Numer. Math. Theor. Meth. Appl. Vol. **17**, No. 1, pp. 37-68
doi: 10.4208/nmtma.OA-2023-0023 February 2024

A Novel Iterative Method to Find the Moore-Penrose Inverse of a Tensor with Einstein Product

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Received 12 February 2023; Accepted (in revised version) 14 July 2023

Abstract. In this study, based on an iterative method to solve nonlinear equations, a third-order convergent iterative method to compute the Moore-Penrose inverse of a tensor with the Einstein product is presented and analyzed. Numerical comparisons of the proposed method with other methods show that the average number of iterations, number of the Einstein products, and CPU time of our method are considerably less than other methods. In some applications, partial and fractional differential equations that lead to sparse matrices are considered as prototypes. We use the iterates obtained by the method as a preconditioner, based on tensor form to solve the multilinear system $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$. Finally, several practical numerical examples are also given to display the accuracy and efficiency of the new method. The presented results show that the proposed method is very robust for obtaining the Moore-Penrose inverse of tensors.

AMS subject classifications: 15A24, 65F10, 65F30

Key words: Tensor, iterative methods, Moore-Penrose inverse, Einstein product.

1. Introduction

Tensors occur in a wide variety of application areas such as document analysis, psychometrics, formulation an n-person noncooperative game, medical engineering, chemometrics, higher-order, and so on [5, 19–21, 26, 29, 34]. In this paper, we denote matrices with uppercase letters $A, B, \ldots$, and tensors are signified by calligraphic font $\mathcal{A}, \mathcal{B}, \ldots$. Suppose that N is a positive integer, and an N-th order tensor $\mathcal{A} = (a_{i_1 \ldots i_N})_{1 \le i_j \le P_j}$ is a multidimensional array with $P_1 \ldots P_N$ entries. The tensor $\mathcal{A}$ is called a hyper-matrix, or tensors are higher-order generalizations of vectors and matrices. Let $\mathbb{R}^{P_1 \times \cdots \times P_N}$ show the space of N-th order tensors.

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In the following, we give some definitions of tensors and the Einstein product.

Definition 1.1 ([6]). *Let N and M be positive integers, also $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times K_1\times\cdots\times K_M}$. Then the Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:*

$$(\mathcal{A} *_N \mathcal{B})_{p_1\ldots p_N k_1\ldots k_M} = \sum_{q_N}^{Q_N} \cdots \sum_{q_1}^{Q_1} a_{p_1\ldots p_N q_1\ldots q_N} b_{q_1\ldots q_N k_1\ldots k_M}, \tag{1.1}$$

therefore, $\mathcal{A} *_N \mathcal{B} \in \mathbb{R}^{P_1\times\cdots\times P_N\times K_1\times\cdots\times K_M}$.

Note that if $N = M = 1$, the Einstein product reduces to the standard matrix multiplication.

Definition 1.2. *Inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is defined as follows:*

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{q_N=1}^{Q_N} \cdots \sum_{q_1=1}^{Q_1} \sum_{p_N=1}^{P_N} \cdots \sum_{p_1=1}^{P_1} x_{p_1\cdots p_N q_1\cdots q_N} y_{q_1\cdots q_N p_1\cdots p_N}.$$

Definition 1.3 ([6]). *Let $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ be a tensor, then transpose and Frobenius norm of the tensor $\mathcal{A}$ are defined as follows:*

$$(\mathcal{A}^T)_{p_1\ldots p_N q_1\ldots q_N} = (\mathcal{A})_{q_1\ldots q_N p_1\ldots p_N},$$

and

$$\|\mathcal{A}\| = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle} = \sqrt{\sum_{q_N=1}^{Q_N} \cdots \sum_{q_1=1}^{Q_1} \sum_{p_N=1}^{P_N} \cdots \sum_{p_1=1}^{P_1} |a_{q_1\ldots q_N p_1\ldots p_N}|^2},$$

respectively.

Definition 1.4 ([6]). *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is called diagonal if for all $p_l \neq q_l$, $l = 1, \ldots, N$ we have $a_{p_1\ldots p_N q_1\ldots q_N} = 0$. A diagonal tensor $\mathcal{I} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is identity if $i_{p_1\ldots p_N q_1\ldots q_N} = \Pi_{l=1}^{N} \delta_{p_l q_l}$, where*

$$\delta_{p_l q_l} = \begin{cases} 1, & p_l = q_l, \\ 0, & p_l \neq q_l. \end{cases}$$

Definition 1.5. *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$, then $\mathcal{B} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is said inverse of $\mathcal{A}$ with the Einstein product if*

$$\mathcal{A} *_N \mathcal{B} = \mathcal{I},$$

therefore $\mathcal{A}^{-1} = \mathcal{B}$.

Proposition 1.1 ([38]). *If $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times K_1\times\cdots\times K_M}$, then*

$$(\mathcal{A} *_N \mathcal{B})^T = \mathcal{B}^T *_N \mathcal{A}^T, \quad \mathcal{I}_N *_N \mathcal{B} = \mathcal{B}, \quad \mathcal{B} *_M \mathcal{I}_M = \mathcal{B},$$

where $\mathcal{I}_N \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{I}_M \in \mathbb{R}^{K_1\times\cdots\times K_M\times K_1\times\cdots\times K_M}$, are identity tensors.

Definition 1.6 ([23]). *The null space and the range of tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ are*

$$\begin{aligned}\mathfrak{N}(\mathcal{A}) &= \left\{\mathcal{X} \in \mathbb{R}^{P_1\times\cdots\times P_N} : \mathcal{A} *_N \mathcal{X} = 0\right\},\\ \mathfrak{R}(\mathcal{A}) &= \left\{\mathcal{A} *_N \mathcal{X} : \mathcal{X} \in \mathbb{R}^{P_1\times\cdots\times P_N}\right\},\end{aligned}$$

respectively.

Definition 1.7. *The orthogonal complement of a subspace $\mathcal{L}$ in $\mathbb{R}^{P_1\times\cdots\times P_N}$ is defined by*

$$\mathcal{L}^\perp = \left\{\mathcal{X} \in \mathbb{R}^{P_1\times\cdots\times P_N} : \langle \mathcal{X}, \mathcal{Y}\rangle = 0,\ \forall \mathcal{Y} \in \mathcal{L}\right\}.$$

Lemma 1.1 ([22]). *Assume that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is a tensor, then*

$$\begin{aligned}&\mathfrak{N}(\mathcal{A})^\perp = \mathfrak{R}(\mathcal{A}^T), &&\mathfrak{R}(\mathcal{A})^\perp = \mathfrak{N}(\mathcal{A}^T),\\ &\mathfrak{R}(\mathcal{A}^T)^\perp = \mathfrak{N}(\mathcal{A}), &&\mathfrak{N}(\mathcal{A}^T)^\perp = \mathfrak{R}(\mathcal{A}),\\ &\mathfrak{N}(\mathcal{A}^T *_N \mathcal{A}) = \mathfrak{N}(\mathcal{A}), &&\mathfrak{R}(\mathcal{A}^T *_N \mathcal{A}) = \mathfrak{R}(\mathcal{A}^T).\end{aligned} \tag{1.2}$$

Note that the dimension of a subspace $\mathcal{L}$ is denoted by $\dim(\mathcal{L})$.

Lemma 1.2 ([23]). *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is a tensor, then we have*

$$\dim\big(\mathfrak{R}(\mathcal{A})\big) = \dim\big(\mathfrak{R}(\mathcal{A}^T)\big).$$

Some literature review a survey of the generalized inverses of a tensor with the Einstein product, for more details see [7, 20, 31, 32, 39].

The Moore-Penrose inverse of the tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ denoted by $\mathcal{A}^\dagger \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times P_1\times\cdots\times P_N}$ is a unique tensor $\mathcal{X}$ satisfying in [38]

$$\begin{aligned}&\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A},\\ &\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X},\\ &(\mathcal{A} *_N \mathcal{X})^T = \mathcal{A} *_N \mathcal{X},\\ &(\mathcal{X} *_N \mathcal{A})^T = \mathcal{X} *_N \mathcal{A},\end{aligned} \tag{1.3}$$

where the above equations are Penrose equations.

Computing inverse tensors has always been a time-consuming and difficult task. Therefore, numerical methods and especially iterative methods have been important to find the tensor inverse. There exist some iterative methods to compute the Moore-Penrose inverse matrix [12, 16, 24, 36]. The most famous method to approximate the matrix A^{-1} is the Newton method, and this method was developed for finding the tensor $\mathcal{A}^{-1}$ as follows [25]:

$$\mathcal{V}_{r+1} = \mathcal{V}_r *_N (2\mathcal{I} - \mathcal{A} *_N \mathcal{V}_r), \quad r = 0, 1, 2, \ldots . \tag{1.4}$$

Proposition 1.2 ([25]). *Suppose that $\mathcal{A}$ is an invertible tensor. Then the method* (1.4) *is convergent to the inverse of $\mathcal{A}$ if*

$$\|\mathcal{E}\| = \|\mathcal{I} - \mathcal{A} *_N \mathcal{V}_0\| \leq q < 1,$$

and then we have

$$\|\mathcal{V}_j - \mathcal{A}^{-1}\| \leq \frac{\|\mathcal{V}_0\|}{1-q}\|\mathcal{I} - \mathcal{A} *_N \mathcal{V}_j\| \leq \frac{\|\mathcal{V}_0\|}{1-q} q^{2^j}.$$

Li *et al.* [28] proposed the third-order Chebyshev method to find the inverse of matrix A as follows:

$$\begin{cases} T_r = AV_r, \\ V_{r+1} = V_r\big(3I - T_r(3I - T_r)\big), \quad r = 0, 1, 2, \ldots, \end{cases} \tag{1.5}$$

and this method works better than Newton's method. In the following, by applying (1.5), the method for finding the inverse of tensor $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times P_1 \times \cdots \times P_N}$ with the Einstein product can be derived

$$\begin{cases} \mathcal{T}_r = \mathcal{A} *_N \mathcal{V}_r, \\ \mathcal{V}_{r+1} = \mathcal{V}_r *_N \big(3\mathcal{I} - \mathcal{T}_r *_N (3\mathcal{I} - \mathcal{T}_r)\big), \quad r = 0, 1, 2, \ldots . \end{cases} \tag{1.6}$$

Dehdezi and Karimi [25] proposed a fast and efficient Newton-Shultz-type iterative method for computing the Moore-Penrose inverse of tensors as follows:

$$\begin{cases} \mathcal{T}_r = \mathcal{A} *_N \mathcal{V}_r, \quad \mathcal{P}_r = \mathcal{T}_r *_N (2\mathcal{I} - \mathcal{T}_r), \\ \mathcal{V}_{r+1} = \mathcal{V}_r *_N (2\mathcal{I} - \mathcal{T}_r) *_N \big(3\mathcal{I} - \mathcal{P}_r *_N (3\mathcal{I} - \mathcal{P}_r)\big), \quad r = 0, 1, 2, \ldots . \end{cases} \tag{1.7}$$

In [9, 25], an initial tensor to start these methods was introduced as follows:

$$\mathcal{V}_0 = \rho \mathcal{A}^T, \quad \rho = \frac{1}{\|\mathcal{A}\|^2}. \tag{1.8}$$

In this study, we focus on presenting and demonstrating a novel method for finding inverse tensors as fast ags possible with close attention to reducing computational time.

Note that throughout this study $\mathcal{X}^n$ means $\overbrace{\mathcal{X} *_N \mathcal{X} *_N \cdots *_N \mathcal{X}}^{n \text{ times}}$.

2. A novel iterative method

Iterative methods to solve nonlinear equations have always been very important, so many researchers studied in this field [10, 13, 14, 35]. In this section, we present a new iterative method to solve nonlinear equations. Assume that $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is sufficiently differentiable in a neighborhood of $\alpha \in I$ and α is a simple root of

$F(v) = 0$, and also v_0 is an initial guess sufficiently close to α. In the following, we propose a method to solve the equation $F(v) = 0$

$$\begin{aligned} w_r &= v_r - \frac{F(v_r)}{F'(v_r)}, \\ v_{r+1} &= w_r + \frac{F(v_r)^2}{2F'(v_r)^3}\left(G(\eta,\mu) - \frac{F(v_r)}{3F'(v_r)^2}H(\zeta,\xi)\right), \end{aligned} \qquad r = 0,1,2,\dots, \tag{2.1}$$

where

$$\eta = F''(v_r), \quad \mu = \frac{F'''(v_r)F(v_r)}{F'(v_r)},$$
$$\zeta = F''(v_r)^2, \quad \xi = F''''(v_r)F(v_r),$$

here G and H are analytic functions in a neighborhood of origin.

In the following theorem, the convergence analysis of this method is studied.

Theorem 2.1. *Suppose that $G(\eta,\mu)$ and $H(\zeta,\xi)$ are analytic functions in a neighborhood of origin and satisfy*

$$\left.\frac{\partial G(\eta,\mu)}{\partial \eta}\right|_{(0,0)} = 1, \quad \left.\frac{\partial G(\eta,\mu)}{\partial \mu}\right|_{(0,0)} = -\frac{1}{9},$$
$$\left.\frac{\partial H(\zeta,\xi)}{\partial \zeta}\right|_{(0,0)} = 1, \quad \left.\frac{\partial H(\zeta,\xi)}{\partial \xi}\right|_{(0,0)} = 2,$$

and $F'(\alpha)$ is continuous in α. Therefore, (2.1) *converges to α with order of convergence three. Moreover, the error equation is*

$$\epsilon_{r+1} = \frac{2}{3}\left(2D_2^2 - D_3\right)(\epsilon_r)^3 + \mathcal{O}((\epsilon_r)^4),$$

where $D_i = F'(\alpha)^{-1}F(\alpha)/i!$, for $i \geq 2$, and $\epsilon_r = v_r - \alpha$.

Proof. Based on the Taylor expansion for F, we have

$$F(v_r) = F'(\alpha)\left((\epsilon_r) + D_2(\epsilon_r)^2 + D_3(\epsilon_r)^3 + D_4(\epsilon_r)^4 + D_5(\epsilon_r)^5 + D_6(\epsilon_r)^6 + \mathcal{O}\big((\epsilon_r)^7\big)\right),$$
$$F'(v_r) = F'(\alpha)\left(1 + 2D_2(\epsilon_r) + 3D_3(\epsilon_r)^2 + 4D_4(\epsilon_r)^3 + 5D_5(\epsilon_r)^4 + 6D_6(\epsilon_r)^5 + \mathcal{O}\big((\epsilon_r)^6\big)\right),$$
$$\vdots$$
$$F''''(v_r) = F'(\alpha)\left(24D_4 + 120D_5(\epsilon_r) + 360D_6(\epsilon_r)^2 + \mathcal{O}\big((\epsilon_r)^3\big)\right).$$

Therefore, according to the method (2.1) we can obtain

$$\epsilon_{r+1} = \frac{2}{3}\left(2D_2^2 - D_3\right)(\epsilon_r)^3 + \mathcal{O}\big((\epsilon_r)^4\big).$$

Now, suppose that $F(v)=1/v-\alpha$, $G(\eta,\mu)=\eta-\mu/9$, and $H(\zeta,\xi)=\zeta+2\xi$, then the iterative method is obtained from (2.1) as follows:

$$v_{r+1}=\frac{1}{3}v_r\left(34-108\alpha v_r+150(\alpha v_r)^2-97(\alpha v_r)^3+24(\alpha v_r)^4\right),\quad r=0,1,2,\ldots. \tag{2.2}$$

In the following, using (2.2) we present the iterative method for computing the Moore-Penrose inverse of a tensor with the Einstein product

$$\begin{aligned}\mathcal{V}_{r+1}=\frac{1}{3}\mathcal{V}_r *_N \big(&34\mathcal{I}-108\mathcal{A}*_N\mathcal{V}_r+150(\mathcal{A}*_N\mathcal{V}_r)^2-97(\mathcal{A}*_N\mathcal{V}_r)^3\\ &+24(\mathcal{A}*_N\mathcal{V}_r)^4\big),\quad r=0,1,2,\ldots,\end{aligned} \tag{2.3}$$

or

$$\begin{cases}\mathcal{T}_r=\mathcal{A}*_N\mathcal{V}_r,\quad \mathcal{Q}_r=\mathcal{T}_r^2,\\ \mathcal{V}_{r+1}=\dfrac{1}{3}\mathcal{V}_r*_N\\ \qquad\times\left(34\mathcal{I}-108\mathcal{T}_r+\mathcal{Q}_r*_N(150\mathcal{I}-97\mathcal{T}_r+24\mathcal{Q}_r)\right),\end{cases}\quad r=0,1,2,\ldots. \tag{2.4}$$

Now, we give some properties of the method (2.4).

Theorem 2.2. *Let $\mathcal{A}$ be a nonsingular tensor, and the initial approximation $\mathcal{V}_0$ satisfies*

$$\|\mathcal{I}-\mathcal{A}*_N\mathcal{V}_0\|<1,$$

then the iterative method (2.4) *converges to $\mathcal{A}^{-1}$ with third-order.*

Proof. Assume that

$$\mathcal{E}_r=\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r, \tag{2.5}$$

then we can write

$$\begin{aligned}\mathcal{E}_{r+1}&=I-\mathcal{A}*_N\mathcal{V}_{r+1}\\ &=(\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r)^3*_N\left(\mathcal{I}-\frac{25}{3}\mathcal{A}*_N\mathcal{V}_r+8(\mathcal{A}*_N\mathcal{V}_r)^2\right)\\ &=(\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r)^3*_N\left(\frac{2}{3}\mathcal{I}-\frac{23}{3}(\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r)+8(\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r)^2\right)\\ &=\frac{2}{3}\mathcal{E}_r^3-\frac{23}{3}\mathcal{E}_r^4+8\mathcal{E}_r^5.\end{aligned} \tag{2.6}$$

So, we have

$$\|\mathcal{E}_{r+1}\|=\left\|\frac{2}{3}\mathcal{E}_r^3-\frac{23}{3}\mathcal{E}_r^4+8\mathcal{E}_r^5\right\|\le\left(\frac{2}{3}+\frac{23}{3}\|\mathcal{E}_r\|+8\|\mathcal{E}_r^2\|\right)\|\mathcal{E}_r\|^3. \tag{2.7}$$

Since $\|\mathcal{E}_0\|<1$, we can get $\mathcal{E}_r\to 0$, therefore $\mathcal{I}-\mathcal{A}*_N\mathcal{V}_r\to 0$, that means $\mathcal{V}_r\to\mathcal{A}^{-1}$.

Now, suppose that $\mathcal{F} = \mathcal{A}^{-1} - \mathcal{V}_r$ is the error tensor, then we have

$$\begin{aligned}
&\mathcal{A} *_N \mathcal{F}_r = \mathcal{I} - \mathcal{A} *_N \mathcal{V}_r = \mathcal{E}_r, \\
&\mathcal{A} *_N \mathcal{F}_{r+1} = \frac{2}{3}(\mathcal{A} *_N \mathcal{F}_r)^3 - \frac{23}{8}(\mathcal{A} *_N \mathcal{F}_r)^4 + 8(\mathcal{A} *_N \mathcal{F}_r)^5.
\end{aligned}$$

Since $\mathcal{A}$ is invertible, we can get

$$\mathcal{F}_{r+1} = \frac{2}{3}\mathcal{F}_r *_N (\mathcal{A} *_N \mathcal{F}_r)^2 - \frac{23}{8}\mathcal{F}_r *_N (\mathcal{A} *_N \mathcal{F}_r)^3 + 8\mathcal{F}_r *_N (\mathcal{A} *_N \mathcal{F}_r)^4.$$

Consequently, we write

$$\|\mathcal{F}_{r+1}\| \leq \left(\frac{2}{3}\|\mathcal{A}\|^2 + \frac{23}{3}\|\mathcal{A}\|^3\|\mathcal{F}_r\| + 8\|\mathcal{A}\|^4\|\mathcal{F}_r\|^2\right)\|\mathcal{F}_r\|^3.$$

Here, the method (2.4) converges to $\mathcal{A}^{-1}$, and the order of convergence is three. □

Theorem 2.3. *Suppose that $\mathcal{A}$ is a nonsingular tensor, and $\mathcal{A} *_N \mathcal{V}_0 = \mathcal{V}_0 *_N \mathcal{A}$. Then for the sequence* (2.4)*, we have*

$$\mathcal{A} *_N \mathcal{V}_i = \mathcal{V}_i *_N \mathcal{A}, \quad i = 1, 2, \ldots .$$

Proof. At first, from

$$\mathcal{A} *_N \mathcal{V}_0 = \mathcal{V}_0 *_N \mathcal{A},$$

and the iterative method (2.4) we can conclude $\mathcal{A} *_N \mathcal{V}_1 = \mathcal{V}_1 *_N \mathcal{A}$. Now, assume that $\mathcal{A} *_N \mathcal{V}_i = \mathcal{V}_i *_N \mathcal{A}$ is true, then from (2.4), we can write

$$\begin{aligned}
\mathcal{A} *_N \mathcal{V}_{i+1} = &\frac{1}{3}\mathcal{A} *_N \mathcal{V}_i *_N \\
&\times \left(34\mathcal{I} - 108\mathcal{A} *_N \mathcal{V}_i + 150(\mathcal{A} *_N \mathcal{V}_i)^2 - 97(\mathcal{A} *_N \mathcal{V}_i)^3 + 24(\mathcal{A} *_N \mathcal{V}_i)^4\right) \\
= &\frac{1}{3}\mathcal{V}_i *_N \mathcal{A} *_N \\
&\times \left(34\mathcal{I} - 108\mathcal{V}_i *_N \mathcal{A} + 150(\mathcal{V}_i *_N \mathcal{A})^2 - 97(\mathcal{V}_i *_N \mathcal{A})^3 + 24(\mathcal{V}_i *_N \mathcal{A})^4\right) \\
= &\mathcal{V}_{i+1} *_N \mathcal{A}.
\end{aligned}$$

The proof is complete. □

Lemma 2.1. *For the sequence $\{\mathcal{V}_i\}_{i=0}^{i=\infty}$ generated by the method* (2.4)*, it holds that*

$$(\mathcal{V}_i *_N \mathcal{A})^T = \mathcal{V}_i *_N \mathcal{A}, \tag{2.8a}$$
$$(\mathcal{A} *_N \mathcal{V}_i)^T = \mathcal{A} *_N \mathcal{V}_i, \tag{2.8b}$$
$$\mathcal{V}_i *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{V}_i, \tag{2.8c}$$
$$\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{V}_i = \mathcal{V}_i. \tag{2.8d}$$

Proof. We prove that by using the principle of mathematical induction on k. For $k = 0$ and according to (1.8), Eqs. (2.8a) and (2.8b) are easily obtained. By applying

$$(\mathcal{A} *_N \mathcal{A}^\dagger)^T = \mathcal{A} *_N \mathcal{A}^\dagger, \quad (\mathcal{A}^\dagger *_N \mathcal{A})^T = \mathcal{A}^\dagger *_N \mathcal{A},$$

we have

$$\begin{aligned}\mathcal{V}_0 *_N \mathcal{A} *_N \mathcal{A}^\dagger = \rho\mathcal{A}^T *_N \mathcal{A} *_N \mathcal{A}^\dagger = \rho\mathcal{A}^T = \mathcal{V}_0,\\ \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{V}_0 = \rho\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^T = \rho\mathcal{A}^T = \mathcal{V}_0.\end{aligned}$$

Now, assume that the conclusion satisfies for $k > 0$. We demonstrate that it holds for $k + 1$. Therefore, we can obtain

$$\begin{aligned}(\mathcal{V}_{k+1} *_N \mathcal{A})^T &= \Big(\frac{1}{3}\mathcal{V}_k *_N \big(34\mathcal{I} - 108\mathcal{A} *_N \mathcal{V}_k + 150(\mathcal{A} *_N \mathcal{V}_k)^2 \\ &\qquad -97(\mathcal{A} *_N \mathcal{V}_k)^3 + 24(\mathcal{A} *_N \mathcal{V}_k)^4\big) *_N \mathcal{A}\Big)^T \\ &= \mathcal{V}_{k+1} *_N \mathcal{A},\end{aligned}$$

and (2.8b) is proved similarly.

It follows from (2.8c) that

$$\mathcal{A} *_N \mathcal{V}_k *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A} *_N \mathcal{V}_k,$$

then we have

$$\begin{aligned}\mathcal{V}_{k+1} *_N \mathcal{A} *_N \mathcal{A}^\dagger &= \frac{1}{3}\mathcal{V}_k *_N \big(34\mathcal{I} - 108\mathcal{A} *_N \mathcal{V}_k + 150(\mathcal{A} *_N \mathcal{V}_k)^2 \\ &\qquad -97(\mathcal{A} *_N \mathcal{V}_k)^3 + 24(\mathcal{A} *_N \mathcal{V}_k)^4\big) \mathcal{A} *_N \mathcal{A}^\dagger \\ &= \frac{1}{3}\mathcal{V}_k *_N \big(34\mathcal{I} - 108\mathcal{A} *_N \mathcal{V}_k + 150(\mathcal{A} *_N \mathcal{V}_k)^2 \\ &\qquad -97(\mathcal{A} *_N \mathcal{V}_k)^3 + 24(\mathcal{A} *_N \mathcal{V}_k)^4\big) = \mathcal{V}_{k+1}.\end{aligned}$$

So, (2.8c) satisfies for $k + 1$. Finally, (2.8d) is proved similarly. □

Theorem 2.4. *Under the assumptions of Theorem* 2.2*, the method* (2.4) *is asymptotically stable.*

Proof. Suppose that $\tilde{\mathcal{V}}_r$ is the approximate value of the exact value $\mathcal{V}_r$, then

$$\Delta_r = \tilde{\mathcal{V}}_r - \mathcal{V}_r$$

is numerical perturbation that occurs at the r-th iteration of the iterative method (2.4). In the following, we verify a first-order error analysis. Ignoring the quadratic terms $(\Delta_r)^2$ and according to (2.5), we have

$$\tilde{\mathcal{E}}_r = \mathcal{I} - \mathcal{A} *_N \tilde{\mathcal{V}}_r, \tag{2.9}$$

therefore,

$$\begin{aligned}\Delta_{r+1} &= \tilde{\mathcal{V}}_{r+1} - \mathcal{V}_{r+1} \\ &= \tilde{\mathcal{V}}_r *_N \left(3\mathcal{I} + 3\tilde{\mathcal{E}}_r + 3\tilde{\mathcal{E}}_r^2 + \tilde{\mathcal{E}}_r^3 + 24\tilde{\mathcal{E}}_r^4\right) \\ &\quad - \mathcal{V}_r *_N \left(3\mathcal{I} + 3\mathcal{E}_r + 3\mathcal{E}_r^2 + \mathcal{E}_r^3 + 24\mathcal{E}_r^4\right) \\ &= (\mathcal{V}_r + \Delta_r) *_N \left(3\mathcal{I} + 3\tilde{\mathcal{E}}_r + 3\tilde{\mathcal{E}}_r^2 + \tilde{\mathcal{E}}_r^3 + 24\tilde{\mathcal{E}}_r^4\right) \\ &\quad - \mathcal{V}_r *_N \left(3\mathcal{I} + 3\mathcal{E}_r + 3\mathcal{E}_r^2 + \mathcal{E}_r^3 + 24\mathcal{E}_r^4\right) \\ &= \Delta_r *_N \left(3\mathcal{I} + 3\tilde{\mathcal{E}}_r + 3\tilde{\mathcal{E}}_r^2 + \tilde{\mathcal{E}}_r^3 + 24\tilde{\mathcal{E}}_r^4\right) \\ &\quad + \mathcal{V}_r *_N \left(3(\tilde{\mathcal{E}}_r - \mathcal{E}_r) + 3(\tilde{\mathcal{E}}_r^2 - \mathcal{E}_r^2) + (\tilde{\mathcal{E}}_r^3 - \mathcal{E}_r^3) + 24(\tilde{\mathcal{E}}_r^4 - \mathcal{E}_r^4)\right). \end{aligned} \tag{2.10}$$

By using tensor norm from (2.10), we can obtain

$$\begin{aligned}\|\Delta_{r+1}\| \leq \|\Delta_r\| &\left(3 + 3\|\tilde{\mathcal{E}}_r\| + 3\|\tilde{\mathcal{E}}_r\|^2 + \|\tilde{\mathcal{E}}_r\|^3 + 24\|\tilde{\mathcal{E}}_r\|^4\right) \\ &+ \|\mathcal{V}_r\| \left(3\|\tilde{\mathcal{E}}_r - \mathcal{E}_r\| + 3\|\tilde{\mathcal{E}}_r^2 - \mathcal{E}_r^2\| + \|\tilde{\mathcal{E}}_r^3 - \mathcal{E}_r^3\| + 24\|\tilde{\mathcal{E}}_r^4 - \mathcal{E}_r^4\|\right). \end{aligned} \tag{2.11}$$

Now according to

$$\tilde{\mathcal{E}}_r = \mathcal{I} - \mathcal{A}\tilde{\mathcal{V}}_r = \mathcal{I} - \mathcal{A}(\mathcal{V}_r + \Delta_r) = \mathcal{E}_r - \mathcal{A}\Delta_r,$$

we can get

$$\begin{aligned}\left\|\tilde{\mathcal{E}}_r^k\right\| = \left\|(\mathcal{E}_r - \mathcal{A}\Delta_r)^k\right\| &\leq \left(\|\mathcal{E}_r\| + \|\mathcal{A}\|\|\Delta_r\|\right)^k \\ &\leq \left(\|\mathcal{E}_r\| + \|\mathcal{A}\|\|\Delta_r\|\right)^k = \eta_0^k, \quad k = 1, 2, 3, 4, \end{aligned} \tag{2.12}$$

where $\eta_0 = \|\mathcal{E}_r\| + \mathcal{O}(\Delta_r)$. Also, we can write

$$\begin{aligned}\left\|\tilde{\mathcal{E}}_r^k - \mathcal{E}_r^k\right\| &= \left\|(\mathcal{E}_r - \mathcal{A} *_N \Delta_r)^k - \mathcal{E}_r^k\right\| \\ &= \left\|\sum_{i=0}^{k}(-1)^i \binom{k}{i} \mathcal{E}_r^i *_N (\mathcal{A} *_N \Delta_r)^{k-i} - \mathcal{E}_r^k\right\| \\ &\leq \|\mathcal{A} *_N \Delta_r\| \sum_{i=0}^{k-1} \binom{k}{k-1-i} \left\|\mathcal{E}_r^{k-1-i}\right\| \left\|\mathcal{A} *_N \Delta_r\right\|^i \\ &\leq \xi_k \|\mathcal{A}\|\|\Delta_r\|, \end{aligned} \tag{2.13}$$

where

$$\xi_k = \sum_{i=0}^{k-1} \binom{k}{k-1-i} \left\|\mathcal{E}_r^{k-1-i}\right\| \left\|\mathcal{A} *_N \Delta_r\right\|^i = k\|\mathcal{E}_r\|^{k-1} + \mathcal{O}\big(\|\Delta_r\|\big).$$

By substituting relations (2.12) and (2.13), in (2.11), we obtain

$$\|\Delta_{r+1}\| \leq \left(3 + 3\eta_0 + 3\eta_0^2 + \eta_0^3 + 24\eta_0^4 + \|\mathcal{V}_r\|\|\mathcal{A}\|(3\xi_1 + 3\xi_2 + \xi_3 + 24\xi_4)\right) \|\Delta_r\|,$$

and subsequently, we have

$$\|\Delta_{r+1}\| \leq \left(3 + 3\eta_0 + 3\eta_0^2 + \eta_0^3 + 24\eta_0^4 + \|\mathcal{V}_r\|\|\mathcal{A}\|(3\xi_1 + 3\xi_2 + \xi_3 + 24\xi_4)\right)^{r+1} \|\Delta_0\|.$$

From the above relation, we can conclude that the perturbation at the iteration $r+1$ is bounded. Then the sequence generated by the iterative method (2.4) is asymptotically stable. As a result, rounding errors will not accumulate much beyond the uncertainty initiated by the ill-conditioning of $\mathcal{A}$. □

The Drazin inverse plays an important role in various applications in difference equations, singular differential, Cryptography, Markov chains, and investigation of Cesaro-Neumann iterations [27, 30, 33]. In [4, 23], the Drazin inverse of an even-order tensor is introduced.

Definition 2.1 ([23]). *Let $k = ind(\mathcal{A})$ be the smallest positive integer that holds*

$$\mathfrak{R}(\mathcal{A}^{k+1}) = \mathfrak{R}(\mathcal{A}^k),$$

k is called index of the tensor $\mathcal{A}$.

Note that the Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{A}^D$.

Definition 2.2 ([23]). *Suppose that $\mathcal{A}$ is a tensor, then the Drazin inverse of $\mathcal{A}$ is $\mathcal{W}$, which holds in the following equations:*

$$\mathcal{A}^{k+1} *_N \mathcal{W} = \mathcal{A}^k, \quad \mathcal{W} *_N \mathcal{A} *_N \mathcal{W} = \mathcal{W}, \quad \mathcal{A} *_N \mathcal{W} = \mathcal{W} *_N \mathcal{A},$$

where $k = ind(\mathcal{A})$.

Now, we develop the method (2.4) to find the Drazin inverse of the tensor, where the initial tensor is

$$\mathcal{W}_0 = \frac{\mathcal{A}^k}{\|\mathcal{A}^{k+1}\|}. \tag{2.14}$$

Let $\mathcal{F}_0 = \mathcal{I} - \mathcal{A} *_N \mathcal{W}_0$, then $\mathcal{F}_r = \mathcal{I} - \mathcal{A} *_N \mathcal{W}_r$. Thus similarly as in (2.6), we can write

$$\begin{aligned}
\mathcal{F}_{r+1} &= \mathcal{I} - \mathcal{A} *_N \mathcal{W}_{r+1} \\
&= (\mathcal{I} - \mathcal{A} *_N \mathcal{W}_r)^3 *_N \left(\frac{2}{3}\mathcal{I} - \frac{23}{3}(\mathcal{I} - \mathcal{A} *_N \mathcal{W}_r) + 8(\mathcal{I} - \mathcal{A} *_N \mathcal{W}_r)^2\right) \\
&= \frac{2}{3}\mathcal{F}_r^3 - \frac{23}{3}\mathcal{F}_r^4 + 8\mathcal{F}_r^5. \qquad (2.15)
\end{aligned}$$

Using the norm of (2.15), we have

$$\|\mathcal{F}_{r+1}\| \le \frac{2}{3}\|\mathcal{F}_r\|^3 + \frac{23}{3}\|\mathcal{F}_r\|^4 + 8\|\mathcal{F}_r\|^5. \tag{2.16}$$

Here, since $\|\mathcal{F}_0\| < 1$ and from the relation (2.16), we can get

$$\|\mathcal{F}_1\| \le \frac{2}{3}\|\mathcal{F}_0\|^3 + \frac{23}{3}\|\mathcal{F}_0\|^4 + 8\|\mathcal{F}_0\|^5 \le \mathcal{O}\big(\|\mathcal{F}_0\|^3\big).$$

By continuing this process, we obtain

$$\|\mathcal{F}_{r+1}\| \le \frac{2}{3}\|\mathcal{F}_r\|^3 + \frac{23}{3}\|\mathcal{F}_r\|^4 + 8\|\mathcal{F}_r\|^5 \le \mathcal{O}\big(\|\mathcal{F}_r\|^3\big).$$

So, we have $\|\mathcal{F}_{r+1}\| \le \mathcal{O}(\|\mathcal{F}_r)\|$, for every $r \ge 0$. It follows from the above expressions

$$\|\mathcal{F}_r\|^3 \le \mathcal{O}\big(\|\mathcal{F}_0\|^{3^r}\big), \quad r \ge 0. \tag{2.17}$$

3. Numerical experiments

In this section, we review and compare the numerical results of the mentioned method with other methods. We show methods (1.4), (1.5), (1.7) and (2.4) with Newton, Chebyshev, KH-K, and OM, respectively. According to [18], we consider the stop criterion as follows:

$$\frac{\|\mathcal{V}_{r+1} - \mathcal{V}_r\|}{1 + \|\mathcal{V}_r\|} < 10^{-10}.$$

Here, the Res is the residual when the process is stopped, and the CPU is the time spent.

The steps of the algorithm to find the inverse of a tensor with the Einstein product are the following:

Algorithm 3.1 The method (2.4) for computing the inverse of a tensor with the Einstein product.

Step 1: Input tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$.
Step 2: Take initial tensor $\mathcal{V}_0 = \mathcal{A}^T/\|\mathcal{A}\|^2$ and tolerance $\varepsilon \ge 0$. Set $r := 0$.
Step 3: Let

$$\mathcal{T} = \mathcal{A} *_N \mathcal{V}_r, \quad \mathcal{Q}_r = \mathcal{T}_r^2,$$
$$\mathcal{V}_{r+1} = \frac{1}{3}\mathcal{V}_r *_N \big(34\mathcal{I} - 108\mathcal{T}_r + \mathcal{Q}_r *_N (150\mathcal{I} - 97\mathcal{T}_r + 24\mathcal{Q}_r)\big).$$

Step 4: Stop if $\|\mathcal{V}_{r+1} - \mathcal{V}_r\|/(1 + \|\mathcal{V}_r\|) \le \varepsilon$. Otherwise, $r := r + 1$ go to Step 3.

Example 3.1. In the first example, we consider the behavior of the mentioned iterative methods with the tensor $\mathcal{A} \in \mathbb{R}^{4,4,4,4}$ as follows:

$$\mathcal{A}(:,:,i,1)=\begin{pmatrix}\begin{pmatrix}2.49&-0.36&3.20&-0.03\\-0.32&2.71&-0.38&1.83\\0.73&-0.40&2.83&-1.12\\2.06&-0.06&1.14&2.24\end{pmatrix}&\begin{pmatrix}0.93&-0.22&0.78&0.88\\2.62&0&1.11&0.18\\0.45&1.50&-0.03&2.38\\0.31&2.01&1.50&2.47\end{pmatrix}\\\begin{pmatrix}0.28&2.05&1.39&0.08\\2.91&2.74&0.68&2.15\\1.75&1.45&1.48&2.15\\-1.38&0.29&0.51&1.37\end{pmatrix}&\begin{pmatrix}0.02&2.58&2.40&2.49\\0.47&-0.58&0.38&3.26\\1.38&-0.48&2.11&-0.72\\-0.57&3.62&-0.85&2.81\end{pmatrix}\end{pmatrix},$$

$$\mathcal{A}(:,:,i,2)=\begin{pmatrix}\begin{pmatrix}-1.48&1.92&0.99&-0.51\\2.37&1.99&-1.32&0.07\\-0.85&-1.91&0.26&0.29\\0.89&-1.65&0.84&-0.21\end{pmatrix}&\begin{pmatrix}1.35&1.44&-0.33&1.73\\2.64&-0.43&-0.49&-0.05\\0.88&-0.86&0.94&2.39\\1.15&-0.06&0.27&2.01\end{pmatrix}\\\begin{pmatrix}0.06&1.16&1.54&3.12\\1.40&-0.25&0.94&2.40\\2.10&1.93&2.31&-0.63\\2.51&0.94&3.19&0.10\end{pmatrix}&\begin{pmatrix}2.12&0.19&1.39&2.22\\3.23&1.72&-0.53&2.57\\0.03&0.92&0.82&2.59\\-0.14&0.53&0.40&0.33\end{pmatrix}\end{pmatrix},$$

$$\mathcal{A}(:,:,i,3)=\begin{pmatrix}\begin{pmatrix}0.63&2.38&2.31&2.68\\-0.48&-0.19&0.39&0.48\\0.89&1.74&2.55&-0.58\\0.39&1.30&-1.08&0.52\end{pmatrix}&\begin{pmatrix}1.32&1.84&2.21&-0.11\\-1.12&0.46&1.47&0.71\\2.67&0.54&0.85&2.16\\-0.40&2.57&3.58&0.23\end{pmatrix}\\\begin{pmatrix}2.81&0.32&-0.66&-0.59\\2.26&3.38&0.71&0.17\\0.38&2.53&3.64&-0.09\\1.08&2.31&-1.12&3.33\end{pmatrix}&\begin{pmatrix}0.36&3.34&1.01&0.99\\-0.44&-0.81&0.52&0.55\\2.99&-0.88&-0.28&-0.77\\2.46&3.7-5&1.04&1.05\end{pmatrix}\end{pmatrix},$$

$$\mathcal{A}(:,:,i,4)=\begin{pmatrix}\begin{pmatrix}-0.20&0.77&1.64&2.89\\0.18&-0.77&1.21&-0.89\\1.77&1.06&1.90&1.74\\-1.11&0.92&0.80&0.47\end{pmatrix}&\begin{pmatrix}2.44&-0.44&-0.45&2.82\\-1.66&2.50&2.24&0.50\\2.49&1.21&-0.93&-1.10\\1.92&1.12&0.31&0.53\end{pmatrix}\\\begin{pmatrix}1.07&1.73&1.51&2.70\\1.72&0.83&-0.68&-1.09\\0.18&-1.63&2.78&0.06\\2.21&2.35&2.87&-0.26\end{pmatrix}&\begin{pmatrix}1.38&-0.82&-0.66&0.55\\3.10&0.08&-0.30&1.19\\1.39&-1.17&2.03&-0.99\\2.11&0.10&2.53&1.82\end{pmatrix}\end{pmatrix},$$

the obtained $\mathcal{A}^{-1}$ is

$$\mathcal{A}^{-1}(:,:,i,1)=\begin{pmatrix}\begin{pmatrix}-0.14&-0.12&0.47&-0.40\\0.04&0.10&0.22&0.12\\0.06&-0.23&-0.11&0.07\\-0.05&-0.02&-0.20&0.29\end{pmatrix}&\begin{pmatrix}-0.43&0.67&0.25&0.26\\-0.06&-0.27&0.34&0.07\\-0.64&-0.17&0.30&-0.47\\0.03&0.40&0&0.53\end{pmatrix}\\\begin{pmatrix}0.41&-0.83&-0.29&0.02\\-0.03&0.08&-0.60&-0.14\\0.95&0.03&-0.34&0.42\\-0.26&-0.26&0.38&-0.23\end{pmatrix}&\begin{pmatrix}-0.51&1.65&-0.05&0.94\\-0.40&-0.32&0.83&0.09\\-2.00&0.06&0.93&-1.28\\0.15&0.98&0.18&0.44\end{pmatrix}\end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,2) = \begin{pmatrix} \begin{pmatrix} -0.02 & -0.27 & 0.35 & -0.56 \\ 0.14 & 0.25 & -0.01 & 0.04 \\ 0.50 & -0.02 & -0.26 & 0.32 \\ -0.01 & -0.39 & -0.18 & -0.08 \end{pmatrix} & \begin{pmatrix} 0.32 & -0.46 & -0.26 & -0.15 \\ 0.08 & 0.04 & -0.41 & 0 \\ 0.69 & 0.05 & -0.21 & 0.46 \\ -0.07 & -0.26 & 0.06 & -0.37 \end{pmatrix} \\ \begin{pmatrix} 0.72 & -1.74 & -0.02 & -0.88 \\ 0.52 & 0.32 & -1.06 & -0.24 \\ 1.98 & 0.23 & -0.85 & 1.20 \\ -0.39 & -0.92 & 0.09 & -0.75 \end{pmatrix} & \begin{pmatrix} 0 & -0.03 & -0.23 & 0.15 \\ -0.01 & -0.18 & -0.06 & -0.08 \\ -0.09 & 0 & 0.12 & 0.05 \\ 0.04 & 0.16 & 0.20 & -0.09 \end{pmatrix} \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,3) = \begin{pmatrix} \begin{pmatrix} -0.16 & 0.71 & 0.27 & 0.40 \\ 0 & -0.34 & 0.35 & 0.02 \\ -0.75 & -0.20 & 0.22 & -0.53 \\ 0 & 0.44 & 0.08 & 0.3743 \end{pmatrix} & \begin{pmatrix} -0.16 & 0.71 & 0.27 & 0.40 \\ 0 & -0.34 & 0.35 & 0.02 \\ -0.75 & -0.20 & 0.22 & -0.53 \\ 0 & 0.44 & 0.08 & 0.37 \end{pmatrix} \\ \begin{pmatrix} -0.41 & 1.04 & -0.06 & 0.70 \\ -0.47 & -0.19 & 0.66 & 0.01 \\ -1.30 & 0.03 & 0.71 & -0.79 \\ 0.21 & 0.59 & -0.06 & 0.37 \end{pmatrix} & \begin{pmatrix} 0.63 & -1.62 & -0.09 & -1.03 \\ 0.50 & 0.43 & -0.81 & -0.12 \\ 1.91 & 0.19 & -0.92 & 1.33 \\ -0.21 & -1.07 & -0.17 & -0.62 \end{pmatrix} \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,4) = \begin{pmatrix} \begin{pmatrix} 0.38 & -0.93 & -0.10 & -0.42 \\ 0.24 & 0.27 & -0.64 & 0 \\ 1.07 & 0.11 & -0.55 & 0.79 \\ -0.06 & -0.52 & -0.06 & -0.45 \end{pmatrix} & \begin{pmatrix} -0.27 & 0.67 & -0.09 & 0.25 \\ -0.27 & -0.09 & 0.46 & 0.05 \\ -0.83 & 0.11 & 0.37 & -0.59 \\ 0.25 & 0.46 & -0.06 & 0.19 \end{pmatrix} \\ \begin{pmatrix} -0.29 & 1.08 & -0.26 & 0.81 \\ -0.35 & -0.11 & 0.62 & 0 \\ -1.36 & 0.05 & 0.66 & -0.88 \\ 0.14 & 0.75 & 0.11 & 0.12 \end{pmatrix} & \begin{pmatrix} 0.78 & -1.59 & -0.16 & -0.87 \\ 0.58 & 0.53 & -0.99 & -0.13 \\ 1.88 & 0.13 & -0.87 & 1.19 \\ -0.14 & -1.11 & -0.06 & -0.72 \end{pmatrix} \end{pmatrix}.$$

Also, for the following tensor $\mathcal{A} \in \mathbb{R}^{6,6,6,6}$

$$\mathcal{A}(:,:,i,1) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

,

$$a_{11} = \begin{pmatrix} -0.27 & 2.21 & -0.75 & 1.78 & 1.98 & 3.32 \\ 1 & 2.65 & 1.66 & -0.32 & 0.03 & 3 \\ 0.95 & 3.25 & -0.40 & 2.63 & 1.42 & 2.08 \\ 1.05 & 3.74 & 2.76 & 1.21 & 0.63 & 0.20 \\ 3.33 & 0.27 & 2.96 & 2.34 & 1.17 & 1.82 \\ 1.36 & 0.64 & 2.84 & 3.54 & 1.97 & 2.76 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 1.87 & 0.18 & 0.10 & 1.47 & 1.25 & 0.11 \\ 3.09 & 2.34 & 1.08 & 1.53 & 1.76 & 2.88 \\ 0.74 & 0.40 & 1.70 & -0.44 & 0.74 & 1.77 \\ 2.32 & 0.33 & 1.64 & -0.75 & 3.67 & 1.41 \\ 2.22 & 0.28 & 0.65 & 0.22 & 0.09 & 3.18 \\ -0.41 & 3.28 & 3.71 & -0.83 & 1.11 & 2.37 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 2.93 & 3.45 & 1.61 & 2.85 & 3.10 & 3.11 \\ 0.37 & 1.80 & 1.92 & 3.24 & -0.65 & 1.85 \\ 0.447 & 2.49 & -0.34 & 1.14 & 3 & 0.92 \\ 2.83 & 2.34 & 3.08 & -0.38 & 0.79 & 3.24 \\ 2.26 & -0.01 & 0 & 0.46 & 2.38 & 0.92 \\ -0.51 & 1.20 & 0.65 & 0.20 & 1.27 & 2.74 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -0.04 & 2.06 & 1.26 & 1.94 & -0.55 & 2.49 \\ 2.78 & 3.24 & -0.34 & 2.94 & 0.46 & 1.21 \\ -0.67 & 2.03 & 0.46 & 1.66 & 1.83 & 1.08 \\ -0.22 & 1.90 & 2.45 & 3.10 & 1.90 & 2.98 \\ 2.61 & 0 & 1.03 & 0.20 & 2.61 & 3.69 \\ 1.83 & 1.11 & 2.83 & 2.95 & 3.57 & 0.45 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 2.94 & 2.46 & 2.67 & -0.10 & 1.77 & 1.85 \\ 1.54 & 2.51 & 0.25 & 2.37 & 1.83 & 3.31 \\ 2.78 & 0.32 & 1.31 & 1.27 & 1.06 & 1.41 \\ -0.69 & 2.40 & -0.21 & 0.96 & 2.31 & 1.60 \\ 0.29 & 1.35 & -0.32 & 1.02 & 0.90 & 1.63 \\ -0.58 & 3.56 & 0.38 & 3.86 & 1.03 & -0.20 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 2.44 & 3.27 & 1.64 & 2.78 & -0.19 & 2.29 \\ 1.91 & 0.55 & 1.26 & 0.33 & 1.75 & 0.90 \\ 2.95 & 0.30 & 1.89 & 2.79 & 0.34 & 0.18 \\ 1.93 & 0 & 0.66 & 1.60 & 1.91 & 0.17 \\ 0.32 & 3.09 & 0.82 & 0.86 & 2.45 & 0.25 \\ 1.34 & 2.51 & 0.70 & 2.19 & -0.23 & 0.76 \end{pmatrix},$$

$$\mathcal{A}(:,:,i,2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} 1.31 & 0.85 & 2.61 & 2.49 & 0.88 & 0.26 \\ 0.83 & 1.81 & 2.89 & 0.55 & 2.63 & 0.92 \\ 3.42 & 1.37 & 1.59 & 3.41 & 3.09 & 2.26 \\ 2.57 & 1.78 & 2.75 & 0.41 & 1.27 & 2.75 \\ -0.02 & 0.60 & -0.22 & -0.03 & -0.34 & 2.65 \\ 0.16 & 0.74 & 1.39 & 2.77 & 3.53 & 3.56 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 1.11 & 0.43 & -0.20 & 1.09 & 0.45 & 3.37 \\ 1.80 & 3.37 & 0.39 & 0.16 & 2.23 & 1.03 \\ -0.24 & -0.02 & 0.96 & 1.30 & 2.05 & 1.28 \\ 3.33 & 0.08 & 1.66 & 0.82 & 2.15 & -0.61 \\ 2.10 & 0.32 & 0.14 & 0.97 & 0.86 & 0.84 \\ 3.10 & 1.02 & 1.44 & -0.09 & 3.23 & 0.95 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 3.98 & 2.16 & 2.68 & 0.64 & 2.51 & 3.02 \\ -0.74 & 0.40 & 0.77 & 0.58 & 0.46 & 1.03 \\ 1.99 & 2.33 & 3.78 & 0.65 & -0.73 & 2.25 \\ 1.48 & 2.09 & 0.30 & 0.58 & 1.51 & 0.55 \\ 1.86 & 1.99 & 1.41 & 2.06 & 3.37 & 0.06 \\ 2.26 & -0.58 & -0.07 & 1.56 & -0.67 & 0.04 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 0.59 & 2.40 & 0.43 & 2.56 & 0.15 & -0.04 \\ 0.21 & 0.54 & 1.86 & 3.42 & 1.97 & 0.37 \\ 2.80 & -0.37 & 1.67 & 2.84 & 3.54 & 0.99 \\ 0.23 & 2.40 & 0.32 & 1.51 & 0.02 & 1.56 \\ 2.45 & -0.63 & 0.24 & 1.41 & 0.44 & 3.34 \\ 0.57 & 2.68 & 1.67 & 0.09 & 1.63 & 1.96 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 1.10 & 2.29 & 2.89 & 3.51 & 2.69 & 0.95 \\ 1.82 & 2.65 & 0.89 & 0.82 & -0.16 & 3.19 \\ 1.06 & 2.64 & -0.78 & -0.62 & 3.54 & 2.09 \\ 2.15 & 0.96 & -0.32 & 2.61 & 3.81 & 2.41 \\ 1.36 & 3.31 & 2.15 & 0.54 & 2.15 & 0.52 \\ -0.04 & 3.23 & 3.35 & 1.76 & 1.92 & 3.63 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 1.96 & 0.90 & 1.25 & 0.75 & 0.88 & 3.41 \\ 1.70 & 1.07 & 0.76 & 1.29 & 3.30 & 3.22 \\ 1.98 & 1 & -0.13 & 3.78 & 3.59 & 0.31 \\ -0.35 & 1.69 & 1.70 & 2.99 & -0.32 & 3.14 \\ 2.87 & 1.25 & 1.57 & 1.22 & 0.62 & 1.53 \\ 3.09 & 1.69 & 1.67 & 0.86 & 1.51 & 3.64 \end{pmatrix},$$

$$\mathcal{A}(:,:,i,3) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -0.20 & -0.06 & 2.89 & 1.53 & 1.67 & -0.22 \\ 1.59 & 1.55 & 2.39 & 1.04 & 2.08 & 1.39 \\ 2.78 & 1.72 & 1.12 & 1.04 & 2 & 0.27 \\ 2.94 & 2.43 & 0.28 & 2.16 & 2.05 & 0.69 \\ 2.65 & 1.40 & 2.69 & 2.67 & 3.51 & 0.19 \\ 1.15 & 2.30 & 2.93 & 1.73 & 1.91 & 1.69 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} -0.56 & 1.43 & 3.42 & 2.18 & 1.22 & 2.69 \\ -0.04 & 2.41 & 2.24 & 0.36 & 2.29 & 1.73 \\ 3.25 & 2.52 & 0.51 & 3.45 & -0.16 & 1.56 \\ -0.33 & 0.50 & 1.48 & 0.09 & 2.76 & 1.07 \\ 3.29 & 3.29 & 1.68 & 1.15 & 0.63 & 1.65 \\ 0.12 & 0.33 & 1.86 & 1.20 & 1 & 2.53 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 0.32 & 3.02 & 0.51 & 2.62 & 2.73 & 3.91 \\ 2.50 & 0.39 & 1.16 & -0.14 & -0.55 & 1.20 \\ 1.14 & 2.02 & -0.53 & 0.34 & 0.51 & 2.94 \\ 1.08 & 0.04 & 0.42 & -0.12 & 2.20 & 1.25 \\ 1.61 & -0.34 & -0.01 & 2.70 & 0.40 & 2.40 \\ -0.11 & 0.04 & 3.44 & 1.26 & -0.07 & 2.15 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 1.80 & 0.29 & 3.48 & 2.15 & 3.76 & 0.32 \\ 2.61 & 3.22 & 2.32 & -0.03 & 0.27 & 2.95 \\ 2.84 & 2.94 & 1.28 & 2.68 & 3.87 & 1.05 \\ 0.39 & 1.79 & 0.60 & 2.75 & 1.69 & 1.04 \\ 2.61 & 2.33 & 1.47 & 0.20 & 0.01 & 1.21 \\ 0.71 & 1.77 & 3.01 & 3.61 & 2.66 & 2.46 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 2.98 & 0.10 & 3.52 & 2.02 & 0.79 & 2.76 \\ 2.52 & 3.27 & -0.34 & 3.32 & 3.46 & 0.70 \\ 1.49 & 2.96 & 1.65 & 1.31 & 0.30 & -0.06 \\ -0.65 & 0.73 & 1.51 & 0.57 & 1.55 & 0.06 \\ 2.37 & -0.17 & 1.06 & -0.14 & 2.90 & 3.87 \\ 1.22 & 2.55 & 2.43 & 3.41 & 3.02 & -0.37 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 3.36 & 0.79 & 1.92 & 0.67 & 3.57 & 2.98 \\ 0.20 & 1.50 & 2.03 & 2.46 & 1.03 & 2.41 \\ 1.48 & 1.28 & 2.17 & 0.88 & 2.16 & 3.38 \\ 2.82 & 1.40 & 1 & 1.36 & 1.73 & 0.21 \\ -0.14 & -0.29 & 1.86 & 1.53 & 1.57 & 1.29 \\ 2.27 & -0.61 & 3.33 & 3.40 & 1.75 & 2.07 \end{pmatrix},$$

$$\mathcal{A}(:,:,i,4) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} 0.04 & 0.83 & 3.24 & 1.04 & 0.20 & 2.47 \\ 3.49 & 2.73 & 0.41 & 2.13 & 1.18 & 2.60 \\ 3.15 & 0.67 & 1.11 & 0.89 & -0.43 & 3.51 \\ 1.07 & -0.36 & 1.39 & 2.80 & 3.50 & 3.02 \\ 0.36 & 2.88 & -0.02 & 0.74 & 1.97 & 2.28 \\ 0.68 & 2.46 & -0.17 & 1.48 & 1.15 & 2.11 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 2.79 & 1.91 & 0.21 & 1.51 & 1.55 & 0.62 \\ 0.80 & 0.67 & 0.65 & 2.47 & 2.34 & 0.84 \\ 0.26 & 0.87 & -0.17 & 3.16 & 0.94 & 0.60 \\ 2.80 & 1.70 & 2.24 & 1.96 & 1.88 & -0.13 \\ 3.29 & 0.92 & 2.35 & 2.39 & 2.37 & 1.38 \\ 0.44 & 2.72 & 3.15 & 1.65 & 0.31 & 0.10 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} -0.75 & 2.20 & 1.48 & 0.31 & 3.56 & 1.23 \\ 1.94 & 2.65 & 0.21 & 0.38 & 0.70 & 2.31 \\ 0.36 & 3.04 & 3.63 & 3.39 & 2.08 & 1.35 \\ 0.51 & 0.20 & -0.63 & 1.84 & 0.23 & 2.93 \\ 1.81 & -0.20 & -0.45 & 0.37 & 1.61 & 2.97 \\ 0.54 & 0.29 & 1.35 & 3.08 & 1.93 & 0.66 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 3.07 & 1.11 & 2.93 & 0.70 & 2.08 & 0.33 \\ 0.82 & 2.31 & 1.47 & 0 & 0.51 & 1.94 \\ 0.81 & 2.92 & 1.62 & 3.15 & 2.06 & 3.13 \\ 1.17 & 3.57 & -0.07 & 1.25 & -0.42 & 1.37 \\ 2.80 & 0.15 & 0.34 & -0.02 & 1.27 & 2.64 \\ 3.71 & 0.86 & -0.18 & 1.12 & 1.28 & 1.94 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 0.091 & 2.36 & 1.18 & 1.23 & 2.46 & 1.37 \\ 3.51 & 2.59 & 3.66 & 2.65 & 2.25 & 0.06 \\ 2.23 & 0.55 & 1.86 & 1.99 & 2.98 & -0.11 \\ 3.19 & 1.57 & -0.05 & 2.65 & 1.19 & 2.15 \\ 3.24 & 0.93 & 1.52 & 3.13 & 3.05 & -0.72 \\ 1.95 & -0.28 & 0.15 & 2.97 & 0.67 & 1.83 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 0.091 & 2.36 & 1.18 & 1.23 & 2.46 & 1.37 \\ 3.51 & 2.59 & 3.66 & 2.65 & 2.25 & 0.06 \\ 2.23 & 0.55 & 1.86 & 1.99 & 2.98 & -0.11 \\ 3.19 & 1.57 & -0.05 & 2.65 & 1.19 & 2.15 \\ 3.24 & 0.93 & 1.52 & 3.13 & 3.05 & -0.72 \\ 1.95 & -0.28 & 0.15 & 2.97 & 0.67 & 1.83 \end{pmatrix},$$

$$\mathcal{A}(:,:,i,5) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -0.21 & 2.31 & 3.28 & 3.20 & 1.77 & 1.61 \\ 2.31 & 3.02 & 1.47 & 2.24 & 3.04 & 0.12 \\ 3.56 & -0.20 & 0.06 & 0.23 & 3.17 & 0.93 \\ 2.07 & 0.28 & 3.31 & 1.97 & 2.35 & 3.30 \\ 0.98 & 2.70 & 1.23 & 2.95 & 3.22 & 0.57 \\ 3.45 & 2.93 & 0.66 & 1.60 & 2.00 & 1.43 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 0.29 & 0.97 & 1.48 & 0.92 & 2.12 & 0.17 \\ 0.08 & 3.62 & -0.51 & -0.01 & 1.83 & 1.67 \\ 2.56 & 0.79 & 2.81 & -0.36 & 0.94 & 2.65 \\ 2.96 & -0.67 & 0.31 & 1.76 & 2.01 & 0.99 \\ 2.41 & 0.52 & 1.68 & 2.13 & 0.49 & 2.84 \\ 3.10 & 0.15 & 2.25 & 2.59 & 2.72 & 2.17 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} -0.01 & -0.60 & 1.42 & 3.50 & 1.35 & -0.14 \\ 1.82 & 2.21 & 1.03 & 1.13 & 0.85 & 3.12 \\ 1.97 & -0.13 & 1.57 & 2.55 & 1.92 & -0.29 \\ 2.77 & 3.13 & 0.48 & 0.43 & 0.40 & 2.89 \\ 2.96 & 0.84 & 0.22 & 2.35 & 1.93 & -0.01 \\ 0.97 & 2.47 & -0.42 & 0.48 & 0.57 & 2.09 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 0.48 & 0.51 & 3.35 & 1.50 & 3.81 & 1.58 \\ 1.63 & 2.72 & 1.22 & 1.16 & 2.08 & 0.01 \\ -0.30 & -0.22 & 0.47 & 1.96 & 0.05 & 0.55 \\ 2.52 & 3.45 & 0.49 & 0.01 & -0.03 & 0.47 \\ 0.64 & 1.67 & 2.85 & 3.07 & 0.82 & 3.57 \\ 0.22 & 0.47 & 3.03 & 0.38 & 2.06 & 2.28 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 3.57 & 0.74 & 0.11 & 0.22 & 1.79 & 1.67 \\ 2.78 & 0.90 & 0.66 & 1.61 & 0.75 & -0.61 \\ 1.02 & 1.59 & -0.27 & 3.36 & 0.68 & 0.68 \\ 3.08 & 2.34 & 0.04 & 2.07 & 3.70 & 1.96 \\ 0.17 & 0.68 & 1.61 & 2.83 & 2.93 & 1.46 \\ 0.20 & 1.36 & 3.52 & 1.45 & 1.86 & 2.63 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 2.29 & 1.64 & 2.07 & 1.86 & -0.79 & -0.17 \\ 1.74 & 2.68 & 1.76 & 2.01 & 3.50 & 3.59 \\ 3.10 & 0.35 & 1.17 & -0.05 & 2.14 & 1.05 \\ 1.63 & 2.45 & 1.70 & 2.11 & 2.68 & 2.30 \\ 2.66 & 2.63 & 1.40 & -0.33 & 3.47 & 1.68 \\ 3.28 & 2.35 & 2.09 & 1.06 & -0.29 & 3.03 \end{pmatrix},$$

$$\mathcal{A}(:,:,i,6) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} 2.17 & 2.58 & 2.46 & 1.76 & 1.19 & 1.33 \\ 2.41 & 0.71 & 2.55 & 2.27 & 2.94 & 3.49 \\ 1.77 & 1.92 & 0.19 & 0.28 & 2.29 & 3.38 \\ 1.80 & 3.51 & 2.36 & -0.42 & 2.91 & -0.08 \\ -0.13 & 2.87 & 1.33 & -0.06 & 2.48 & 1.82 \\ 3.45 & 1.96 & 2.87 & -0.73 & 2.46 & 3.04 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 2.26 & 2.35 & -0.30 & -0.06 & 2.82 & 1.09 \\ 0.33 & 0.62 & -0.64 & 2.83 & 1.06 & 1.53 \\ 0.68 & 2.98 & -0.05 & 1.70 & 3.22 & 0.66 \\ 0.34 & 2.70 & 2.22 & 2.77 & 2.03 & 0.29 \\ 1.07 & 1.22 & 2.73 & 3.18 & 2.47 & 3.00 \\ 1.47 & 0.56 & 1.26 & 3.11 & 1.27 & 2.19 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 0.93 & 1.37 & 3.51 & 2.95 & 3.12 & 0.62 \\ 0.53 & 0.65 & 1.04 & 2.11 & 3.22 & 2.89 \\ 3.23 & 1.97 & 3.52 & 0.62 & 0.37 & 3.66 \\ 0.10 & 3.42 & 0.56 & -0.13 & 2.82 & 2.04 \\ 1.18 & 3.36 & 1.13 & 0.92 & 0.93 & -0.02 \\ 0.89 & 2.54 & 2.76 & 1.48 & 0.76 & 0.62 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -0.04 & 3.64 & 2.82 & 0.42 & 0.55 & 2.36 \\ 2.20 & 2.55 & 0.15 & 1.61 & 0.54 & 1.96 \\ 3.09 & 2.39 & -0.18 & -0.56 & 0.10 & 1.25 \\ -0.32 & 1.11 & 0.53 & 1.20 & -0.46 & 0.43 \\ 2.67 & -0.05 & 0.12 & 2.72 & 1.20 & -0.26 \\ -0.17 & 1.07 & 0.35 & -0.04 & 1.53 & 0 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 3.20 & 2.96 & 3.38 & 3.59 & 3.24 & 3.06 \\ 1.79 & -0.33 & 1.12 & 3.38 & 2.80 & 1.53 \\ -0.18 & 3.89 & 0.32 & 1.82 & 1.41 & 1.86 \\ 1.22 & 1.77 & 1.54 & 0.82 & 2.11 & 2.24 \\ 1.34 & 1.56 & 0.84 & 1.79 & 0.73 & 2.77 \\ 0.29 & 3.28 & 1.23 & 2.29 & -0.06 & 1.63 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 1.93 & 2.60 & 0.41 & 0.58 & -0.39 & 2.24 \\ 2.02 & 1.94 & 3.25 & 0.25 & 0.20 & 2.14 \\ 3.28 & 1.66 & 0.57 & 0.71 & 0.13 & 1.81 \\ 0.53 & 2.35 & 1.52 & 0.70 & 3.28 & 2.43 \\ -0.12 & 2.10 & 2.17 & 3.42 & 1.34 & 0.85 \\ 2.15 & -0.17 & 2.60 & 0.27 & 3.43 & 0 \end{pmatrix},$$

by applying the mentioned methods, we obtain

$$\mathcal{A}^{-1}(:,:,i,1) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} 0.08 & 0.09 & -0.32 & -0.03 & 0.07 & 0.16 \\ -0.02 & -0.06 & -0.05 & 0.16 & 0 & -0.20 \\ -0.20 & 0.36 & -0.09 & 0.05 & 0.10 & -0.24 \\ -0.07 & 0.11 & 0.13 & -0.20 & -0.03 & 0.04 \\ 0 & 0.13 & 0.06 & 0 & 0.11 & 0.08 \\ -0.26 & 0 & -0.02 & 0 & 0.04 & 0 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} -0.32 & 0.06 & 0.33 & 0.05 & -0.22 & -0.37 \\ 0.28 & 0 & -0.07 & -0.12 & -0.10 & 0.54 \\ 0.25 & -0.60 & 0.30 & -0.31 & -0.13 & 0.49 \\ 0.11 & -0.40 & -0.30 & 0.60 & 0.08 & -0.02 \\ -0.07 & -0.04 & 0.10 & 0.22 & -0.33 & -0.37 \\ 0.58 & 0.15 & 0 & -0.06 & -0.23 & 0.10 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} -0.69 & -0.01 & 1.00 & 0.22 & -0.56 & -1.06 \\ 0.25 & 0.18 & -0.07 & -0.32 & -0.02 & 1.23 \\ 0.95 & -1.54 & 0.66 & -0.65 & -0.43 & 1.22 \\ 0 & -0.68 & -0.65 & 1.31 & 0.17 & -0.18 \\ -0.12 & -0.25 & 0.24 & 0.21 & -0.70 & -0.77 \\ 1.27 & 0.25 & 0.01 & -0.16 & -0.24 & 0.36 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 0.97 & -0.26 & -1.08 & -0.14 & 0.64 & 1.58 \\ -0.86 & -0.10 & -0.04 & 0.37 & 0.35 & -1.98 \\ -0.84 & 2 & -0.82 & 1.08 & 0.40 & -1.62 \\ -0.22 & 1.14 & 1.23 & -2.14 & -0.23 & 0.12 \\ 0.22 & -0.02 & -0.37 & -0.49 & 1.15 & 1.12 \\ -1.78 & -0.36 & -0.17 & 0.49 & 0.60 & -0.53 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 0.62 & -0.09 & -0.80 & -0.11 & 0.41 & 1.10 \\ -0.43 & -0.18 & 0.01 & 0.38 & 0.24 & -1.27 \\ -0.65 & 1.49 & -0.46 & 0.56 & 0.31 & -1.09 \\ -0.06 & 0.73 & 0.79 & -1.34 & -0.21 & 0.05 \\ 0.19 & 0.09 & -0.16 & -0.16 & 0.65 & 0.61 \\ -1.28 & -0.13 & -0.25 & 0.23 & 0.21 & -0.37 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 0.38 & -0.25 & -0.44 & -0.13 & 0.36 & 0.66 \\ -0.28 & -0.10 & 0.05 & 0.03 & 0.23 & -0.76 \\ -0.34 & 0.67 & -0.35 & 0.35 & 0.19 & -0.62 \\ 0 & 0.42 & 0.51 & -0.70 & -0.15 & 0.02 \\ 0.08 & -0.12 & -0.17 & -0.20 & 0.51 & 0.49 \\ -0.61 & -0.14 & 0 & 0.14 & 0.20 & -0.19 \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -1.47 & 0.24 & 1.85 & 0.17 & -1.09 & -2.21 \\ 1.04 & 0.24 & 0 & -0.59 & -0.22 & 2.73 \\ 1.57 & -3.08 & 1.20 & -1.48 & -0.76 & 2.38 \\ 0.38 & -1.68 & -1.74 & 3.02 & 0.39 & -0.19 \\ -0.22 & -0.18 & 0.32 & 0.64 & -1.62 & -1.51 \\ 2.79 & 0.49 & 0.17 & -0.56 & -0.77 & 0.72 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 0.70 & -0.13 & -0.98 & -0.16 & 0.58 & 0.79 \\ -0.37 & -0.03 & 0.09 & 0.18 & -0.01 & -1.14 \\ -0.73 & 1.22 & -0.64 & 0.71 & 0.30 & -1.00 \\ -0.21 & 0.73 & 0.70 & -1.20 & -0.11 & 0.24 \\ 0.06 & 0.14 & -0.16 & -0.30 & 0.77 & 0.65 \\ -1.15 & -0.36 & 0.06 & 0.23 & 0.49 & -0.29 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 1.62 & -0.32 & -1.96 & -0.31 & 1.17 & 2.36 \\ -1.11 & -0.25 & 0.04 & 0.50 & 0.29 & -2.91 \\ -1.62 & 3.18 & -1.39 & 1.70 & 0.70 & -2.53 \\ -0.41 & 1.79 & 1.85 & -3.21 & -0.46 & 0.36 \\ 0.18 & 0.14 & -0.43 & -0.73 & 1.84 & 1.79 \\ -2.86 & -0.62 & -0.18 & 0.63 & 0.92 & -0.76 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -0.95 & 0.12 & 1.38 & 0.19 & -0.82 & -1.57 \\ 0.55 & 0.32 & -0.08 & -0.50 & -0.21 & 1.86 \\ 1.14 & -2.10 & 0.86 & -1.04 & -0.53 & 1.74 \\ 0.24 & -1.09 & -1.07 & 2.00 & 0.33 & -0.20 \\ -0.13 & -0.25 & 0.20 & 0.36 & -1.04 & -0.99 \\ 1.80 & 0.29 & 0.04 & -0.34 & -0.36 & 0.48 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 4.74 & -0.78 & -6.16 & -0.82 & 3.51 & 7.43 \\ -3.29 & -0.99 & 0.04 & 1.98 & 1.00 & -8.85 \\ -4.99 & 9.98 & -3.97 & 4.82 & 2.28 & -7.75 \\ -1.02 & 5.38 & 5.57 & -9.77 & -1.20 & 0.77 \\ 0.91 & 0.65 & -1.31 & -1.95 & 5.22 & 4.89 \\ -8.69 & -1.62 & -0.77 & 1.79 & 2.40 & -2.38 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 1.28 & -0.26 & -1.37 & -0.22 & 0.98 & 1.75 \\ -0.63 & -0.27 & 0 & 0.38 & 0.23 & -2.20 \\ -1.21 & 2.38 & -1.08 & 1.15 & 0.50 & -1.90 \\ -0.25 & 1.35 & 1.29 & -2.27 & -0.35 & 0.20 \\ 0.28 & 0.13 & -0.36 & -0.54 & 1.33 & 1.27 \\ -2.02 & -0.39 & -0.14 & 0.40 & 0.49 & -0.62 \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,3) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -0.85 & 0.14 & 1.04 & 0 & -0.47 & -1.08 \\ 0.58 & 0.09 & 0.12 & -0.36 & -0.17 & 1.31 \\ 0.68 & -1.44 & 0.58 & -0.78 & -0.31 & 1.08 \\ 0.36 & -0.89 & -0.83 & 1.52 & 0.20 & -0.02 \\ -0.06 & -0.13 & 0.05 & 0.23 & -0.80 & -0.70 \\ 1.37 & 0.26 & 0.21 & -0.25 & -0.43 & 0.25 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 0.29 & 0 & -0.25 & 0.04 & 0.15 & 0.29 \\ -0.10 & -0.06 & 0 & 0.13 & -0.01 & -0.42 \\ -0.24 & 0.43 & -0.20 & 0.20 & 0.07 & -0.43 \\ -0.14 & 0.28 & 0.24 & -0.38 & -0.05 & -0.02 \\ 0.01 & 0.04 & -0.01 & 0 & 0.14 & 0.30 \\ -0.44 & -0.13 & -0.01 & -0.01 & 0.13 & 0.03 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 0.38 & -0.01 & -0.47 & -0.14 & 0.31 & 0.51 \\ -0.01 & -0.08 & 0.02 & 0 & 0 & -0.63 \\ -0.50 & 0.90 & -0.43 & 0.38 & 0.19 & -0.59 \\ 0.03 & 0.44 & 0.40 & -0.76 & -0.10 & 0.15 \\ 0.01 & 0.04 & -0.18 & -0.15 & 0.40 & 0.40 \\ -0.56 & -0.14 & 0.03 & 0.11 & 0.12 & -0.24 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -0.46 & 0.17 & 0.62 & 0.08 & -0.26 & -0.80 \\ 0.48 & 0.07 & 0 & -0.17 & -0.17 & 1.00 \\ 0.51 & -0.98 & 0.37 & -0.54 & -0.24 & 0.80 \\ 0.18 & -0.64 & -0.52 & 1.01 & 0.12 & -0.01 \\ -0.14 & -0.16 & 0.06 & 0.14 & -0.58 & -0.52 \\ 0.93 & 0.15 & 0.10 & -0.22 & -0.26 & 0.21 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} -2.72 & 0.37 & 3.59 & 0.47 & -2.04 & -4.27 \\ 1.95 & 0.48 & -0.01 & -1.14 & -0.58 & 5.09 \\ 2.92 & -5.74 & 2.21 & -2.78 & -1.43 & 4.52 \\ 0.59 & -3.10 & -3.29 & 5.65 & 0.76 & -0.54 \\ -0.53 & -0.24 & 0.78 & 1.20 & -3.11 & -2.94 \\ 5.11 & 1.07 & 0.43 & -0.90 & -1.54 & 1.38 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 2.34 & -0.42 & -3.01 & -0.55 & 1.82 & 3.63 \\ -1.56 & -0.51 & 0.07 & 0.93 & 0.46 & -4.46 \\ -2.51 & 4.89 & -1.96 & 2.43 & 1.11 & -3.83 \\ -0.41 & 2.66 & 2.78 & -4.85 & -0.61 & 0.49 \\ 0.42 & 0.24 & -0.73 & -1.08 & 2.71 & 2.44 \\ -4.28 & -0.82 & -0.21 & 0.88 & 1.28 & -1.23 \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,1,4) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -2.08 & 0.39 & 2.52 & 0.44 & -1.61 & -3.29 \\ 1.30 & 0.47 & -0.09 & -0.76 & -0.45 & 3.93 \\ 2.16. & -4.39 & 1.82 & -2.15 & -0.87 & 3.47 \\ 0.35 & -2.32 & -2.41 & 4.25 & 0.57 & -0.39 \\ -0.54 & -0.15 & 0.73 & 0.88 & -2.38 & -2.15 \\ 3.81 & 0.65 & 0.27 & -0.82 & -1.00 & 1.14 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 1.86 & -0.45 & -2.47 & -0.33 & 1.39 & 3.00 \\ -1.31 & -0.44 & 0.10 & 0.72 & 0.46 & -3.55 \\ -1.91 & 3.84 & -1.68 & 1.89 & 0.94 & -3.07 \\ -0.34 & 2.20 & 2.22 & -3.91 & -0.53 & 0.36 \\ 0.34 & 0.15 & -0.54 & -0.75 & 2.16 & 2.02 \\ -3.44 & -0.69 & -0.14 & 0.73 & 0.94 & -0.98 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} -0.49 & 0.04 & 0.52 & 0.02 & -0.29 & -0.68 \\ 0.33 & 0.06 & 0.13 & -0.14 & -0.15 & 0.84 \\ 0.43 & -1.01 & 0.33 & -0.40 & -0.15 & 0.73 \\ 0.13 & -0.52 & -0.55 & 0.99 & 0.09 & -0.05 \\ -0.08 & -0.07 & 0.07 & 0.15 & -0.43 & -0.51 \\ 0.90 & 0.15 & 0.16 & -0.14 & -0.30 & 0.19 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -2.73 & 0.46 & 3.56 & 0.73 & -2.17 & -4.43 \\ 1.70 & 0.72 & -0.18 & -1.05 & -0.56 & 5.20 \\ 2.95 & -5.81 & 2.31 & -2.79 & -1.40 & 4.57 \\ 0.47 & -3.05 & -3.06 & 5.59 & 0.78 & -0.47 \\ -0.65 & -0.40 & 0.76 & 1.11 & -3.07 & -2.76 \\ 5.00 & 0.89 & 0.33 & -1.11 & -1.20 & 1.49 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 2.27 & -0.24 & -3.06 & -0.53 & 1.83 & 3.51 \\ -1.38 & -0.53 & 0.11 & 0.95 & 0.36 & -4.17 \\ -2.70 & 4.91 & -1.95 & 2.29 & 1.29 & -3.88 \\ -0.43 & 2.57 & 2.56 & -4.65 & -0.62 & 0.56 \\ 0.42 & 0.39 & -0.68 & -0.96 & 2.53 & 2.43 \\ -4.24 & -0.83 & -0.19 & 0.79 & 1.08 & -1.13 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 1.89 & -0.23 & -2.36 & -0.40 & 1.38 & 2.90 \\ -1.22 & -0.50 & 0.03 & 0.75 & 0.45 & -3.45 \\ -1.99 & 3.86 & -1.51 & 1.78 & 0.92 & -3.05 \\ -0.28 & 2.00 & 2.09 & -3.75 & -0.53 & 0.27 \\ 0.47 & 0.25 & -0.50 & -0.68 & 2.01 & 1.90 \\ -3.33 & -0.62 & -0.25 & 0.67 & 0.84 & -0.98 \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,5) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} -2.56 & 0.33 & 3.31 & 0.62 & -1.95 & -3.99 \\ 1.62 & 0.56 & -0.14 & -0.96 & -0.42 & 4.83 \\ 2.90 & -5.47 & 2.23 & -2.59 & -1.30 & 4.32 \\ 0.36 & -2.92 & -2.87 & 5.26 & 0.72 & -0.58 \\ -0.51 & -0.41 & 0.81 & 1.10 & -2.87 & -2.70 \\ 4.66 & 0.89 & 0.23 & -1.01 & -1.30 & 1.36 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} -1.51 & 0.38 & 1.88 & 0.31 & -1.19 & -2.40 \\ 0.94 & 0.41 & -0.08 & -0.50 & -0.36 & 2.86 \\ 1.53 & -3.18 & 1.30 & -1.48 & -0.76 & 2.55 \\ 0.15 & -1.74 & -1.86 & 3.07 & 0.43 & -0.26 \\ -0.32 & -0.10 & 0.54 & 0.70 & -1.73 & -1.56 \\ 2.66 & 0.55 & 0.13 & -0.60 & -0.65 & 0.81 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} 1.33 & -0.20 & -1.78 & -0.40 & 1.15 & 2.20 \\ -0.92 & -0.20 & 0.06 & 0.51 & 0.10 & -2.62 \\ -1.59 & 3.08 & -1.15 & 1.53 & 0.66 & -2.38 \\ -0.24 & 1.67 & 1.64 & -2.94 & -0.39 & 0.27 \\ 0.32 & 0.27 & -0.47 & -0.63 & 1.53 & 1.41 \\ -2.64 & -0.46 & -0.11 & 0.65 & 0.72 & -0.80 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} -3.46 & 0.49 & 4.43 & 0.58 & -2.54 & -5.24 \\ 2.30 & 0.80 & 0.07 & -1.45 & -0.67 & 6.35 \\ 3.58 & -7.08 & 2.85 & -3.47 & -1.63 & 5.53 \\ 0.80 & -3.83 & -3.87 & 6.89 & 0.83 & -0.66 \\ -0.56 & -0.51 & 0.84 & 1.39 & -3.68 & -3.44 \\ 6.17 & 1.09 & 0.45 & -1.25 & -1.69 & 1.69 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} -0.05 & 0.13 & 0.03 & 0.06 & -0.01 & -0.06 \\ 0.05 & -0.05 & -0.03 & 0.12 & -0.14 & 0.18 \\ -0.01 & -0.02 & 0.12 & -0.01 & 0.04 & 0.05 \\ -0.06 & -0.10 & -0.23 & 0.17 & 0 & -0.04 \\ -0.05 & 0.17 & 0.18 & 0.07 & -0.14 & -0.25 \\ 0.05 & 0.02 & -0.02 & -0.09 & -0.07 & 0.05 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} -0.65 & 0.22 & 0.79 & 0.19 & -0.58 & -0.84 \\ 0.40 & 0.07 & -0.09 & -0.13 & 0.031 & 1.19 \\ 0.70 & -1.28 & 0.52 & -0.74 & -0.25 & 1.02 \\ 0.13 & -0.74 & -0.77 & 1.28 & 0.14 & -0.17 \\ -0.09 & 0.02 & 0.31 & 0.37 & -0.78 & -0.74 \\ 1.12 & 0.30 & -0.08 & -0.33 & -0.56 & 0.41 \end{pmatrix},$$

$$\mathcal{A}^{-1}(:,:,i,6) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

$$a_{11} = \begin{pmatrix} 0.25 & -0.13 & -0.14 & 0.01 & 0.11 & 0.28 \\ -0.23 & 0.09 & -0.02 & -0.03 & 0.01 & -0.45 \\ -0.10 & 0.46 & -0.13 & 0.23 & 0 & -0.31 \\ -0.06 & 0.29 & 0.27 & -0.51 & -0.02 & -0.02 \\ 0.08 & -0.03 & -0.06 & -0.14 & 0.24 & 0.25 \\ -0.37 & -0.01 & -0.03 & 0.13 & 0.14 & -0.12 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} -1.09 & 0.17 & 1.34 & 0.22 & -0.88 & -1.55 \\ 0.66 & 0.19 & -0.01 & -0.33 & -0.18 & 1.97 \\ 1.10 & -2.22 & 0.89 & -1.01 & -0.36 & 1.65 \\ 0.18 & -1.24 & -1.22 & 2.08 & 0.26 & -0.18 \\ -0.17 & -0.11 & 0.30 & 0.40 & -1.26 & -1.07 \\ 1.89 & 0.39 & 0.19 & -0.41 & -0.51 & 0.60 \end{pmatrix},$$

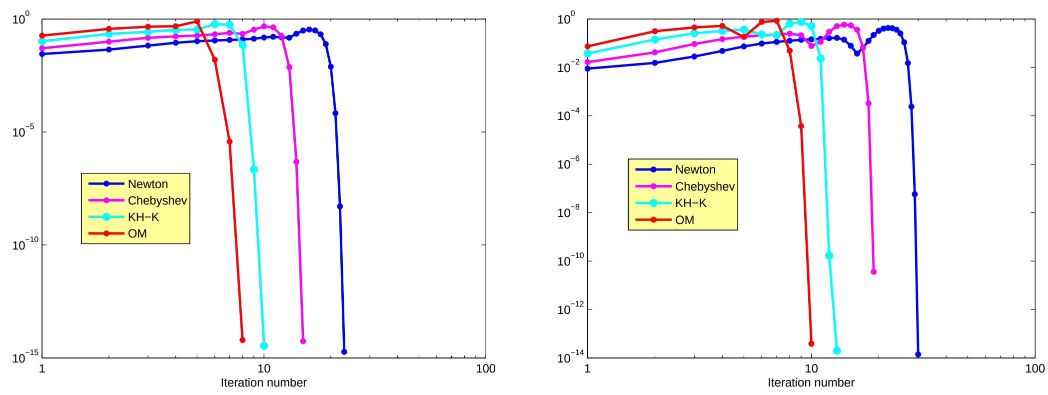
$$a_{21} = \begin{pmatrix} 1.01 & -0.01 & -1.40 & -0.11 & 0.78 & 1.51 \\ -0.59 & -0.24 & 0 & 0.56 & 0.06 & -1.82 \\ -1.20 & 2.13 & -0.77 & 0.97 & 0.46 & -1.55 \\ -0.23 & 1.11 & 0.97 & -1.87 & -0.28 & 0.20 \\ 0.15 & 0.34 & -0.21 & -0.33 & 1.00 & 0.86 \\ -1.88 & -0.31 & -0.08 & 0.37 & 0.38 & -0.52 \end{pmatrix},$$

$$a_{22} = \begin{pmatrix} 0.31 & -0.05 & -0.40 & -0.07 & 0.27 & 0.46 \\ -0.23 & -0.13 & -0.02 & 0.13 & 0.11 & -0.70 \\ -0.27 & 0.68 & -0.27 & 0.38 & 0.12 & -0.49 \\ -0.04 & 0.34 & 0.32 & -0.67 & -0.09 & 0 \\ 0.08 & 0.05 & -0.10 & -0.13 & 0.39 & 0.34 \\ -0.65 & -0.02 & -0.09 & 0.15 & 0.17 & -0.11 \end{pmatrix},$$

$$a_{31} = \begin{pmatrix} 0.17 & -0.08 & -0.16 & 0.07 & 0.07 & 0.33 \\ -0.26 & 0 & -0.06 & 0.09 & 0.09 & -0.37 \\ -0.12 & 0.42 & -0.06 & 0.27 & 0.03 & -0.39 \\ -0.11 & 0.32 & 0.30 & -0.47 & 0.02 & -0.10 \\ 0.06 & -0.04 & 0 & -0.15 & 0.15 & 0.23 \\ -0.42 & -0.09 & -0.14 & 0.12 & 0.20 & -0.04 \end{pmatrix},$$

$$a_{32} = \begin{pmatrix} 0.49 & 0.04 & -0.64 & -0.13 & 0.34 & 0.67 \\ -0.05 & -0.22 & 0.09 & 0.05 & 0.11 & -0.69 \\ -0.56 & 0.92 & -0.44 & 0.29 & 0.28 & -0.75 \\ -0.01 & 0.42 & 0.38 & -0.78 & -0.13 & 0.17 \\ 0.04 & 0.13 & -0.14 & -0.09 & 0.54 & 0.43 \\ -0.68 & -0.11 & 0 & 0.04 & 0.10 & -0.32 \end{pmatrix}.$$

The results of Example 3.1 are presented in Fig. 1.



(a) (b)

Figure 1: (a) and (b), respectively, show the Res of the results of the iterative methods for $\mathcal{A} \in \mathbb{R}^{4,4,4,4}$ and $\mathcal{A} \in \mathbb{R}^{6,6,6,6}$ of Example (3.1).

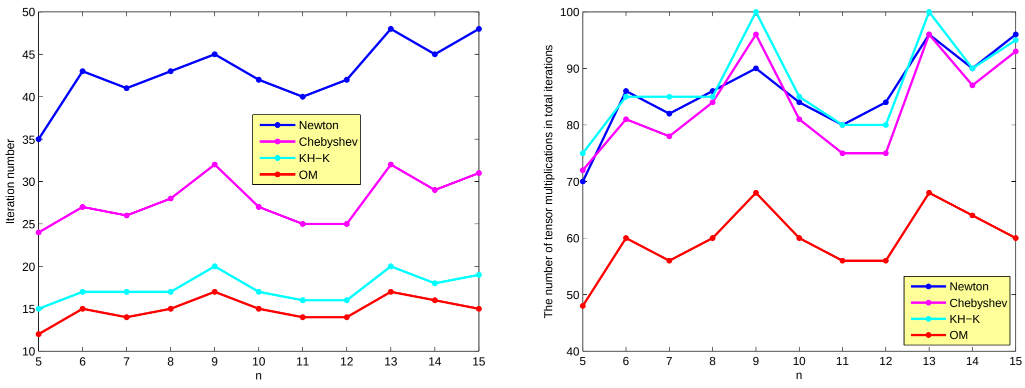
Example 3.2. In this example, we consider the random tensor $\mathcal{A} \in \mathbb{R}^{n,n,n,n,n,n}$ generated in Matlab by

$$\mathcal{A} = \texttt{tenrand}(n, n, n, n, n, n).$$

The results of Example 3.2 are displayed in Fig. 2 and Table 1.

Table 1: The average CPU time spent on three complete executions of different methods for Example 3.2.

Method	Newton	Chebyshev	KH-K	OM
$n = 5$	0.1401	0.1357	0.1353	0.1025
$n = 6$	0.3325	0.3354	0.2737	0.1842
$n = 7$	0.6450	0.5585	0.6174	0.4607
$n = 8$	1.7913	1.6679	1.6605	1.2499
$n = 9$	4.7081	4.4130	4.5434	3.2146
$n = 10$	9.0462	8.6501	9.8953	7.7623
$n = 11$	18.0487	16.3592	18.5172	13.2875
$n = 12$	34.8504	33.0318	34.9995	25.4372
$n = 13$	96.5779	88.4262	88.0699	60.7782
$n = 14$	145.9944	149.6369	159.3744	107.7250
$n = 15$	287.0627	275.6962	285.3082	179.0688



(a) (b)

Figure 2: (a) and (b), respectively, represent the average three number of iterations and whole number of the Einstein products of Example 3.2.

Example 3.3. Consider multi diagonal tensor, where diagonals are

$$\begin{aligned}
&(1,1,1,1,1,1) = 0, \quad &&(1,1,1,2,1,1) = 1, \quad &&(1,1,1,1,2,1) = 1,\\
&(1,1,1,1,1,2) = 1, \quad &&(2,1,1,1,1,1) = -1, \quad &&(1,2,1,1,1,1) = -1,\\
&(1,1,2,1,1,1) = -1.
\end{aligned}$$

Here dimension is odd, and tensors are singular with $ind(\mathcal{A}) = 1$. The numerical results to compute Drazin inverse tensors are shown in Fig. 3 and Table 2.

Table 2: Number of the Einstein products/CPU time spend of Example 3.3.

Method	Newton	Chebyshev	KH-K	OM
$n = 11$	52/10.4407	51/10.4866	55/10.8991	40/8.2667
$n = 15$	56/150.0999	54/143.9156	60/161.7676	44/121.9163

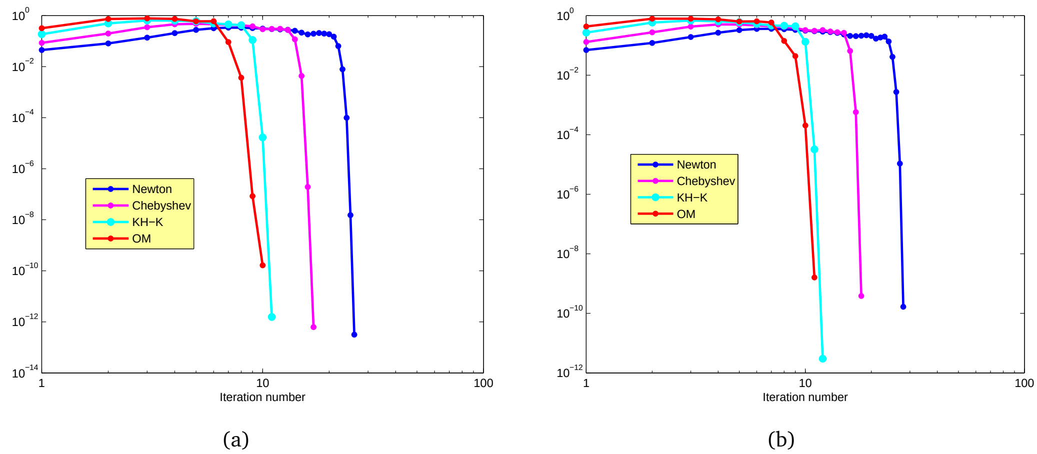


Figure 3: (a) and (b), respectively, display the Res of the results of the iterative methods of Example (3.3) for $n = 11, 15$.

4. Applications

In recent years, tensors have been extensively investigated. They represent a mathematical tool that has a wide variety of applications for several problems [1–3]. In the last section, as in some applications, partial and fractional differential equations that lead to sparse matrices are considered prototypes. We use the iterates obtained by the new method as a preconditioner-based tensor form to solve the multilinear system $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$. Therefore, in the following, two examples are given to display the accuracy and efficiency of the new method.

Example 4.1 ([37]). Consider the three-dimensional Poisson problem

$$\begin{aligned} &\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2} \\ &= F(x,y,z) && \text{in } \Omega=[0,1]\times[0,1]\times[0,1], \\ &\psi(x,y,z)=0 && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

For solving Eq. (4.1), spatial discretization is one way and continues with finite differences. We use the center finite difference for the second derivative as follows:

$$\frac{\partial^2\psi(x,y,z)}{\partial x^2} \approx \frac{\psi_{i-1,j,l}-2\psi_{i,j,l}+\psi_{i+1,j,l}}{h^2},$$

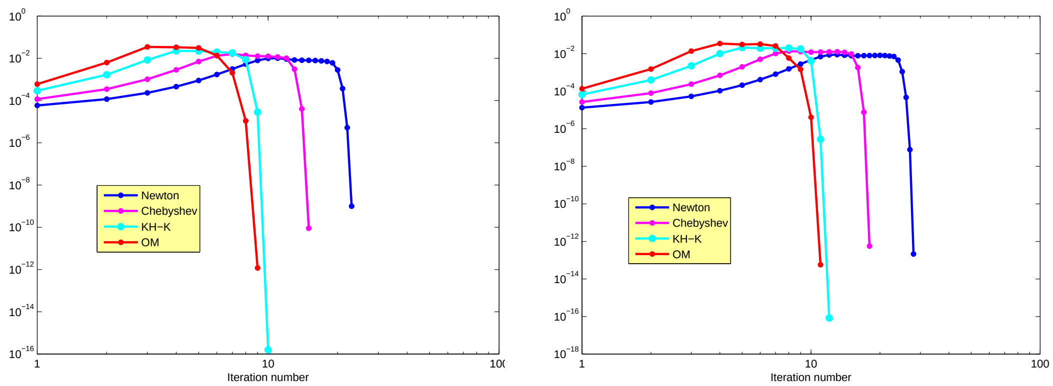
$$\frac{\partial^2\psi(x,y,z)}{\partial y^2}\approx\frac{\psi_{i,j-1,l}-2\psi_{i,j,l}+\psi_{i,j+1,l}}{h^2},$$
$$\frac{\partial^2\psi(x,y,z)}{\partial z^2}\approx\frac{\psi_{i,j,l-1}-2\psi_{i,j,l}+\psi_{i,j,l+1}}{h^2},$$

where h is the step size along the space x, y and z.

The results of Example 4.1 are presented in Fig. 4 and Table 3.

Table 3: Number of the Einstein products/CPU time spend of the Example 4.1.

Method	Newton	Chebyshev	KH-K	OM
$h=1/10$	48/2.6207	45/2.4285	50/2.5388	36/1.9087
$h=1/15$	56/86.8038	54/89.2903	60/95.1278	44/72.8544



(a) (b)

Figure 4: (a) and (b), respectively, display numerical results of Example 4.1, for $h=1/10,1/15$.

Example 4.2 ([11, 40]). In the last example, consider the two-dimensional time fractional diffusion equation

$$\begin{aligned}&\frac{\partial^\theta\psi(x,y,t)}{\partial t^\theta}-\frac{\partial^2\psi(x,y,t)}{\partial x^2}-\frac{\partial^2\psi(x,y,t)}{\partial y^2}=G(x,y,t), && (x,y)\in\Omega\times[0,T],\\ &\psi(x,y,t)=0 && \text{on}\quad \partial\Omega,\\ &\psi(x,y,0)=H(x,y), && (x,y)\in\Omega,\quad 0<\theta\le 1,\end{aligned} \tag{4.2}$$

where $\partial^\theta\psi(\tau)/\partial t^\theta$ is a fractional derivative in the Caputo sense which is defined as follows:

$$\frac{\partial^\theta\psi(\tau)}{\partial t^\theta}=\begin{cases}\dfrac{1}{\Gamma(n-\theta)}\displaystyle\int_0^\tau\frac{\psi^{(n)}(u)}{(\tau-u)^{\theta-1+n}}du, & n-1<\theta<n,\quad n\in\mathbb{N},\\ \dfrac{d^n}{d\tau^n}\psi(\tau), & \theta=n\in\mathbb{N}.\end{cases} \tag{4.3}$$

For more details about the properties of Caputo derivative see [8, 17].

Similar to Example 4.1, for solving Eq. (4.2), by using spatial discretization we can get

$$\frac{\partial^2\psi(x,y,t)}{\partial x^2} \approx \frac{\psi_{i-1,j,k} - 2\psi_{i,j,k} + \psi_{i+1,j,k}}{h^2},$$
$$\frac{\partial^2\psi(x,y,t)}{\partial y^2} \approx \frac{\psi_{i,j-1,k} - 2\psi_{i,j,k} + \psi_{i,j+1,k}}{h^2},$$

and according to [15], we have

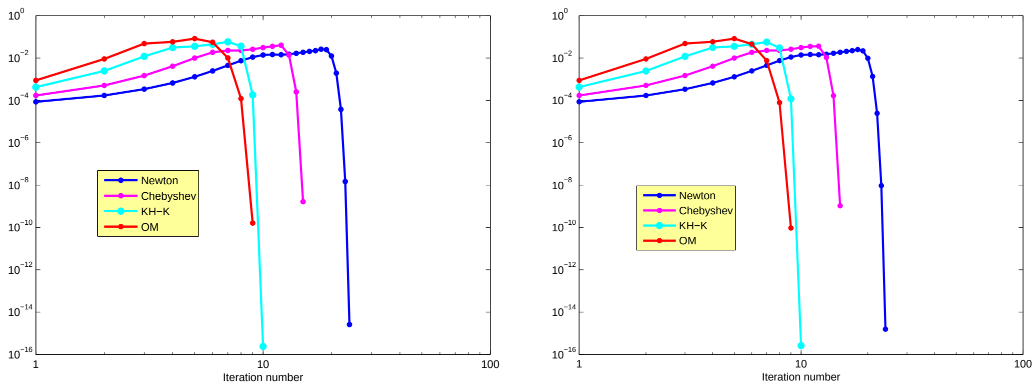
$$\frac{\partial^\theta\psi(x,y,t)}{\partial t^\theta} \approx \frac{\theta!(\psi_{i,j,k+1} - \psi_{i,j,k})}{h^\theta},$$

again h is the step size along the time t.

The results of Example 4.2 are displayed in Fig. 5 and Table 4.

Table 4: Number of the Einstein products/CPU time spend of the Example 4.2.

Method	Newton	Chebyshev	KH-K	OM
$\theta = 0.6$	48/2.0608	45/2.0507	50/2.2365	36/1.7730
$\theta = 0.9$	48/2.1881	45/1.9035	50/2.12548	36/1.6499



(a) (b)

Figure 5: (a) and (b), respectively, depict numerical results of Example 4.2, for $\theta = 0.6, 0.9$.

5. Conclusions

In this study, we have presented the iterative method to compute the Moore-Penrose inverse of a tensor, based on the iterative method to solve nonlinear equations. Analysis of the convergence error shows that the convergence order of the method is three.

Numerical comparisons of the proposed method with other methods in a wide range of random tensors have presented that the average number of iterations, number of Einstein products, and CPU time of our method were considerably less than other methods. Finally, as some applications, we also have used the obtained method to solve partial and fractional differential equations.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgments

The authors are very much indebted to the editor and anonymous referees for their valuable comments and careful reading of the manuscript.

This work is based upon research funded by Iran National Science Foundation (INSF) under Project No. 4013447.

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