.21), yields the desired estimate for $\|\boldsymbol{\varepsilon_h}(\xi_{h0})\|_0^2$. This finishes the proof. □

Applying Lemmas 3.8, 3.2 and the triangle inequality gives the following error estimate for the semi-discrete WG scheme.

Theorem 3.2. *Let* $(\boldsymbol{\sigma},\boldsymbol{v})\in C^1(\boldsymbol{L}^2_{\boldsymbol{S}}(\Omega)\bigcap[H^{k+1}(\Omega)]^{d\times d})\times C^1([H_0^1(\Omega)\bigcap H^{k+2}(\Omega)]^d)$ *be the weak solution of system* (2.2) *and* $(\boldsymbol{\sigma}_h,\boldsymbol{v}_h)\in C^1(\Sigma_h)\times C^1(V_h^0)$ *be the solution of the WG scheme* (3.6)*. Then*

$$\begin{aligned} &\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0+\|\boldsymbol{\varepsilon}(\boldsymbol{v})-\varepsilon_h(\boldsymbol{v}_{h0})\|_0\\ \lesssim\;&h^{k+1}\big(\tilde{M}_0(\boldsymbol{\sigma},\boldsymbol{v})+\tilde{M}_1(\boldsymbol{\sigma},\boldsymbol{v})+\tilde{M}_2(\boldsymbol{\sigma},\boldsymbol{v})\big)^{\frac12}, \end{aligned} \tag{3.33}$$

where $\tilde{M}_0(\boldsymbol{\sigma},\boldsymbol{v}),\tilde{M}_1(\boldsymbol{\sigma},\boldsymbol{v})$ *and* $\tilde{M}_2(\boldsymbol{\sigma},\boldsymbol{v})$ *are defined in Lemma* 3.8*.*

4. Fully discrete weak Galerkin method

4.1. Backward Euler fully discrete scheme

We consider a full discretization of the quasistatic viscoelastic Maxwell model based on backward Euler scheme. Given a positive integer N, let $0=t_0<t_1<\cdots<t_N=T$

be a uniform division of time domain $[0,T]$, with $t_n = n\Delta t$ and $\Delta t = \frac{T}{N}$. For any vector or tensor-valued function $g(t)$ and any n, we set

$$g^n := g(t_n), \quad \overline{\partial}_t g^n := \frac{g^n - g^{n-1}}{\Delta t}.$$

Based on the semi-discrete scheme (3.6), the backward Euler fully discrete WG scheme is given as follows: for $n = 1, \dots, N$, find $(\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n) = (\boldsymbol{\sigma}_h^n, \{\boldsymbol{v}_{h0}^n, \boldsymbol{v}_{hb}^n\}) \in \Sigma_h \times V_h^0$ such that

$$a_h(\overline{\partial}_t \boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) + a_h(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) - b_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h^n) = 0, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \tag{4.1a}$$

$$b_h(\boldsymbol{\sigma}_h^n, \boldsymbol{w_h}) + s_h(\boldsymbol{v}_h^n, \boldsymbol{w_h}) = (\boldsymbol{f}^n, \boldsymbol{w}_{h0}), \quad \forall \boldsymbol{w_h} \in V_h^0, \tag{4.1b}$$

$$\boldsymbol{\sigma}_h^0 = Q_k^0 \psi_0. \tag{4.1c}$$

Theorem 4.1. *The backward Euler fully discrete WG scheme* (4.1) *has a unique solution* $(\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n), n = 1, \dots, N$.

Proof. Since this is a square system, it suffices to show that the homogeneous system

$$\begin{cases} a_h(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau_h}) + \Delta t a_h(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau_h}) - \Delta t b_h(\tau_h, \boldsymbol{v}_h^n) = 0, & \forall \boldsymbol{\tau_h} \in \Sigma_h, \quad (4.2a) \\ b_h(\boldsymbol{\sigma}_h^n, \boldsymbol{w_h}) + s_h(\boldsymbol{v}_h^n, \boldsymbol{w_h}) = 0, & \forall \boldsymbol{w_h} \in V_h^0 \quad (4.2b) \end{cases}$$

only admits a zero solution. In fact, taking $(\tau_h, w_h) = (\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n)$ and summing up the above two equations, we obtain

$$(1 + \Delta t) a_h(\boldsymbol{\sigma}_h^n, \boldsymbol{\sigma}_h^n) + \Delta t s_h(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n) = 0, \tag{4.3}$$

which gives $\boldsymbol{\sigma}_h^n = 0$ and $s_h(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n) = 0$. Then, take $\tau_h = \boldsymbol{\varepsilon}_{w,k}(\boldsymbol{v}_h^n)$ in Eq. (4.2a) leads to $\varepsilon_{w,k}(\boldsymbol{v}_h^n) = 0$, which, together with $s_h(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n) = 0$ and (3.15), implies $\boldsymbol{v}_h^n = \{\boldsymbol{v}_{h0}^n, \boldsymbol{v}_{hb}^n\} = \mathbf{0}$. This completes the proof. $\square$

We have the following stability results for the fully-discrete WG scheme (4.1).

Theorem 4.2. *Assume that* $\Delta t < 1$, *then for any* $1 \le n \le j \le N$ *it holds*

$$\begin{aligned} &\sum_{n=1}^{j} \|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}\|_a^2 + \|\boldsymbol{\sigma}_h^j\|_a^2 + 2\sum_{n=1}^{j} \Delta t \|\boldsymbol{\sigma}_h^n\|_a^2 + 2\Delta t \sum_{n=1}^{j} s_h(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n) \\ = &\|\boldsymbol{\sigma}_h^0\|_a^2 + \sum_{n=1}^{j} (\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\Delta t \sum_{n=1}^{j} \|\boldsymbol{\varepsilon_h}(\boldsymbol{v_{h0}^n})\|_0^2 + \sum_{n=1}^{j} s_h(\boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}, \boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}) + s_h(\boldsymbol{v}_h^j, \boldsymbol{v}_h^j) \\ \lesssim &\|\boldsymbol{\sigma_h^0}\|_a^2 + s_h(\boldsymbol{v}_h^0, \boldsymbol{v}_h^0) + \sum_{n=1}^{j} (\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n) + \sum_{n=1}^{j} (\overline{\partial}_t \boldsymbol{f}^n, \boldsymbol{v}_{h0}^n). \end{aligned} \tag{4.5}$$

Proof. Taking $(\tau_h, w_h) = (\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n)$ in the scheme (4.1), we get

$$\begin{cases} a_h\big(\overline{\partial_t}\boldsymbol{\sigma}_h^n, \boldsymbol{\sigma}_h^n\big) + a_h\big(\boldsymbol{\sigma}_h^n, \boldsymbol{\sigma}_h^n\big) - b_h\big(\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n\big) = 0, \\ b_h\big(\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n\big) + s_h\big(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n\big) = \big(\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n\big). \end{cases} \tag{4.6}$$

Applying the relationship

$$2(p-q,p) = (p-q,p+q) + (p-q,p-q)$$

and adding the above two equalities, we have

$$\begin{aligned} &\frac{1}{2\Delta t}\big\|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}\big\|_a^2 + \frac{1}{2\Delta t}\big(\|\boldsymbol{\sigma}_h^n\|_a^2 - \|\boldsymbol{\sigma}_h^{n-1}\|_a^2\big) \\ &\quad + \|\boldsymbol{\sigma}_h^n\|_a^2 + s_h\big(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n\big) = \big(\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n\big). \end{aligned} \tag{4.7}$$

For any $j \le N$, summing up the above inequality with $n = 1, \ldots, j$, we finally obtain the desired result (4.4). Applying (3.15), we get

$$\big\|\boldsymbol{\varepsilon_h}\big(\boldsymbol{v_{h0}^n}\big)\big\|_0^2 \lesssim \big\|\boldsymbol{\varepsilon_{w,k}}\big(\boldsymbol{v_h^n}\big)\big\|_0^2 + s_h\big(\boldsymbol{v_h^n}, \boldsymbol{v_h^n}\big). \tag{4.8}$$

Using the inf-sup condition (3.13) and Eq. (4.1a), we obtain

$$\begin{aligned} \big\|\boldsymbol{\varepsilon_{w,k}}\big(\boldsymbol{v_h^n}\big)\big\|_0 &\lesssim \sup_{\tau_h \in \Sigma_h} \frac{b_h(\tau_h, \boldsymbol{v_h^n})}{\|\tau_h\|_0} = \sup_{\tau_h \in \Sigma_h} \frac{a_h(\overline{\partial_t}\boldsymbol{\sigma}_h^n, \tau_h) + a_h(\boldsymbol{\sigma}_h^n, \tau_h)}{\|\tau_h\|_0} \\ &\lesssim \big\|\overline{\partial_t}\boldsymbol{\sigma}_h^n\big\|_a + \big\|\boldsymbol{\sigma}_h^n\big\|_a, \end{aligned}$$

which, together with (4.8), yields

$$\big\|\boldsymbol{\varepsilon_h}\big(\boldsymbol{v_{h0}^n}\big)\big\|_0^2 \lesssim \big\|\overline{\partial_t}\boldsymbol{\sigma}_h^n\big\|_a^2 + \big\|\boldsymbol{\sigma}_h^n\big\|_a^2 + s_h\big(\boldsymbol{v_h^n}, \boldsymbol{v_h^n}\big). \tag{4.9}$$

In light of (4.6), we have

$$\begin{aligned} &a_h\big(\overline{\partial_t}\boldsymbol{\sigma}_h^n, \overline{\partial_t}\boldsymbol{\sigma}_h^n\big) + a_h\big(\boldsymbol{\sigma}_h^n, \overline{\partial_t}\boldsymbol{\sigma}_h^n\big) - b_h\big(\overline{\partial_t}\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n\big) = 0, \\ &b_h\big(\overline{\partial_t}\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n\big) + s_h\big(\overline{\partial_t}\boldsymbol{v}_h^n, \boldsymbol{v}_h^n\big) = \big(\overline{\partial_t}\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n\big). \end{aligned}$$

Summing up these two equalities and using the identity

$$2p(p-q) = (p-q)^2 + p^2 - q^2,$$

we arrive at

$$\begin{aligned} &\big\|\overline{\partial_t}\boldsymbol{\sigma_h^n}\big\|_a^2 + \frac{1}{2\Delta t}\Big(\big\|\boldsymbol{\sigma_h^n} - \boldsymbol{\sigma_h^{n-1}}\big\|_a^2 + \big\|\boldsymbol{\sigma_h^n}\big\|_a^2 - \big\|\boldsymbol{\sigma_h^{n-1}}\big\|_a^2\Big) \\ &\quad + \frac{1}{2\Delta t}\Big(s_h\big(\boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}, \boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}\big) + s_h\big(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n\big) - s_h\big(\boldsymbol{v}_h^{n-1}, \boldsymbol{v}_h^{n-1}\big)\Big) = \big(\overline{\partial_t}\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n\big). \end{aligned}$$

This identity plus (4.9) implies

$$\begin{aligned}\|\varepsilon_{\boldsymbol{h}}(\boldsymbol{v}_{h0}^n)\|_0^2 \le C\Bigg(&\|\boldsymbol{\sigma}_h^n\|_a^2 + s_h(\boldsymbol{v}_h^n,\boldsymbol{v}_h^n) - \frac{1}{2\Delta t}\|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n}} - \boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n-1}}\|_a^2 \\ &- \frac{1}{2\Delta t}\left(\|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n}}\|_a^2 - \|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n-1}}\|_a^2\right) - \frac{1}{2\Delta t}\Big(s_h(\boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}, \boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}) \\ &+ s_h(\boldsymbol{v}_h^n,\boldsymbol{v}_h^n) - s_h(\boldsymbol{v}_h^{n-1},\boldsymbol{v}_h^{n-1})\Big)\Bigg) + (\overline{\partial}_t\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n)\end{aligned}$$

for $n = 1,\ldots,j$, where C is positive constant independent of $h, \Delta t$ and n. Thus, we have

$$\begin{aligned}&\Delta t\sum_{n=1}^{j}\|\varepsilon_{\boldsymbol{h}}(\boldsymbol{v}_{\boldsymbol{h0}}^{\boldsymbol{n}})\|_0^2 + \sum_{n=1}^{j}\|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n}} - \boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n-1}}\|_a^2 + \|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{j}}\|_a^2 \\ &\quad+ \sum_{n=1}^{j} s_h(\boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}, \boldsymbol{v}_h^n - \boldsymbol{v}_h^{n-1}) + s_h(\boldsymbol{v}_h^j,\boldsymbol{v}_h^j) \\ \lesssim\ &\Delta t\sum_{n=1}^{j}\|\boldsymbol{\sigma}_h^n\|_a^2 + \Delta t\sum_{n=1}^{j} s_h(\boldsymbol{v}_{\boldsymbol{h}}^{\boldsymbol{n}},\boldsymbol{v}_{\boldsymbol{h}}^{\boldsymbol{n}}) + \|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{0}}\|_a^2 + s_h(\boldsymbol{v}_h^0,\boldsymbol{v}_h^0) \\ &\quad+ \sum_{n=1}^{j}(\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n) + \sum_{n=1}^{j}(\overline{\partial}_t\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n) \\ \lesssim\ &\|\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{0}}\|_a^2 + s_h(\boldsymbol{v}_h^0,\boldsymbol{v}_h^0) + \sum_{n=1}^{j}(\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n) + \sum_{n=1}^{j}(\overline{\partial}_t\boldsymbol{f}^n, \boldsymbol{v}_{h0}^n),\end{aligned}$$

where in the second estimate we have used the stability result (4.4). Hence, the desired result (4.5) follows. □

4.2. Error estimation

By following the same line as in the proof of Lemma 3.6, we can derive the following lemma.

Lemma 4.1. *Let* $(\boldsymbol{\sigma},\boldsymbol{v}) \in C^1(\boldsymbol{L}_{\boldsymbol{S}}^{\boldsymbol{2}}(\Omega)\bigcap \boldsymbol{H}(\mathbf{div},\Omega)) \times C^0([H_0^1(\Omega)]^d)$ *be weak solution of system* (2.2)*, then for all* $\boldsymbol{\tau}_{\boldsymbol{h}} \in \Sigma_h$ *and* $\boldsymbol{w}_{\boldsymbol{h}} = \{\boldsymbol{w}_{\boldsymbol{h0}}, \boldsymbol{w}_{\boldsymbol{hb}}\} \in V_h^0$*, it holds*

$$\begin{aligned}&a_h(\overline{\partial}_t Q_k^0\boldsymbol{\sigma}^n, \boldsymbol{\tau}_{\boldsymbol{h}}) + a_h(Q_k^0\boldsymbol{\sigma}^n, \boldsymbol{\tau}_{\boldsymbol{h}}) - b_h(\boldsymbol{\tau}_h, \{Q_{k+1}^0\boldsymbol{v}^n, Q_k^b\boldsymbol{v}^n\}) \\ &= a_h(\overline{\partial}_t Q_k^0\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_t^n, \boldsymbol{\tau}_{\boldsymbol{h}}), \qquad (4.10a)\\ &b_h(Q_k^0\boldsymbol{\sigma}^n, \boldsymbol{w}_{\boldsymbol{h}}) + s_h(\{Q_{k+1}^0\boldsymbol{v}^n, Q_k^b\boldsymbol{v}^n\}, \boldsymbol{w}_{\boldsymbol{h}}) \\ &= (\boldsymbol{f}^n, \boldsymbol{w}_{\boldsymbol{h}}) + l_1(\boldsymbol{\sigma}^n, \boldsymbol{w}_{\boldsymbol{h}}) + l_2(\boldsymbol{v}^n, \boldsymbol{w}_{\boldsymbol{h}}) \qquad (4.10b)\end{aligned}$$

for $n = 1,\ldots,N$*, where the bilinear forms* $l_1(), l_2()$ *are defined in Lemma* 3.6.

Lemma 4.2. *Let* $(\boldsymbol{\sigma},\boldsymbol{v}) \in C^2(\boldsymbol{L}^{\mathbf{2}}_{\boldsymbol{S}}(\Omega)\cap[H^{k+1}(\Omega)]^{d\times d})\times C^1([H_0^1(\Omega)\cap H^{k+2}(\Omega)]^d)$ *be the solution of* (2.2), *and let* $(\boldsymbol{\sigma}_h^n,\boldsymbol{v}_h^n)=(\boldsymbol{\sigma}_h^n,\{\boldsymbol{v}_{h0}^n,\boldsymbol{v}_{hb}^n\})$ *be the solution of* (4.1) *for* $n=1,\dots,N$. *Then it holds*

$$
\begin{aligned}
&\|\zeta_h^n\|_0^2+2\Delta t\sum_{j=1}^n\|\zeta_h^j\|_0^2+s_h\big(\xi_h^n,\xi_h^n\big)+2\Delta t\sum_{j=1}^n s_h\big(\xi_h^j,\xi_h^j\big)\\
\lesssim\ & h^{2k+2}\big(\tilde M_0(0)+\tilde M_1(t_n)+\tilde M_2(t_n)\big)+\Delta t^2\tilde M_3(t_n),
\end{aligned}
\tag{4.11}
$$

$$
\Delta t\sum_{j=1}^n\big\|\boldsymbol{\varepsilon}_h\big(\xi_{h0}^j\big)\big\|^2\lesssim h^{2k+2}\big(\tilde M_0(0)+\tilde M_1(t_n)+\tilde M_2(t_n)\big)+\Delta t^2\tilde M_3(t_n),\tag{4.12}
$$

where

$$\zeta_h^n:=Q_k^0\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n,\quad \xi_h^n:=\big\{\xi_{h0}^n,\xi_{hb}^n\big\}$$

with

$$\xi_{h0}^n=Q_{k+1}^0\boldsymbol{v}^n-\boldsymbol{v}_{h0}^n,\quad \xi_{hb}^n=Q_k^b\boldsymbol{v}^n-\boldsymbol{v}_{hb}^n,$$

and

$$
\begin{aligned}
\tilde M_0(0)&:=|\boldsymbol{\sigma}(0)|_{k+1}^2+|\boldsymbol{v}(0)|_{k+2}^2+|\boldsymbol{\sigma}_t(0)|_{k+1}^2,\\
\tilde M_1(t_n)&:=\max_{t_j\in[0,T],1\le j\le n}\big(|\boldsymbol{\sigma}^j|_{k+1}^2+|\boldsymbol{v}^j|_{k+2}^2\big),\\
\tilde M_2(t_n)&:=\int_0^{t_n}\big(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2\big)ds,\\
\tilde M_3(t_n)&:=\int_0^{t_n}\|\boldsymbol{\sigma}_{tt}\|_0^2ds.
\end{aligned}
$$

Proof. The proof is similar to that of Lemma 3.8 for the semi-discrete scheme. For completeness, we show it as following. We mention that the notation C_i in this proof for any i denotes a generic positive constant independent of h and Δt.

Our proof mainly divides into 4 steps.

Step 1. From (4.1) and (4.10) it follows, for any $1\le j\le n$,

$$
\begin{aligned}
&a_h\big(\overline{\partial}_t\zeta_h^j,\tau_h\big)+a_h\big(\zeta_h^j,\tau_h\big)-b_h\big(\tau_h,\xi_h^j\big)=a_h\big(\overline{\partial}_tQ_k^0\boldsymbol{\sigma}^j-\boldsymbol{\sigma}_t^j,\tau_h\big), &\text{(4.13a)}\\
&b_h\big(\zeta_h^j,w_h\big)+s_h\big(\xi_h^j,w_h\big)=l_1\big(\boldsymbol{\sigma}^j,w_h\big)+l_2\big(\boldsymbol{v}^j,w_h\big). &\text{(4.13b)}
\end{aligned}
$$

Taking $\tau_h=\zeta_h^j$ and $w_h=\xi_h^j$, and summing up the above two equations, we obtain

$$
\begin{aligned}
&a_h\big(\overline{\partial}_t\zeta_h^j,\zeta_h^j\big)+a_h\big(\zeta_h^j,\zeta_h^j\big)+s_h\big(\xi_h^j,\xi_h^j\big)\\
=\ &l_1\big(\boldsymbol{\sigma}^j,\xi_h^j\big)+l_2\big(\boldsymbol{v}^j,\xi_h^j\big)+a_h\big(\overline{\partial}_tQ_k^0\boldsymbol{\sigma}^j-\boldsymbol{\sigma}_t^j,\zeta_h^j\big)\\
=:\ &E_1^j+E_2^j+E_3^j.
\end{aligned}
\tag{4.14}
$$

Taking $\tau_h = \overline{\partial_t}\zeta_h^j$ in equality (4.13a) gives

$$a_h\big(\overline{\partial_t}\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) + a_h\big(\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) - b_h\big(\overline{\partial_t}\zeta_h^j, \xi_h^j\big) = a_h\big(\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big), \overline{\partial_t}\zeta_h^j\big).$$

In light of (4.10b) and the fact that $-\nabla\cdot\overline{\partial_t}\boldsymbol{\sigma}^n = \overline{\partial_t}f^n$, we have

$$b_h\big(\overline{\partial_t}\zeta_h^j, \xi_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) = l_1\big(\overline{\partial_t}\boldsymbol{\sigma}^j, \xi_h^j\big) + l_2\big(\overline{\partial_t}\boldsymbol{v}^j, \xi_h^j\big).$$

Summing up the above two equalities, we obtain

$$\begin{aligned}
&a_h\big(\overline{\partial_t}\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) + a_h\big(\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) \\
&= a_h\big(\overline{\partial_t}(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j), \overline{\partial_t}\zeta_h^j\big) + l_1\big(\overline{\partial_t}\boldsymbol{\sigma}^j, \xi_h^j\big) + l_2\big(\overline{\partial_t}\boldsymbol{v}^j, \xi_h^j\big),
\end{aligned} \tag{4.15}$$

which shows

$$\begin{aligned}
&\big\|\overline{\partial_t}\zeta_h^j\big\|_a^2 + a_h\big(\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) \\
\le\; &\frac{1}{2}\big\|\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_a^2 + \frac{1}{2}\big\|\overline{\partial_t}\zeta_h^j\big\|_a^2 + \big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}^j - \xi_{h0}^j\big)\big\|_{\partial\mathscr{T}_h} \\
&\times \big\|\alpha^{-\frac{1}{2}}\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_{\partial\mathscr{T}_h} + \big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}^j - \xi_{hb}^j\big)\big\|_{\partial\mathscr{T}_h} \\
&\times \Big(\big\|\alpha^{-\frac{1}{2}}\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_{\partial\mathscr{T}_h} + \big\|\alpha^{\frac{1}{2}}Q_k^b\overline{\partial_t}\big(Q_{k+1}^0\boldsymbol{v}^j - \boldsymbol{v}^j\big)\big\|_{\partial\mathscr{T}_h}\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\big\|\overline{\partial_t}\zeta_h^j\big\|_a^2 + a_h\big(\zeta_h^j, \overline{\partial_t}\zeta_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) \\
\le\; &\frac{1}{2}\big\|\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_a^2 + \big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}^j - \xi_{h0}^j\big)\big\|_{\partial\mathscr{T}_h}\big\|\alpha^{-\frac{1}{2}}\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_{\partial\mathscr{T}_h} \\
&+ \big\|\alpha^{\frac{1}{2}}\big(Q_k^b\xi_{h0}^j - \xi_{hb}^j\big)\big\|_{\partial\mathscr{T}_h}\Big(\big\|\alpha^{-\frac{1}{2}}\overline{\partial_t}\big(Q_k^0\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\big)\big\|_{\partial\mathscr{T}_h} \\
&\qquad + \big\|\alpha^{\frac{1}{2}}Q_k^b\overline{\partial_t}\big(Q_{k+1}^0\boldsymbol{v}^j - \boldsymbol{v}^j\big)\big\|_{\partial\mathscr{T}_h}\Big).
\end{aligned} \tag{4.16}$$

By the projection properties of Q_k^0, we obtain

$$\begin{aligned}
&\big\|\overline{\partial_t}Q_k^0\boldsymbol{\sigma}^j - \overline{\partial_t}\boldsymbol{\sigma}^j\big\|_{0,\mathscr{T}_h} \\
&= \big\|Q_k^0\overline{\partial_t}\boldsymbol{\sigma}^j - \overline{\partial_t}\boldsymbol{\sigma}^j\big\|_{0,\mathscr{T}_h} = \frac{1}{\Delta t}\int_{t_{j-1}}^{t_j}\big|Q_k^0\boldsymbol{\sigma}_t(s) - \boldsymbol{\sigma}_t(s)\big|ds \\
&\lesssim \frac{h^{k+1}}{\Delta t}\int_{t_{j-1}}^{t_j}|\boldsymbol{\sigma}_t(s)|_{k+1}ds \lesssim \frac{h^{k+1}}{\sqrt{\Delta t}}\bigg(\int_{t_{j-1}}^{t_j}|\boldsymbol{\sigma}_t(s)|_{k+1}^2ds\bigg)^{\frac{1}{2}},
\end{aligned} \tag{4.17}$$

$$\big\|\overline{\partial_t}Q_k^0\boldsymbol{\sigma}^j - \overline{\partial_t}\boldsymbol{\sigma}^j\big\|_{\partial\mathscr{T}_h} \lesssim \frac{h^{k+\frac{1}{2}}}{\sqrt{\Delta t}}\bigg(\int_{t_{j-1}}^{t_j}|\boldsymbol{\sigma}_t(s)|_{k+1}^2ds\bigg)^{\frac{1}{2}}. \tag{4.18}$$

Similarly, we have

$$\|\overline{\partial}_t Q_k^0 \boldsymbol{v}^j - \overline{\partial}_t \boldsymbol{v}^j\|_{\partial\mathscr{T}_h} \lesssim \frac{h^{k+\frac{3}{2}}}{\sqrt{\Delta t}} \left(\int_{t_{j-1}}^{t_j} |\boldsymbol{v}_t(s)|_{k+2}^2 ds \right)^{\frac{1}{2}}. \tag{4.19}$$

In light of (4.17)-(4.19), the projection properties of Q_k^b, the Young's inequality, the norm equivalence (2.3) and Lemma 3.5, we further apply (4.16) to get

$$\begin{aligned}
&\frac{1}{2}\|\overline{\partial}_t \zeta_h^j\|_a^2 + a_h(\zeta_h^j, \overline{\partial}_t \zeta_h^j) + s_h(\overline{\partial}_t \xi_h^j, \xi_h^j) \\
\leq& \frac{1}{p}\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h^j)\|_0^2 + \frac{1}{p} s_h(\xi_h^j, \xi_h^j) + C_1 \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} \left(|\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |\boldsymbol{v}_t(s)|_{k+2}^2 \right) ds,
\end{aligned} \tag{4.20}$$

where $p > 0$ is an arbitrary positive constant.

A similar proof of $\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h^j)$ as that of (3.27) implies

$$\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h^j)\|_0^2 \leq C_2 \left(\|\overline{\partial}_t \zeta_h^j\|_a^2 + \|\zeta_h^j\|_a^2 + \frac{h^{2k+2}}{\sqrt{\Delta t}} \int_{t_{j-1}}^{t_j} |\boldsymbol{\sigma}_t|_{k+1}^2 \right). \tag{4.21}$$

Hence, if we choose p sufficiently large such that $p > 4C_2$, then the above two inequalities give

$$\begin{aligned}
&\frac{1}{4}\|\overline{\partial}_t \zeta_h^j\|_a^2 + a_h(\zeta_h^j, \overline{\partial}_t \zeta_h^j) + s_h(\overline{\partial}_t \xi_h^j, \xi_h^j) \\
\leq& \frac{1}{p} s_h(\xi_h^j, \xi_h^j) + \frac{C_2}{p}\|\zeta_h^j\|_a^2 + \left(C_1 + \frac{C_2}{p} \right) \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} \left(|\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |\boldsymbol{v}_t(s)|_{k+2}^2 \right),
\end{aligned} \tag{4.22}$$

and from (4.21) we get

$$\begin{aligned}
\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h^j)\|_0^2 \leq C_3 \Big(& s_h(\xi_h^j, \xi_h^j) + \|\zeta_h^j\|_a^2 + \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} (|\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |\boldsymbol{v}_t(s)|_{k+2}^2) ds \\
& - a_h(\zeta_h^j, \overline{\partial}_t \zeta_h^j) - s_h(\overline{\partial}_t \xi_h^j, \xi_h^j) \Big).
\end{aligned} \tag{4.23}$$

Step 2. The next thing is to estimate the terms E_1^j, E_2^j and E_3^j in (4.14), respectively. From Lemma 3.7 and 3.5, it follows:

$$E_1^j = l_1(\boldsymbol{\sigma}^j, \xi_h^j) \lesssim h^{k+1} |\boldsymbol{\sigma}^j|_{k+1} \left(\|\boldsymbol{\varepsilon}_{\boldsymbol{w},\boldsymbol{k}}(\xi_h^j)\|_0 + s_h(\xi_h^j, \xi_h^j)^{\frac{1}{2}} \right),$$

which, together with the Cauchy inequality, the Young's inequality and (4.23), indicates

$$\begin{aligned}
E_1^j \leq& C_4 \Big(h^{2k+2} |\boldsymbol{\sigma}^j|_{k+1}^2 + \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} (|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2) ds - a_h(\overline{\partial}_t \zeta_h^j, \zeta_h^j) - s_h(\overline{\partial}_t \xi_h^j, \xi_h^j) \Big) \\
&+ \frac{1}{p'}\|\zeta_h^j\|_a^2 + \frac{1}{p'} s_h(\xi_h^j, \xi_h^j),
\end{aligned} \tag{4.24}$$

where p' is an arbitrary positive constant. For the term E_2^j, by Lemma 3.7 and the Young's inequality we also have

$$E_2^j = l_2\big(\boldsymbol{v}^j, \xi_h^j\big) \le C_5 h^{2k+2} |\boldsymbol{v}^j|_{k+2}^2 + \frac{1}{p'} s_h\big(\xi_h^j, \xi_h^j\big). \tag{4.25}$$

Applying the Taylor formula and the Cauchy inequality gives

$$\big\|\overline{\partial_t}\sigma^j - \sigma_t^j\big\|_{0,\mathscr{T}_h} = \frac{1}{\Delta t}\int_{t_{j-1}}^{t_j} (s-t_{j-1})\|\sigma_{tt}(s)\|_0 ds \le \sqrt{\Delta t}\bigg(\int_{t_{j-1}}^{t_j} \|\boldsymbol{\sigma}_{tt}(s)\|_0^2 ds\bigg)^{\frac{1}{2}},$$

which, together with (4.17), shows

$$\begin{aligned} E_3^j &= a_h\big(\overline{\partial_t} Q_k^0 \boldsymbol{\sigma}^j - \boldsymbol{\sigma}_t^j, \zeta_h^j\big) = a_h\big(\overline{\partial_t} Q_k^0 \boldsymbol{\sigma}^j - \overline{\partial_t}\boldsymbol{\sigma}^j, \zeta_h^j\big) + a_h\big(\overline{\partial_t}\boldsymbol{\sigma}^j - \boldsymbol{\sigma}_t^j, \zeta_h^j\big) \\ &\le M_1 \left(\big\|\overline{\partial_t} Q_k^0 \boldsymbol{\sigma}^j - \overline{\partial_t}\boldsymbol{\sigma}^j\big\|_{0,\mathscr{T}_h} + \big\|\overline{\partial_t}\boldsymbol{\sigma}^j - \boldsymbol{\sigma}_t^j\big\|_{0,\mathscr{T}_h} \right) \|\zeta_h^j\|_{0,\mathscr{T}_h} \\ &\le C_6 \bigg(\frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} |\boldsymbol{\sigma}_t|_{k+1}^2 ds + \Delta t \int_{t_{j-1}}^{t_j} \|\boldsymbol{\sigma}_{tt}\|_0^2 ds \bigg) + \frac{1}{p'} \big\|\zeta_h^j\big\|_a^2. \end{aligned} \tag{4.26}$$

Step 3. The equality (4.14) plus the estimates (4.24)-(4.26) implies

$$\begin{aligned} & a_h\big(\overline{\partial_t}\zeta_h^j, \zeta_h^j\big) + a_h\big(\zeta_h^j, \zeta_h^j\big) + s_h\big(\xi_h^j, \xi_h^j\big) \\ \le\, & C_7 \bigg(h^{2k+2}\big(|\boldsymbol{\sigma}^j|_{k+1}^2 + |\boldsymbol{v}^j|_{k+2}^2\big) + \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} \big(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\big) ds \\ & \quad + \Delta t \int_{t_{j-1}}^{t_j} \|\boldsymbol{\sigma}_{tt}\|_0^2 ds \bigg) + \frac{2}{p'} \big\|\zeta_h^j\big\|_a^2 + \frac{2}{p'} s_h\big(\xi_h^j, \xi_h^j\big) \\ & - C_4 \left(a_h\big(\overline{\partial_t}\zeta_h^j, \zeta_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) \right). \end{aligned}$$

Taking $p' = 4$ in this inequality, we further obtain

$$\begin{aligned} & (1+C_4) a_h\big(\overline{\partial_t}\zeta_h^j, \zeta_h^j\big) + C_4 s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) + \frac{1}{2}\big\|\zeta_h^j\big\|_a^2 + \frac{1}{2} s_h\big(\xi_h^j, \xi_h^j\big) \\ \le\, & C_7 \bigg(h^{2k+2}\big(|\boldsymbol{\sigma}^j|_{k+1}^2 + |\boldsymbol{v}^j|_{k+2}^2\big) + \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} \big(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\big) ds \\ & \quad + \Delta t \int_{t_{j-1}}^{t_j} \|\boldsymbol{\sigma}_{tt}\|_0^2 ds \bigg), \end{aligned}$$

which means

$$\begin{aligned} & a_h\big(\overline{\partial_t}\zeta_h^j, \zeta_h^j\big) + s_h\big(\overline{\partial_t}\xi_h^j, \xi_h^j\big) + \big\|\zeta_h^j\big\|_a^2 + s_h\big(\xi_h^j, \xi_h^j\big) \\ \le\, & C_8 \bigg(h^{2k+2}\big(|\boldsymbol{\sigma}^j|_{k+1}^2 + |\boldsymbol{v}^j|_{k+2}^2\big) + \frac{h^{2k+2}}{\Delta t} \int_{t_{j-1}}^{t_j} \big(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\big) ds \\ & \quad + \Delta t \int_{t_{j-1}}^{t_j} \|\boldsymbol{\sigma}_{tt}\|_0^2 ds \bigg). \end{aligned}$$

This estimate and the equations

$$a_h(\overline{\partial}_t\zeta_h^j,\zeta_h^j)=\frac{1}{2\Delta t}\left(\|\zeta_h^j-\zeta_h^{j-1}\|_a^2+\|\zeta_h^j\|_a^2-\|\zeta_h^{j-1}\|_a^2\right),$$
$$s_h(\overline{\partial}_t\xi_h^j,\xi_h^j)=\frac{1}{2\Delta t}\left(s_h(\xi_h^j-\xi_h^{j-1},\xi_h^j-\xi_h^{j-1})+s_h(\xi_h^j,\xi_h^j)-s_h(\xi_h^{j-1},\xi_h^{j-1})\right)$$

yield

$$\begin{aligned}&\|\zeta_h^j\|_a^2-\|\zeta_h^{j-1}\|_a^2+s_h(\xi_h^j,\xi_h^j)-s_h(\xi_h^{j-1},\xi_h^{j-1})+2\Delta t\|\zeta_h^j\|_a^2+2\Delta t s_h(\xi_h^j,\xi_h^j)\\ &\le 2C_8\bigg(\Delta t h^{2k+2}(|\boldsymbol{\sigma}^j|_{k+1}^2+|\boldsymbol{v}^j|_{k+2}^2)+h^{2k+2}\int_{t_{j-1}}^{t_j}(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2)ds\\ &\qquad+(\Delta t)^2\int_{t_{j-1}}^{t_j}\|\boldsymbol{\sigma}_{tt}\|_0^2ds\bigg).\end{aligned}$$

Summing up the above inequality for $j=1,\dots,n$, we arrive at

$$\begin{aligned}&\|\zeta_h^n\|_a^2+s_h(\xi_h^n,\xi_h^n)+2\Delta t\sum_{j=1}^n\left(\|\zeta_h^j\|_a^2+s_h(\xi_h^j,\xi_h^j)\right)\\ &\le\|\zeta_h^0\|_a^2+s_h(\xi_h^0,\xi_h^0)+2C_8\bigg(h^{2k+2}\max_{t_j\in[0,T]}(|\boldsymbol{\sigma}^j|_{k+1}^2+|\boldsymbol{v}^j|_{k+2}^2)\\ &\qquad+h^{2k+2}\int_0^{t_n}(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2)ds+(\Delta t)^2\int_0^{t_n}\|\boldsymbol{\sigma}_{tt}\|_0^2ds\bigg),\end{aligned}\tag{4.27}$$

which, together with (4.1c) and (3.32), leads to desired estimate (4.11).

Step 4. Finally, let us prove (4.12). From inequalities (3.15) and (4.23), we get

$$\begin{aligned}\|\boldsymbol{\varepsilon_h}(\xi_{h0}^j)\|_0^2&\lesssim\|\boldsymbol{\varepsilon_{w,k}}(\xi_h^j)\|_0^2+s_h(\xi_h^j,\xi_h^j)\\ &\lesssim C_3\bigg(s_h(\xi_h^j,\xi_h^j)+\|\zeta_h^j\|_a^2+\frac{h^{2k+2}}{\Delta t}\int_{t_{j-1}}^{t_j}(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2)ds\\ &\qquad-a_h(\overline{\partial}_t\zeta_h^j,\zeta_h^j)-s_h(\overline{\partial}_t\xi_h^j,\xi_h^j)\bigg)+s_h(\xi_h^j,\xi_h^j),\end{aligned}$$

which implies

$$\begin{aligned}&\|\varepsilon_h(\xi_{h0}^j)\|_0^2+\frac{1}{2\Delta t}\left(\|\zeta_h^j\|_a^2-\|\zeta_h^{j-1}\|_a^2+s_h(\xi_h^j,\xi_h^j)-s_h(\xi_h^{j-1},\xi_h^{j-1})\right)\\ &\le C_9\bigg(\frac{h^{2k+2}}{\Delta t}\int_{t_{j-1}}^{t_j}(|\boldsymbol{\sigma}_t|_{k+1}^2+|\boldsymbol{v}_t|_{k+2}^2)ds+s_h(\xi_h^j,\xi_h^j)+\|\zeta_h^j\|_a^2\bigg).\end{aligned}$$

Summing up the above inequality for $j=1,\dots,n$, we have

$$2\Delta t\sum_{j=1}^n\|\boldsymbol{\varepsilon_h}(\xi_{h0}^j)\|_0^2+\|\zeta_h^n\|_a^2+s_h(\xi_h^n,\xi_h^n)$$

$$\lesssim \|\zeta_h^0\|_a^2 + s_h(\xi_h^0, \xi_h^0) + h^{2k+2}\int_0^{t_n} \left(|\boldsymbol{\sigma}_t|_{k+1}^2 + |\boldsymbol{v}_t|_{k+2}^2\right)ds$$
$$+ 2\Delta t \sum_{j=1}^{n} \left(\|\zeta_h^j\|_a^2 + s_h(\xi_h^j, \xi_h^j)\right), \tag{4.28}$$

which, together with (4.1c), (3.32) and (4.11), yields the desired result (4.12). □

Applying Lemmas 4.2, 3.2 and the triangle inequality leads to the following error estimate for the fully discrete scheme.

Theorem 4.3. *Let* $(\boldsymbol{\sigma}, \boldsymbol{v}) \in C^2(\boldsymbol{L}_{\boldsymbol{S}}^{\boldsymbol{2}}(\Omega) \cap [H^{k+1}(\Omega)]^{d\times d}) \times C^1([H_0^1(\Omega) \cap H^{k+2}(\Omega)]^d)$ *be the solution of* (2.2)*, and let* $(\boldsymbol{\sigma}_h^n, \boldsymbol{v}_h^n) = (\boldsymbol{\sigma}_h^n, \{\boldsymbol{v}_{h0}^n, \boldsymbol{v}_{hb}^n\})$ *be the solution of* (4.1) *for* $n = 1, \ldots, N$*. Then it holds*

$$\|\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^n\|_0^2 + \Delta t \|\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{t_n})) - \boldsymbol{\varepsilon_h}(\boldsymbol{v_{h0}^n})\|^2$$
$$\lesssim h^{2k+2}\left(\tilde{M}_0(0) + \tilde{M}_1(t_n) + \tilde{M}_2(t_n)\right) + \Delta t^2 \tilde{M}_3(t_n), \tag{4.29}$$

where $\tilde{M}_0(0), \tilde{M}_1(t_n), \tilde{M}_2(t_n)$ *and* $\tilde{M}_3(t_n)$ *are defined in Lemma* 4.2*.*

5. Numerical examples

In this section, we provide two 2-dimensional examples and one 3-dimensional example to verify the performance of the proposed fully discrete weak Galerkin method (4.1) with $k = 1, 2$. In all the examples, we take $T = 1$ and assume the elastic medium to be isotropic with $\mu = 1$ and $\lambda = 1$. For the spatial domain, we take $\Omega = [0,1]^2$ in the first two examples with $M \times M$ uniform triangular meshes and $\Omega = [0,1]^3$ in the third example with $M \times M \times M$ uniform tetrahedral meshes; see Fig. 1 for the meshes with $M = 4$.

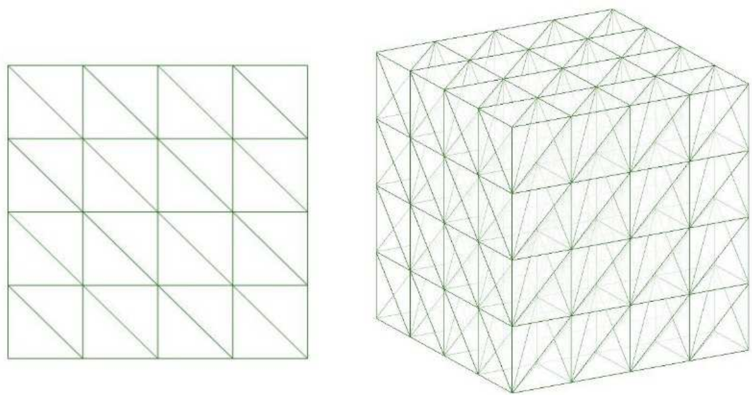

Figure 1: The domains: 4×4 mesh (left) for $\Omega = [0,1]^2$ and $4 \times 4 \times 4$ mesh (right) for $\Omega = [0,1]^3$.

Example 5.1. The exact displacement field $\boldsymbol{u}(x,t)$ and symmetric stress tensor $\boldsymbol{\sigma}(x,t) = (\sigma_{ij})_{2\times 2}$ are respectively given by

$$\boldsymbol{u} = \begin{pmatrix} -e^{-t}\big(x_1^4 - 2x_1^3 + x_1^2\big)\big(4x_2^3 - 6x_2^2 + 2x_2\big) \\ -e^{-t}\big(x_2^4 - 2x_2^3 + x_2^2\big)\big(4x_1^3 - 6x_1^2 + 2x_1\big) \end{pmatrix}$$

and

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} 16te^{-t}\big(2x_1^3 - 3x_1^2 + x_1\big)(2x_2^3 - 3x_2^2 + x_2\big) \\ 2te^{-t}\left[\big(x_1^4 - 2x_1^3 + x_1^2\big)\big(6x_2^2 - 6x_2 + 1\big) + \big(x_2^4 - 2x_2^3 + x_2^2\big)\big(6x_1^2 - 6x_1 + 1\big)\right] \\ 16te^{-t}\big(2x_1^3 - 3x_1^2 + x_1\big)\big(2x_2^3 - 3x_2^2 + x_2\big) \end{pmatrix}.$$

Notice that the velocity field $\boldsymbol{v} = \boldsymbol{u}_t$.

To verify the spatial accuracy, we take the time step $\Delta t = 0.0005$ for $k = 1$ and $\Delta t = 0.00005$ for $k = 2$, respectively. Numerical results of relative errors for the discrete stress $\boldsymbol{\sigma}_h$ and discrete strain $\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)$ at the final time $T = 1$ are presented in the Tables 1 and 2. We can see that spatial convergence orders of the stress and strain are $(k+1)$-th, as is conformable to the theoretical prediction in Theorem 4.3.

Table 1: Convergence rates for Example 5.1 with $\Delta t = 0.0005$: spatial accuracy.

	mesh	$\frac{\Vert\boldsymbol{\sigma}(T)-\boldsymbol{\sigma}_h(T)\Vert_0}{\Vert\boldsymbol{\sigma}(T)\Vert_0}$	order	$\frac{\Vert\boldsymbol{\varepsilon}(\boldsymbol{v}(T))-\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h(T))\Vert_0}{\Vert\boldsymbol{\varepsilon}(\boldsymbol{v}(T))\Vert_0}$	order
$k=1$	2×2	4.8559e-01	–	2.4777	–
	4×4	1.6332e-01	1.57	6.3838e-01	1.96
	8×8	4.6361e-02	1.82	1.7095e-01	1.90
	16×16	1.2528e-02	1.89	4.6659e-02	1.88
	32×32	3.2939e-03	1.93	1.2930e-02	1.86

Table 2: Convergence rates for Example 5.1 with $\Delta t = 0.00005$: spatial accuracy.

	mesh	$\frac{\Vert\boldsymbol{\sigma}(T)-\boldsymbol{\sigma}_h(T)\Vert_0}{\Vert\boldsymbol{\sigma}(T)\Vert_0}$	order	$\frac{\Vert\boldsymbol{\varepsilon}(\boldsymbol{v}(T))-\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h(T))\Vert_0}{\Vert\boldsymbol{\varepsilon}(\boldsymbol{v}(T))\Vert_0}$	order
$k=2$	2×2	1.3765e-01	–	1.0564	–
	4×4	3.0684e-02	2.17	1.8662e-01	2.50
	8×8	4.3824e-03	2.81	2.5445e-02	2.87
	16×16	5.6970e-04	2.94	3.2646e-03	2.96
	32×32	7.2100e-05	2.98	4.1377e-04	2.98

To test the temporal accuracy, we use a very fine spatial mesh with $M = 64$. Numerical results of the errors at the final time $T = 1$ are listed in Table 3. We can observe the first order temporal convergence rate for the stress approximation, as is consistent with Theorem 4.3, and a better rate than first order for the strain approximation.

Table 3: Convergence rates for Example 5.1 with $M = 64$: temporal accuracy.

	Δt	$\frac{\|v(T)-v_{h0}(T)\|_0}{\|v(T)\|_0}$	order	$\frac{\sqrt{\Delta t}\|\varepsilon(v(T))-\varepsilon_h(v_h(T))\|_0}{\|\varepsilon(v(T))\|_0}$	order
$k=1$	0.5	3.5128e-01	–	2.4956e-01	–
	0.25	1.4792e-01	1.25	7.4798e-02	1.74
	0.125	6.7960e-02	1.12	2.4631e-02	1.60
	0.0625	3.2583e-02	1.06	8.5896e-03	1.52
	0.03125	1.5954e-02	1.03	3.1565e-03	1.44
	0.015625	7.8951e-03	1.01	1.2511e-03	1.34
$k=2$	0.5	3.5128e-01	–	2.4839e-01	–
	0.25	1.4792e-01	1.25	7.3962e-02	1.75
	0.125	6.7961e-02	1.12	2.4028e-02	1.62
	0.0625	3.2583e-02	1.06	8.1460e-03	1.56
	0.03125	1.5954e-02	1.03	2.8205e-03	1.53
	0.015625	7.8944e-03	1.02	9.8692e-04	1.52

Example 5.2. The exact displacement field u and symmetric stress tensor σ are of the following forms:

$$\boldsymbol{u} = \begin{pmatrix} -e^{-t}\sin(\pi x_1)\sin(\pi x_2) \\ -e^{-t}\sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}$$

and

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} \pi t e^{-t}\big(3\cos(\pi x_1)\sin(\pi x_2) + \sin(\pi x_1)\cos(\pi x_2)\big) \\ \pi t e^{-t}\big(\sin(\pi x_1)\cos(\pi x_2) + \cos(\pi x_1)\sin(\pi x_2)\big) \\ \pi t e^{-t}\big(3\sin(\pi x_1)\cos(\pi x_2) + \cos(\pi x_1)\sin(\pi x_2)\big) \end{pmatrix}.$$

Tables 4 and 5 show that the scheme (4.1) yields the $(k+1)$-th spatial convergence orders for the stress and strain approximations, and Table 6 shows the first order temporal convergence rate for the stress approximation. These are conformable to Theorem 4.3. In particular, Table 6 also shows a better convergence rate than first order for the strain approximation.

Table 4: Convergence rates for Example 5.2 with $\Delta t = 0.0005$: spatial accuracy.

	mesh	$\frac{\|\boldsymbol{\sigma}(T)-\boldsymbol{\sigma}_h(T)\|_0}{\|\boldsymbol{\sigma}(T)\|_0}$	order	$\frac{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))-\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h(T))\|_0}{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))\|_0}$	order
$k=1$	2×2	1.2181e-01	–	5.8905e-01	–
	4×4	3.3882e-02	1.85	1.5300e-01	1.95
	8×8	8.7967e-03	1.95	3.9273e-02	1.96
	16×16	2.2206e-03	1.99	1.0160e-02	1.95
	32×32	5.5614e-04	2.00	2.7133e-03	1.91

Table 5: Convergence rates for Example 5.2 with $\Delta t = 0.00005$: spatial accuracy.

	mesh	$\frac{\|\boldsymbol{\sigma}(T)-\boldsymbol{\sigma}_h(T)\|_0}{\|\boldsymbol{\sigma}(T)\|_0}$	order	$\frac{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))-\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h(T))\|_0}{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))\|_0}$	order
$k=2$	2×2	24575e-02	–	1.5894e-01	–
	4×4	3.3115e-03	2.89	2.1071e-02	2.92
	8×8	4.2371e-04	2.97	2.7122e-03	2.96
	16×16	5.3390e-05	2.99	3.4421e-04	2.98
	32×32	6.6913e-06	3.00	4.3500e-05	2.98

Table 6: Convergence rates for Example 5.2 with $M = 64$: temporal accuracy.

	Δt	$\frac{\|v(T)-v_{h0}(T)\|_0}{\|v(T)\|_0}$	order	$\frac{\sqrt{\Delta t}\|\varepsilon(v(T))-\varepsilon_h(v_h(T))\|_0}{\|\varepsilon(v(T))\|_0}$	order
$k=1$	0.5	3.5128e-01	–	2.4872e-01	–
	0.25	1.4792e-01	1.25	7.4196e-02	1.75
	0.125	6.7961e-02	1.12	2.4194e-02	1.62
	0.0625	3.2583e-02	1.06	8.2635e-03	1.55
	0.03125	1.5954e-02	1.03	2.9041e-03	1.51
	0.015625	7.8944e-03	1.02	1.0467e-03	1.47
$k=2$	0.5	3.5128e-01	–	2.4839e-01	–
	0.25	1.4792e-01	1.25	7.3962e-02	1.75
	0.125	6.7961e-02	1.12	2.4028e-02	1.62
	0.0625	3.2583e-02	1.06	8.1460e-03	1.56
	0.03125	1.5954e-02	1.03	2.8205e-03	1.53
	0.015625	7.8944e-03	1.02	9.8680e-04	1.52

Example 5.3. This is a 3-dimensional example, and the domain and mesh are shown in Fig. 1. The exact displacement field $\boldsymbol{u}(x,t)$ and symmetric stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^3$ are respectively given by

$$\boldsymbol{u} = \begin{pmatrix} -e^{-t}\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -e^{-t}\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -e^{-t}\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \end{pmatrix}$$

and

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} \pi t e^{-t}\big(3\cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) + \sin(\pi x_1)\cos(\pi(x_2+x_3))\big) \\ \pi t e^{-t}\big(3\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) + \sin(\pi x_2)\sin(\pi(x_1+x_3))\big) \\ \pi t e^{-t}\big(3\sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) + \sin(\pi x_3)\sin(\pi(x_1+x_2))\big) \\ \pi t e^{-t}\sin\big(\pi(x_1+x_2)\big)\sin(\pi x_3) \\ \pi t e^{-t}\sin\big(\pi(x_1+x_3)\big)\sin(\pi x_2) \\ \pi t e^{-t}\sin(\pi x_1)\sin\big(\pi(x_2+x_3)\big) \end{pmatrix}.$$

Numerical results are presented in Tables 7 and 8 for $k = 1$. We can observe that the scheme (4.1) yields the second order accuracy for the stress and strain approximations,

Table 7: Convergence rates for Example 5.3 with $\Delta t = 0.001$: spatial accuracy.

	mesh	$\frac{\|\boldsymbol{\sigma}(T)-\boldsymbol{\sigma}_h(T)\|_0}{\|\boldsymbol{\sigma}(T)\|_0}$	order	$\frac{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))-\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h(T))\|_0}{\|\boldsymbol{\varepsilon}(\boldsymbol{v}(T))\|_0}$	order
$k=1$	$1\times1\times1$	6.0913e-01	–	3.2443e+00	–
	$2\times2\times2$	2.5567e-01	1.25	7.4786e-01	2.18
	$4\times4\times4$	7.2373e-02	1.82	1.6944e-01	2.14
	$8\times8\times8$	1.18782e-02	1.95	4.1371e-02	2.03

Table 8: Convergence rates for Example 5.3 with $M = 16$: temporal accuracy.

	Δt	$\frac{\|\boldsymbol{v}(T)-\boldsymbol{v}_{h0}(T)\|_0}{\|\boldsymbol{v}(T)\|_0}$	order	$\frac{\sqrt{\Delta t}\|\varepsilon(v(T))-\varepsilon_h(v_h(T))\|_0}{\|\varepsilon(v(T))\|_0}$	order
$k=1$	1	1.0000e+00	–	4.2956e+00	–
	0.5	3.5131e-01	1.51	3.5815e-01	1.49
	0.25	1.4796e-01	1.25	1.5490e-01	1.21
	0.125	6.7998e-02	1.12	7.5133e-02	1.04

and the first order temporal accuracy for the stress approximation. These are also conformable to the theoretical results.

6. Conclusion

In this paper, we have developed a class of semi-discrete and fully-discrete WG finite element methods for the quasistatic Maxwell viscoelastic model, and shown theoretically and numerically that the methods are of optimal convergence rates.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 12171340).

References

[1] E. BÉCACHE, P. JOLY, AND C. TSOGKA, *A new family of mixed finite elements for the linear elastodynamic problem*, SIAM J.Numer.Anal. 39 (2002), 2109–2132.
[2] D. BLAND, *The Theory of Linear Viscoelasticity*, Pergamon Press, 1960.
[3] J. BURKARDT, M. GUNZBURGER, AND W. ZHAO, *High-precision computation of the weak Galerkin methods for the fourth-order problem*, Numer. Algor. 84 (2020), 181–205.
[4] W. CAO AND C. WANG, *New primal-dual weak Galerkin finite element methods for convection-diffusion problems*, Appl. Numer. Math. 162 (2021), 171–191.
[5] J. CARCIONE, D. KOSLOFF, AND R. KOSLOFF, *Wave propagation simulation in a linear viscoacoustic medium*, Geophys. J. Int. 93 (1988), 393–407.
[6] J. CARCIONE, D. KOSLOFF, AND R. KOSLOFF, *Wave propagation simulation in a linear viscoelastic medium*, Geophys. J. Int. 95 (1988), 597–611.

[7] G. Chen, M. Feng, and X. Xie, *Robust globally divergence-free weak Galerkin methods for Stokes equations*, J. Comput. Math. 34 (2016), 549–572.

[8] G. Chen, M. Feng, and X. Xie, *A class of robust WG finite element method for convection-diffusion-reaction equations*, J. Comput. Appl. Math. 315 (2017), 107–125.

[9] G. Chen and X. Xie, *A robust weak Galerkin finite element for linear elasticity with strong symmetric stresses*, Comput. Methods Appl. Math. 16 (2016), 389–408.

[10] L. Chen, J. Wang, Y. Wang, and X. Ye, *An auxiliary space multigrid preconditioner for the weak Galerkin method*, Comput. Math. Appl. 70 (2015), 330–344.

[11] Y. Chen, G. Chen, and X. Xie, *Weak Galerkin finite element method for Biot's consolidation problem*, J. Comput. Appl. Math. 330 (2018), 398–416.

[12] R. M. Christensen, *Theory of Viscoelasticity: An Introduction*, Academic Press, 1982.

[13] E. Dill, *Continuum Mechanics: Elasticity, Plasticity, Viscoelasticity*, CRC Press, 2007.

[14] A. Drozdov, *Mechanics of Viscoelastic Solids*, Wiley, 1998.

[15] A. Earl and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill New York, 1955.

[16] Y. C. Fung, *Foundations of Solid Mechanics*, in: *International Series in Dynamics*, Prentice Hall, 1965.

[17] F. Gao, X. Wang, and L. Mu, *A modified weak Galerkin finite element methods for convection-diffusion problems in 2D*, J. Appl. Math. Comput. 49 (2015), 493–511.

[18] J. Golden and G. Graham, *Boundary Value Problems in Linear Viscoelasticity*, Springer, 1988.

[19] M. Gurtin and E. Sternberg, *On the linear theory of viscoelasticity*, Arch. Ration. Mech. Anal. 11 (1962), 291–356.

[20] T. Ha, J. Santos, and D. Sheen, *Nonconforming finite element methods for the simulation of waves in viscoelastic solids*, Comput. Meth. Appl. Mech. Eng. 191 (2002), 5647–5670.

[21] Y. Han, H. Li, and X. Xie, *Robust globally divergence-free weak Galerkin finite element methods for unsteady natural convection problems*, Numer. Math. Thero. Meth. Appl. 12 (2019), 1266–1308.

[22] Y. Han and X. Xie, *Robust globally divergence-free weak Galerkin finite element methods for natural convection problems*, Commun. Comput. Phys. 26 (2019), 1039–1070.

[23] G. Harper, J. Liu, S. Tavener, and B. Zheng, *Lowest-order weak Galerkin finite element methods for linear elasticity on rectangular and brick meshes*, J. Sci. Comput. 78 (2019), 1917–1941.

[24] X. Hu, L. Mu, and X. Ye, *Weak Galerkin method for the Biot's consolidation model*, Comput. Math. Appl. 75 (2018), 2017–2030.

[25] V. Janovsky, S. Shaw, M. K. Warby, and J. R. Whiteman, *Numerical methods for treating problems of viscoelastic isotropic solid deformation*, J. Comput. Appl. Math. 63 (1995), 91–107.

[26] J. Lee, *Mixed Methods with Weak Symmetry for Time Dependent Problems of Elasticity and Viscoelasticity*, PhD Thesis, University of Minnesota, 2012.

[27] B. Li and X. Xie, *A two-level algorithm for the weak Galerkin discretization of diffusion problems*, J. Comput. Appl. Math. 287 (2015), 179–195.

[28] B. Li and X. Xie, *BPX preconditioner for nonstandard finite element methods for diffusion problems*, SIAM J. Numer. Anal. 54 (2016), 1147–1168.

[29] B. Li, X. Xie, and S. Zhang, *BPS preconditioners for a WG method for diffusion problems with stongly discontinuous coefficients*, Comput. Math. Appl. 76 (2018), 701–724.

[30] G. Li, Y. Chen, and Y. Huang, *A robust modified weak Galerkin finite element method for reaction-diffusion equations*, Numer. Math. Theor. Meth. Appl. 15 (2022), 68–90.

[31] H. Li, Z. Zhao, and Z. Luo, *A space-time continuous finite element method for 2D viscoelastic wave equation*, Bound. Value Probl. 53 (2016), 1–17.
[32] R. Lin, X. Ye, S. Zhang, and P. Zhu, *A weak Galerkin finite element method for singularly perturbed convection-diffusion-reaction problems*, SIAM J. Numer. Anal. 56 (2018), 1482–1497.
[33] S. Marques and G. J. Creus, *Computational Viscoelasticity*, Springer-Verlag, 2012.
[34] L. Mu, J. Wang, X. Ye, and S. Zhang, *A C^0-weak Galerkin finite element method for the biharmonic equation*, J. Sci. Comput. 59 (2014), 473–495.
[35] L. Mu, J. Wang, X. Ye, and S. Zhang, *A weak Galerkin finite element method for the Maxwell equations*, J. Sci. Comput. 65 (2015), 363–386.
[36] B. Rivière and S. Shaw, *Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers*, SIAM J. Numer. Anal. 44 (2006), 2650–2670.
[37] B. Rivière, S. Shaw, M. F. Wheeler, and J. R. Whiteman, *Discontinuous Galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity*, Numer. Math. 95 (2003), 347–376.
[38] B. Rivière, S. Shaw, and J. R. Whiteman, *Discontinuous Galerkin finite element methods for dynamic linear solid viscoelasticity problems*, Numer. Meth. Part. D. E. 23 (2007), 1149–1166.
[39] M. Rognes and R. Winther, *Mixed finite element methods for linear viscoelasticity using weak symmetry*, Math. Mod. Meth. Appl. S. 20 (2010), 955–985.
[40] J. Salencon, *Viscoélasticité pour le Calcul des structures*, Éditions de l'École polytechnique, 2016.
[41] R. Schapery, *Nonlinear viscoelastic solids*, Int. J. Solids Struct. 37 (2000), 359–366.
[42] D. Shi and B. Zhang, *High accuracy analysis of the finite element method for nonlinear viscoelastic wave equations with nonlinear boundary conditions*, J. Syst. Sci. Complex, 24 (2011), 795–802.
[43] Z. Shi and M. Wang, *Finite Element Methods*, Science Press, 2013.
[44] S. Shields, J. Li, and E. A. Machorro, *Weak Galerkin methods for time-dependent Maxwell's equations*, Comput. Math. Appl. 74 (2017), 2106–2124.
[45] C. Wang, J. Wang, R. Wang, and R. Zhang, *A locking-free weak Galerkin finite element method for elasticity problems in the primal formulation*, J. Comput. Appl. Math. 307 (2016), 346–366.
[46] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comput. Appl. Math. 241 (2013), 103–115.
[47] J. Wang and X. Ye, *A weak Galerkin mixed finite element method for second-order elliptic problems*, Math. Comp. 83 (2014), 2101–2126.
[48] J. Wang and X. Ye, *A weak Galerkin finite element method for the Stokes equation*, Adv. Comput. Math. 42 (2016), 155–174.
[49] R. Wang and R. Zhang, *A weak Galerkin finite element method for the linear elasticity problem in mixed form*, J. Comput. Math. 36 (2018), 469–491.
[50] S. Wang and X. Xie, *Semi-discrete and fully discrete hybrid stress finite element methods for Maxwell viscoelastic model of wave propagation*, Numer. Math. J. Chinese Universities, 43 (2020), 28–58.
[51] X. Ye and S. Zhang, *A stabilizer free weak Galerkin method for the biharmonic equation on polytopal meshes*, SIAM J. Numer. Anal. 58 (2020), 2572–2588.
[52] X. Ye and S. Zhang, *Low regularity error analysis for weak Galerkin finite element methods for second order elliptic problems*, Numer. Math. Theor. Meth. Appl. 14 (2021), 613–623.

[53] X. Ye and S. Zhang, *A weak Galerkin finite element method for p-Laplacian problem*, East Asian J. Appl. Math. 11 (2021), 219–233.

[54] H. Yuan and X. Xie, *Mixed finite element discretization for Maxwell viscoelastic model of wave propagation*, Adv. Appl. Math. Mech. 14 (2022), 344–364.

[55] Q. Zhai, R. Zhang, and X. Wang, *A hybridized weak Galerkin finite element scheme for the Stokes equations*, Sci. China Math. 58 (2015), 2455–2472.

[56] T. Zhang and Y. Chen, *An analysis of the weak Galerkin finite element method for convection-diffusion equations*, Appl. Math. Comput. 346 (2019), 612–621.

[57] X. Zheng, G. Chen, and X. Xie, *A divergence-free weak Galerkin method for quasi-Newtonian Stokes flows*, Sci. China Math. 60 (2017), 1515–1528.

[58] X. Zheng and X. Xie, *A posteriori error estimation for a weak Galerkin finite element discretization of stokes equations*, East Asian J. Appl. Math. 7 (2017), 508–529.