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On Polar Decomposition of Tensors with Einstein Product and a Novel Iterative Parametric Method

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Abstract. This study aims to investigate the polar decomposition of tensors with the Einstein product for the first time. The polar decomposition of tensors can be computed using the singular value decomposition of the tensors with the Einstein product. In the following, some iterative methods for finding the polar decomposition of matrices have been developed into iterative methods to compute the polar decomposition of tensors. Then, we propose a novel parametric iterative method to find the polar decomposition of tensors. Under the obtained conditions, we prove that the proposed parametric method has the order of convergence four. In every iteration of the proposed method, only four Einstein products are required, while other iterative methods need to calculate multiple Einstein products and one tensor inversion in each iteration. Thus, the new method is superior in terms of efficiency index. Finally, the numerical comparisons performed among several well-known methods, show that the proposed method is remarkably efficient and accurate.

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Key words: Iterative methods, Einstein product, polar decomposition of a tensor, polar factor, order of convergence.

1. Introduction

Throughout this study, matrices are denoted by uppercase letters $A, B, \dots,$ and tensors are written in calligraphic font $\mathcal{A}, \mathcal{B}, \dots$. Tensors occur in a wide variety of application areas such as in document analysis, psychometrics, formulation an n-person noncooperative game, medical engineering, chemometrics, higher-order, and so on [6, 15, 18, 24, 25]. Suppose that N is a positive integer, an N-th order tensor $\mathcal{A} = (a_{i_1\dots i_N})_{1\le i_j\le P_j}$ is a multidimensional array with $P_1 \dots P_N$ entries. The tensor $\mathcal{A}$ is

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called a hyper-matrix or the tensor is the higher-order generalization of vectors and matrices.

In the following, we give some definitions of tensors and the Einstein product which are used in the body of this manuscript [3, 4, 12, 13, 20].

Definition 1.1 ([5]). *Let N and M be positive integers, also $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times K_1\times\cdots\times K_M}$ are tensors. Then the Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:*

$$(\mathcal{A} *_N \mathcal{B})_{p_1\ldots p_N k_1\ldots k_M} := \sum_{q_N}^{Q_N}\cdots\sum_{q_1}^{Q_1} a_{p_1\ldots p_N q_1\ldots q_N} b_{q_1\ldots q_N k_1\ldots k_M}, \tag{1.1}$$

therefore, $\mathcal{A} *_N \mathcal{B} \in \mathbb{R}^{P_1\times\cdots\times P_N\times K_1\times\cdots\times K_M}$.

Note that if $N = M = 1$, the Einstein product reduces to the standard matrix multiplication.

Definition 1.2 ([5]). *Let $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ be a tensor, then transpose and Frobenius norm of the tensor $\mathcal{A}$ are defined as follows:*

$$(\mathcal{A}^T)_{p_1\ldots p_N q_1\ldots q_N} := (\mathcal{A})_{q_1\ldots q_N p_1\ldots p_N},$$

and

$$\|\mathcal{A}\| := \sqrt{\sum_{q_1\ldots q_N, p_1\ldots p_N} (a_{q_1\ldots q_N p_1\ldots p_N})^2},$$

respectively.

Definition 1.3. *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is a symmetric tensor if $\mathcal{A} = \mathcal{A}^T$, it means*

$$a_{p_1,p_2,\ldots,p_n,q_1,q_2,\ldots,q_n} = a_{q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n}.$$

Definition 1.4 ([5]). *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is said to be a diagonal tensor if $a_{p_1\ldots p_N q_1\ldots q_N} = 0$ for $p_l \neq q_l$, $l = 1, \ldots, N$. A diagonal tensor $\mathcal{I} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is identity if*

$$i_{p_1\ldots p_N q_1\ldots q_N} = \Pi_{k=1}^N \delta_{p_l q_l},$$

where

$$\delta_{p_l q_l} = \begin{cases} 1, & p_l = q_l, \\ 0, & p_l \neq q_l. \end{cases}$$

Definition 1.5. *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is an orthogonal tensor if*

$$\mathcal{A}^T *_N \mathcal{A} = \mathcal{I}.$$

Definition 1.6. *For $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$, a tensor $\mathcal{B} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is called inverse of $\mathcal{A}$ with Einstein product if*

$$\mathcal{A} *_N \mathcal{B} = \mathcal{I},$$

therefore $\mathcal{A}^{-1} = \mathcal{B}$.

The literature on the tensors is large and still growing. Miao *et al.* [22] investigated the tensor similarity and proposed the T-Jordan canonical form and its properties. In [21], a semi-tensor product of matrices was proposed as a generalization of the usual matrix product in the case where the dimensions of two-factor matrices do not match. Also, the properties of the semi-tensor product of tensors and swap tensors based on the Einstein product were studied. In 2021, He *et al.* [13] investigated and discussed in detail the structures of the quotient singular value decomposition and product singular value decomposition for two tensors. Mo and Wei proved that the multi-linear system with strong Mz-tensors always has a nonnegative solution under certain conditions by the fixed point theory. Also, it was proved that the zero solution is the only solution of the homogeneous multi-linear system for some structured tensors, such as strong M-tensors, $H+$-tensors, and strictly diagonally dominant tensors with positive diagonal elements [23].

In the last years, many works have investigated tensors and their applications to economic equilibrium problems. For example, Barbagallo and Guarino Lo Bianco studied variational inequalities defined on a class of structured tensors. Moreover, they introduced the generalized tensor change inequalities in the Hilbert tensor space containing the inner product between two tensors and used this new tool to analyze the extension of the monopoly market equilibrium problem [2]. In [1], inverse tensor variational inequalities were introduced and analyzed for their application to an economic control equilibrium model.

One of the most important and practical matrix decompositions is the polar decomposition of the matrix, for which various numerical methods have long been proposed [7, 10, 16, 19]. In this study, we present the polar decomposition of the tensors with the Einstein product. Suppose that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is a tensor, then the polar decomposition $\mathcal{A}$ defined as follows:

$$\mathcal{A} = \mathcal{U} *_N \mathcal{H},$$

where $\mathcal{H} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times Q_1\times\cdots\times Q_N}$ is the symmetric tensor and $\mathcal{U} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is orthogonal tensor, such that

$$\mathcal{U}^T *_N \mathcal{U} = \mathcal{I}.$$

We only need methods for finding the orthogonal coefficient of the tensor $\mathcal{U}$ because the symmetric coefficient $\mathcal{H}$ can be obtained from the following formula:

$$\mathcal{H} = \frac{1}{2}\left(\mathcal{U}^T *_N \mathcal{A} + \mathcal{A}^T *_N \mathcal{U}\right).$$

2. Preliminaries

At first, we show that the polar decomposition of the tensor $\mathcal{A}$ can be computed using the singular value decomposition of the tensor with the Einstein product. For this purpose, we give the singular value decomposition of a tensor in the following lemma [5, 26].

Lemma 2.1 ([5]). *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$, $P_i \geq Q_i$, for $1 \leq i \leq N$, is a tensor, the singular value decomposition of $\mathcal{A}$ has the form as follows:*

$$\mathcal{A} = \mathcal{V} *_N \mathcal{S} *_N \mathcal{W}^T, \tag{2.1}$$

where $\mathcal{V} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ and $\mathcal{W} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times Q_1\times\cdots\times Q_N}$ are orthogonal tensors and $\mathcal{S} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is a diagonal tensor its entries are called the singular values of the tensor $\mathcal{A}$.

Let $\mathcal{V}$ be a partition tensor of the form $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$, where $\mathcal{V}_1 \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{V}_2 \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1-Q_1\times\cdots\times P_N-Q_N}$, also $\mathcal{V}_1$ is an orthogonal tensor, that means $\mathcal{A}$ has the polar decomposition

$$\mathcal{A} = \mathcal{U} *_N \mathcal{H},$$

where

$$\mathcal{U} = \mathcal{V}_1 *_N \mathcal{W}^T, \quad \mathcal{H} = \mathcal{W} *_N \mathcal{S} *_N \mathcal{W}^T.$$

Note that throughout this study $\mathcal{X}^n$ means $\overbrace{\mathcal{X} *_N \mathcal{X} *_N \cdots *_N \mathcal{X}}^{n \text{ times}}$.

In the following, we are interested in computing the polar decomposition of tensors by iterative methods. There are several numerical methods for finding the polar decomposition of matrices, and we review some of them for computing the polar decomposition of tensors. Assume that $A \in \mathbb{R}^{P_1\times Q_1}$ is a nonsingular matrix, then the Kovaric method (KM) for finding the polar decomposition of a matrix is as follows [17]:

$$\begin{aligned} V_k &= U_k^T U_k, \\ K_k &= (I - V_k)(I + V_k)^{-1}, \\ U_{k+1} &= U_k(I + K_k), \quad k \geq 0, \end{aligned} \tag{2.2}$$

where I is the identity matrix. Based on Newton's method (NM) for finding a square root of a positive number, Higham [14] presented another method for finding the factor U. The method is

$$U_{k+1} = \frac{1}{2}\left(U_k + U_k^{-T}\right), \quad k \geq 0. \tag{2.3}$$

Using a simple modification of the iterative method (2.3) by Gander, the following method was also presented (GM) [9]:

$$U_{k+1} = \frac{1}{2}U_k\left(I + V_k^{-1}\right), \quad k \geq 0. \tag{2.4}$$

Like Newton's method, we have the following cubically convergent Halley method (HM) to obtain the factor U [8]:

$$U_{k+1} = U_k(V_k + 3I)(3V_k + I)^{-1}, \quad k \geq 0. \tag{2.5}$$

Now, assume that $\mathcal{A}$ is a nonsingular tensor, then the Kovaric method to find the polar decomposition of tensors is as follows:

$$\begin{aligned} \mathcal{V}_k &= \mathcal{U}_k^T *_N \mathcal{U}_k, \\ \mathcal{K}_k &= (\mathcal{I} - \mathcal{V}_k) *_N (\mathcal{I} + \mathcal{V}_k)^{-1}, \\ \mathcal{U}_{k+1} &= \mathcal{U}_k *_N (\mathcal{I} + \mathcal{K}_k), \quad k \geq 0. \end{aligned} \tag{2.6}$$

The initial approximation used in the methods of this article to find the $\mathcal{U}$ factor is as follows:

$$\mathcal{U}_0 = \alpha \mathcal{A}, \quad \alpha = \frac{1}{\|\mathcal{A}\| + 1}. \tag{2.7}$$

Theorem 2.1. *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$, $P_i \geq Q_i$, for $1 \leq i \leq N$, is an arbitrary tensor. Then the sequence of the tensor iterates $\{\mathcal{U}_k\}_{k \geq 0}$ generated by* (2.6) *converges to the unitary factor $\mathcal{U}$, for $\mathcal{U}_0 = \alpha \mathcal{A}$.*

Proof. By using the singular value decomposition of the tensor $\mathcal{A}$, we have

$$\mathcal{A} = \mathcal{V} *_N \mathcal{S} *_N \mathcal{W}^T, \tag{2.8}$$

where $\mathcal{S} = \operatorname{diag}(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{Q_r})$, $Q_r = \min\{Q_1, Q_2, \ldots, Q_N\}$, is a diagonal tensor, such that

$$\mathcal{S}_1 \geq \mathcal{S}_2 \geq \cdots \geq \mathcal{S}_{Q_r} \geq 0$$

are the singular values of tensor $\mathcal{A}$, and $\mathcal{V}, \mathcal{W}$ are orthogonal tensors. In the following, we define the sequence of tensors

$$\overline{\mathcal{D}}_k = \begin{pmatrix} \mathcal{D}_k \\ 0 \end{pmatrix} = \mathcal{V}^T *_N \mathcal{U}_k *_N \mathcal{W},$$

where $\mathcal{D}_k \in \mathbb{R}^{\overbrace{Q_r \times \cdots \times Q_r}^{N \text{ times}} \times Q_1 \times \ldots \times Q_N}$, and $0 \in \mathbb{R}^{P_1 - Q_r \times \ldots \times P_N - Q_r \times Q_1, \ldots \times Q_N}$. Therefore, from (2.6), we can get

$$\begin{aligned} \mathcal{D}_0 &= \mathcal{S}, \\ \mathcal{D}_{k+1} &= \mathcal{D}_k *_N \left(\mathcal{I} + (\mathcal{I} - \mathcal{D}_k^2) *_N (\mathcal{I} + \mathcal{D}_k^2)^{-1}\right). \end{aligned} \tag{2.9}$$

As $\mathcal{D}_0$ is a diagonal tensor with positive diagonal entries, it follows by induction that sequence $\{\mathcal{D}_k\}_{k \geq 0}$ is defined as

$$\mathcal{D}_k = \operatorname{diag}\left(d_1^{(k)}, d_2^{(k)}, \ldots, d_{Q_r}^{(k)}, 0, \ldots, 0\right).$$

Note that (2.9) represents k uncoupled scalar iterations

$$\begin{aligned} &d_i^{(0)} = \mathcal{S}_i, \quad 1 \le i \le Q_r, \\ &d_i^{(k+1)} = d_i^{(k)} \left(1 + \left(1 - d_i^{(k)^2}\right)\left(1 + d_i^{(k)^2}\right)^{-1}\right). \end{aligned}$$

Some algebraic manipulations give the following relation between $d_i^{(k+1)}$ and $d_i^{(k)}$:

$$\frac{d_i^{(k+1)} - 1}{d_i^{(k+1)} + 1} = -\frac{(d_i^{(k)} - 1)^2}{(d_i^{(k)} + 1)^2}.$$

By repeating this step, we can get

$$\frac{d_i^{(k+1)} - 1}{d_i^{(k+1)} + 1} = -\left(\frac{d_i^{(0)} - 1}{d_i^{(0)} + 1}\right)^{2^{k+1}}.$$

By taking $d_i^{(0)} > 0$, we have

$$\lim_{k \to +\infty} \left| \frac{d_i^{(k+1)} - 1}{d_i^{(k+1)} + 1} \right| = 0,$$

that means

$$\begin{aligned} &\mathcal{D}_k \to \mathcal{I}_{Q_r}, \\ &\overline{\mathcal{D}}_k \to \begin{pmatrix} \mathcal{I}_{Q_r} \in \mathbb{R}^{\overbrace{Q_r \times \cdots \times Q_r}^{N \text{ times}} \times \overbrace{Q_r \times \cdots \times Q_r}^{N \text{ times}}} & 0 \in \mathbb{R}^{\overbrace{Q_r \times \cdots \times Q_r}^{N \text{ times}} \times Q_1 - Q_r \times \cdots \times Q_N - Q_r} \\ 0 \in \mathbb{R}^{P_1 - Q_r \times \cdots \times P_N - Q_r \times \overbrace{Q_r \times \cdots \times Q_r}^{N \text{ times}}} & 0 \in \mathbb{R}^{P_1 - Q_r \times \cdots \times P_N - Q_r \times Q_1 - Q_r \times \cdots \times Q_N - Q_r} \end{pmatrix}. \end{aligned}$$

Then, we have $\lim_{k \to +\infty} \mathcal{U}_k = \mathcal{U}$, as a result $\mathcal{H} = \mathcal{U}^T *_N \mathcal{A}$. □

Theorem 2.2. *Let* $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$ *be an arbitrary tensor. Then the iterative method generated by* (2.6) *has fourth order of convergence for finding the unitary factor* $\mathcal{U}$.

Proof. According to Theorem 2.1, the iterative method (2.6) transforms the singular values of $\mathcal{U}_k$ as follows:

$$\mathcal{S}_i^{(k+1)} = \mathcal{S}_i^{(k)} \left(1 + \frac{1 - \mathcal{S}_i^{(k)^2}}{1 + \mathcal{S}_i^{(k)^2}}\right), \quad 1 \le i \le Q_r,$$

and leaves the singular vectors invariant. Now, similar to the previous theorem, we can obtain

$$\frac{\mathcal{S}_i^{(k+1)} - 1}{\mathcal{S}_i^{(k+1)} + 1} = -\frac{\left(\mathcal{S}_i^{(k)} - 1\right)^2}{\left(\mathcal{S}_i^{(k)} + 1\right)^2}.$$

Therefore, we have

$$\left|\frac{\mathcal{S}_i^{(k+1)}-1}{\mathcal{S}_i^{(k+1)}+1}\right| = \left|\frac{\mathcal{S}_i^{(k)}-1}{\mathcal{S}_i^{(k)}+1}\right|^2.$$

This means the convergence order of the method (2.6) is two. □

In the following, Newton's method for computing the polar decomposition of tensors is

$$\mathcal{U}_{k+1} = \frac{1}{2}\left(\mathcal{U}_k + \mathcal{U}_k^{-T}\right), \quad k \geq 0. \tag{2.10}$$

Also, Gander's method is

$$\mathcal{U}_{k+1} = \frac{1}{2}\mathcal{U}_k *_N \left(\mathcal{I} + \mathcal{V}_k^{-1}\right), \quad k \geq 0, \tag{2.11}$$

and finally, Hally's method is as follows:

$$\mathcal{U}_{k+1} = \mathcal{U}_k *_N (\mathcal{V}_k + 3\mathcal{I}) *_N (3\mathcal{V}_k + \mathcal{I})^{-1}, \quad k \geq 0. \tag{2.12}$$

3. A new parametric iterative method

Most of the methods in the previous section are in the form

$$\mathcal{U}_{k+1} = \mathcal{U}_k *_N h\left(\mathcal{U}_k^T *_N \mathcal{U}_k\right), \quad k \geq 0, \tag{3.1}$$

where $h(t)$ is a function. Using $\mathcal{L}_k = \mathcal{U}_k^T *_N \mathcal{U}_k$, $\mathcal{U}_k = \mathcal{U}_k^{-T} *_N \mathcal{L}_k$, and (3.1) can be rewritten as

$$\mathcal{U}_{k+1} = \mathcal{U}_k^{-T} *_N \mathcal{L}_k *_N h(\mathcal{L}_k).$$

If $\lim_{k\to\infty} \mathcal{L}_k *_N h(\mathcal{L}_k) \to \mathcal{I}$, then $\mathcal{U}_k \to \mathcal{U}^{-T}$. To have $\mathcal{L}_k *_N h(\mathcal{L}_k) \to \mathcal{I}$, 1 must be a fixed point of the function

$$g(x) = xh(x^2),$$

for example $g(1) = 1$. On the other hand, if

$$g^{(i)}(x) = 0, \quad i = 1, \ldots, p-1,$$

then the method is p-th order convergence.

For this purpose, we define a function as follows:

$$g(x) = ax + bx^3 + cx^5 + dx^7 + ex^9.$$

By applying these relationships

$$g(1) = 1, \quad g'(1) = 0, \quad g''(1) = 0, \quad g'''(1) = 0,$$

therefore, we have

$$a = \frac{1}{16}(35 + 16e), \quad b = -\frac{1}{16}(35 + 64e), \quad c = \frac{3}{16}(7 + 32e), \quad d = -\frac{1}{16}(5 + 64e),$$

finally, we can write

$$\begin{aligned} g(x) = & \frac{1}{16}(35+16e)x - \frac{1}{16}(35+64e)x^3 \\ & + \frac{3}{16}(7+32e)x^5 - \frac{1}{16}(5+64e)x^7 + ex^9. \end{aligned} \tag{3.2}$$

Now, we can find the fixed and critical points of $g(x)$ as follows:

$$\begin{aligned} & g(x) = x \Rightarrow x = 0, 1, 1+\zeta, \\ & g'(x) = 0 \Rightarrow x = 1, \frac{\sqrt{35+16e}}{12\sqrt{e}}, \end{aligned}$$

where

$$\begin{aligned} \zeta = \Bigg(& 1 + \frac{5}{48e} - \frac{6}{(125-2160e+27648e^2+48\sqrt{3}\sqrt{775e^2-13824e^3+110592e^4})^{1/3}} \\ & + \frac{25}{48e(125-2160e+27648e^2+48\sqrt{3}\sqrt{775e^2-13824e^3+110592e^4})^{1/3}} \\ & + \frac{(125-2160e+27648e^2+48\sqrt{3}\sqrt{775e^2-13824e^3+110592e^4})^{1/3}}{48e} \Bigg)^{1/2}. \end{aligned}$$

In the following, we show that $[0, 1+\zeta]$ is the optimal range. Note that $x = 0, 1$ and $\overline{x} = \sqrt{35+16e}/(12\sqrt{e}), 1+\zeta$, are minimizers and maximizers of $g(x)$ in $[0, 1+\zeta]$. Now, we want $[0, 1+\zeta]$ to be written to itself by the function $g(x)$, for this we need

$$g\left(\frac{\sqrt{35+16e}}{12\sqrt{e}}\right) < g(1+\zeta) = 1+\zeta.$$

Then, we have $0.2 \leq e \leq 2$, so for $e \in [0.2, 2]$, we obtain

$$0 = g(0) \leq g(x) \leq g(1+\zeta) = 1+\zeta.$$

Fig. 1 shows $y = x$ and the function (3.2), for several e. By choosing $e = 3/2$, we have

$$g(x) = \frac{59}{16}x - \frac{131}{16}x^3 + \frac{165}{16}x^5 - \frac{101}{16}x^7 + \frac{3}{2}x^9. \tag{3.3}$$

Using the iteration function (3.3), we present the following iterative method to find the factor $\mathcal{U}$:

$$\begin{aligned} \mathcal{U}_{k+1} = \mathcal{U}_k *_N \bigg(& \frac{59}{16}\mathcal{I} - \frac{131}{16}\left(\mathcal{U}_k^T *_N \mathcal{U}_k\right) + \frac{165}{16}\left(\mathcal{U}_k^T *_N \mathcal{U}_k\right)^2 \\ & - \frac{101}{16}\left(\mathcal{U}_k^T *_N \mathcal{U}_k\right)^3 + \frac{3}{2}\left(\mathcal{U}_k^T *_N \mathcal{U}_k\right)^4 \bigg). \end{aligned} \tag{3.4}$$

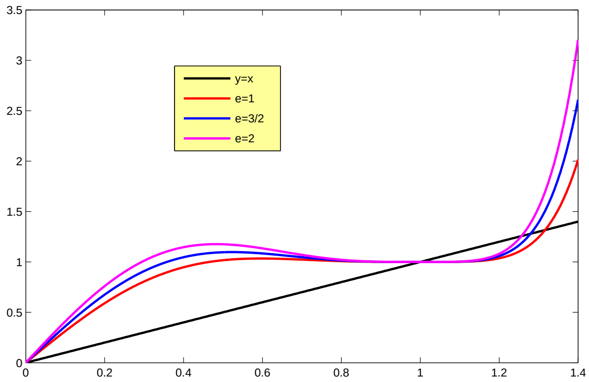


Figure 1: Plot $y = x$ and $g(x)$ for several e.

Now, we prove that the method (3.4) is fourth-order convergent. So, the iterative method (3.4) can be written as follows:

$$\begin{aligned} &\mathcal{V}_k = \mathcal{U}_k^T *_N \mathcal{U}_k, \quad \mathcal{W}_k = \mathcal{V}_k^2, \\ &\mathcal{U}_{k+1} = \mathcal{U}_k *_N \left(\frac{59}{16}\mathcal{I} - \frac{131}{16}\mathcal{V}_k + \frac{165}{16}\mathcal{W}_k + \mathcal{W}_k *_N \left(\frac{3}{2}\mathcal{W} - \frac{101}{16}\mathcal{V}_k \right) \right), \quad k \geq 0. \end{aligned} \tag{3.5}$$

Theorem 3.1. *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is an arbitrary tensor. Then the sequence of the tensor iterates $\{\mathcal{U}_k\}_{k\geq 0}$ generated by* (3.4) *converges to the unitary factor $\mathcal{U}$, for $\mathcal{U}_0 = \alpha\mathcal{A}$.*

Proof. Similar to Theorem 2.1 for the method (3.4), we have

$$\begin{aligned} &\mathcal{D}_0 = \mathcal{S}, \\ &\mathcal{D}_{k+1} = \mathcal{D}_k *_N \left(\frac{59}{16}\mathcal{I} - \frac{131}{16}\mathcal{D}_k^2 + \frac{165}{16}\mathcal{D}_k^4 + \frac{3}{2}\mathcal{D}_k^6 - \frac{101}{16}\mathcal{D}_k^8 \right). \end{aligned} \tag{3.6}$$

Again, it follows by induction that the sequence $\{\mathcal{D}_k\}_{k\geq 0}$ is defined as

$$\mathcal{D}_k = \operatorname{diag}\left(d_1^{(k)}, d_2^{(k)}, \ldots, d_{Q_r}^{(k)}, 0, \ldots, 0\right),$$

therefore, we have

$$\begin{aligned} &d_i^{(0)} = \mathcal{S}_i, \quad 1 \leq i \leq Q_r, \\ &d_i^{(k+1)} = \frac{59}{16}d_i^{(k)} - \frac{131}{16}d_i^{(k)^3} + \frac{165}{16}d_i^{(k)^5} + \frac{3}{2}d_i^{(k)^7} - \frac{101}{16}d_i^{(k)^9}. \end{aligned}$$

Some algebraic manipulations give the following relation between $d_i^{(k+1)}$ and $d_i^{(k)}$

$$\frac{d_i^{(k+1)} - 1}{d_i^{(k+1)} + 1} = -\left(d_i^{(k)} - 1\right)^4 \left(1 + \frac{5}{16}d_i^{(k)} - \frac{19}{4}d_i^{(k)^2} - \frac{139}{16}d_i^{(k)^3} - 6d_i^{(k)^4} - \frac{3}{2}d_i^{(k)^5}\right)$$

$$\times\left(\left(d_i^{(k)}+1\right)^4\left(1-\frac{5}{16}d_i^{(k)}-\frac{19}{4}d_i^{(k)^2}+\frac{139}{16}d_i^{(k)^3}-6d_i^{(k)^4}+\frac{3}{2}d_i^{(k)^5}\right)\right)^{-1},$$

by repeating this step, we can get

$$\frac{d_i^{(k+1)}-1}{d_i^{(k+1)}+1}=-\left(\frac{d_i^{(0)}-1}{d_i^{(0)}+1}\right)^{4^{k+1}}G\left(d_i^{(k)},d_i^{(k-1)},\ldots,d_i^{(0)}\right).$$

Considering $G(d_i^{(k)},d_i^{(k-1)},\ldots,d_i^{(0)})<1$ and $d_i^{(0)}>0$, we have

$$\lim_{k\to+\infty}\left|\frac{d_i^{(k+1)}-1}{d_i^{(k+1)}+1}\right|=0.$$

Finally, like Theorem 2.1, we can obtain $\lim_{k\to+\infty}\mathcal{U}_k=\mathcal{U}$, as a result, $\mathcal{H}=\mathcal{U}^T*_N\mathcal{A}$. $\square$

Theorem 3.2. *Let $\mathcal{A}\in\mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ be an arbitrary tensor. Then the iterative method generated by* (3.4) *has fourth order of convergence for finding the unitary factor $\mathcal{U}$.*

Proof. According to Theorem 3.1, the iterative method (3.4) transforms the singular values of $\mathcal{U}_k$ as follows:

$$\mathcal{S}_i^{(k+1)}=\frac{59}{16}\mathcal{S}_i^{(k)}-\frac{131}{16}\mathcal{S}_i^{(k)^3}+\frac{165}{16}\mathcal{S}_i^{(k)^5}+\frac{3}{2}\mathcal{S}_i^{(k)^7}-\frac{101}{16}\mathcal{S}_i^{(k)^9},\quad 1\le i\le Q_r,$$

and leaves the singular vectors invariant. Now, similar to the previous theorem, we can obtain

$$\begin{aligned}\frac{\mathcal{S}_i^{(k+1)}-1}{\mathcal{S}_i^{(k+1)}+1}=&-\left(\mathcal{S}_i^{(k)}-1\right)^4\left(1+\frac{5}{16}\mathcal{S}_i^{(k)}-\frac{19}{4}\mathcal{S}_i^{(k)^2}-\frac{139}{16}\mathcal{S}_i^{(k)^3}-6\mathcal{S}_i^{(k)^4}-\frac{3}{2}\mathcal{S}_i^{(k)^5}\right)\\&\times\left(\left(\mathcal{S}_i^{(k)}+1\right)^4\left(1-\frac{5}{16}\mathcal{S}_i^{(k)}-\frac{19}{4}\mathcal{S}_i^{(k)^2}+\frac{139}{16}\mathcal{S}_i^{(k)^3}-6\mathcal{S}_i^{(k)^4}+\frac{3}{2}\mathcal{S}_i^{(k)^5}\right)\right)^{-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|\frac{\mathcal{S}_i^{(k+1)}-1}{\mathcal{S}_i^{(k+1)}+1}\right|\le&\left|\frac{\mathcal{S}_i^{(k)}-1}{\mathcal{S}_i^{(k)}+1}\right|^4\left|1+\frac{5}{16}\mathcal{S}_i^{(k)}-\frac{19}{4}\mathcal{S}_i^{(k)^2}-\frac{139}{16}\mathcal{S}_i^{(k)^3}-6\mathcal{S}_i^{(k)^4}-\frac{3}{2}\mathcal{S}_i^{(k)^5}\right|\\&\times\left|1-\frac{5}{16}\mathcal{S}_i^{(k)}-\frac{19}{4}\mathcal{S}_i^{(k)^2}+\frac{139}{16}\mathcal{S}_i^{(k)^3}-6\mathcal{S}_i^{(k)^4}+\frac{3}{2}\mathcal{S}_i^{(k)^5}\right|^{-1}.\end{aligned}$$

This means the convergence order of the method (3.4) is four. $\square$

4. Efficiency index

In this section, we want to analyze the efficiency index (EI) of the new parametric method. Since a high-speed method may cost a lot, we have to define the efficiency index for the proposed methods. Therefore, to get better implementation times when solving problems, the most important goal is to create a theoretical discussion on practical issues. In [11], the efficiency index of iterative methods for finding the polar decomposition of matrices was presented. In the following, similarly, we generalize the efficiency index of iterative methods for calculating the polar decomposition of tensors with the Einstein product.

Definition 4.1. *The efficiency of iterative methods to find $\mathcal{U}$ is defined as follows:*

$$EI = \mathcal{O}^{\frac{1}{s(m+2c)}}, \tag{4.1}$$

where s, m, and c, show the whole number of iterations needed to obtain $\mathcal{U}$, the number of the Einstein products and the tensor inversion in every cycle, respectively. Also, $\mathcal{O}$ is the order of convergence methods.

Note that to obtain a fair comparison, if the cost of m is β, then the cost of c would be more than 2β.

Now, the approximate value EI for the proposed methods would be in Table 1, wherein s_i, is all the number of iterations required for the convergence of the methods, respectively, in the same environment. For example, if s_i for the method KM is s, then for the method HM is $(2/3)s$. Fig. 2 shows $\gamma = EI - 1$ of each method for different dimensions.

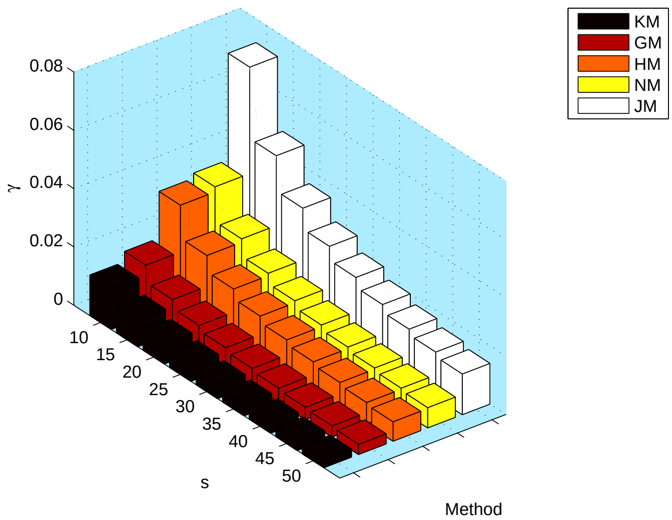


Figure 2: The EI of several methods for various s.

Table 1: The process of calculating the EI.

Method	s_i	m	c	$\mathcal{O}$	EI
KM	s	3	1	2	$2^{1/5s}$
NM	s	0	1	2	$2^{1/2s}$
GM	s	2	1	2	$2^{1/4s}$
HM	$(2/3)s$	3	1	3	$3^{3/10s}$
JM	$(1/2)s$	4	0	4	$4^{1/2s}$

The EI supports the superiority of the new method for solving some numerical examples in the next section.

5. Numerical experiments

In this section, we give some numerical experiments to illustrate the performance of the proposed methods. All of the programs have been done in Matlab code. We compare our method, denoted by JM, which is free from tensor inversion computation, with several well-known iterative methods such as KM, NM, GM, and HM, which require an inversion per iteration. The stopping criterion is as follows:

$$Res = \frac{\|\mathcal{U}_{k+1} - \mathcal{U}_k\|}{\|\mathcal{U}_{k+1}\|} < 10^{-10}.$$

Example 5.1. In the first example, we consider the behavior of the mentioned iterative methods with the tensor $\mathcal{A} \in \mathbb{R}^{3,3,3,3}$ as follows:

$$\mathcal{A}(:,:,1,1) = \begin{pmatrix} 0.0301 & 0.4130 & 0.4640 \\ 0.3292 & 0.2561 & -0.1517 \\ 0.1736 & -0.3315 & 0.0985 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1) = \begin{pmatrix} 0.2314 & -0.0370 & -0.0436 \\ 0.3708 & -0.3331 & -0.3722 \\ -0.1666 & -0.0336 & 0.1399 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,1) = \begin{pmatrix} 0.3965 & -0.2051 & -0.4733 \\ -0.0726 & -0.4413 & 0.0505 \\ -0.3242 & -0.4398 & 0.0631 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,2) = \begin{pmatrix} -0.1954 & 0.4398 & 0.2892 \\ 0.2545 & -0.0148 & 0.0712 \\ 0.1711 & 0.0181 & 0.2665 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,2) = \begin{pmatrix} 0.4202 & -0.0343 & 0.1776 \\ -0.2897 & 0.1167 & -0.4827 \\ 0.1081 & -0.2969 & -0.2734 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,2) = \begin{pmatrix} 0.4145 & -0.4316 & 0.4666 \\ 0.2684 & -0.3004 & 0.3322 \\ 0.0333 & -0.4880 & -0.0547 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,3) = \begin{pmatrix} 0.4905 & -0.3694 & -0.1762 \\ 0.2943 & -0.2878 & 0.2388 \\ -0.0114 & 0.3311 & -0.3401 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,3) = \begin{pmatrix} 0.2315 & -0.3783 & -0.2305 \\ -0.4463 & 0.4840 & -0.0680 \\ 0.2144 & 0.0931 & 0.4305 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,3) = \begin{pmatrix} 0.0079 & -0.0508 & 0.0313 \\ -0.4359 & -0.3528 & -0.4121 \\ 0.1591 & 0.2752 & 0.4005 \end{pmatrix},$$

and the obtained factor $\mathcal{U}$ is

$$\mathcal{U}(:,:,1,1) = \begin{pmatrix} 0.0226 & 0.2854 & 0.0423 \\ 0.4577 & 0.1772 & -0.2263 \\ 0.6499 & -0.4446 & 0.0631 \end{pmatrix},$$

$$\mathcal{U}(:,:,2,1) = \begin{pmatrix} 0.1765 & -0.1499 & 0.0219 \\ 0.5411 & -0.1323 & -0.5822 \\ -0.4477 & 0.1026 & 0.2928 \end{pmatrix},$$

$$\mathcal{U}(:,:,3,1) = \begin{pmatrix} 0.3345 & 0.1279 & -0.5839 \\ -0.1235 & -0.3842 & 0.1380 \\ -0.1758 & -0.5415 & 0.1575 \end{pmatrix},$$

$$\mathcal{U}(:,:,1,2) = \begin{pmatrix} 0.2649 & 0.6728 & 0.3542 \\ 0.0372 & -0.0504 & 0.3170 \\ -0.1249 & 0.2145 & 0.4309 \end{pmatrix},$$

$$\mathcal{U}(:,:,2,2) = \begin{pmatrix} 0.5492 & 0.1636 & 0.2932 \\ -0.4138 & 0.1557 & -0.4471 \\ -0.0851 & -0.1565 & -0.3981 \end{pmatrix},$$

$$\mathcal{U}(:,:,3,2) = \begin{pmatrix} 0.2048 & -0.5080 & 0.6010 \\ 0.1048 & -0.2437 & 0.3338 \\ 0.0130 & -0.3865 & 0.0865 \end{pmatrix},$$

$$\mathcal{U}(:,:,1,3) = \begin{pmatrix} 0.5650 & -0.1265 & -0.2104 \\ 0.3182 & -0.2025 & 0.2146 \\ 0.3120 & 0.4742 & -0.3315 \end{pmatrix},$$

$$\mathcal{U}(:,:,2,3) = \begin{pmatrix} 0.3444 & -0.3486 & -0.1829 \\ -0.2186 & 0.5791 & 0.0241 \\ 0.1600 & 0.0827 & 0.5570 \end{pmatrix},$$

$$\mathcal{U}(:,:,3,3) = \begin{pmatrix} -0.0696 & -0.0678 & 0.0805 \\ -0.3870 & -0.5839 & -0.3639 \\ 0.4475 & 0.2168 & 0.3371 \end{pmatrix}.$$

Also, the tensor $\mathcal{A} \in \mathbb{R}^{5,5,5,5}$ is

$$\mathcal{A}(:,:,1,1) = \begin{pmatrix} 0.2513 & 0.3087 & 0.3598 & 0.4061 & 0.3053 \\ -0.3611 & 0.1329 & 0.1270 & -0.4227 & 0.0308 \\ -0.1507 & 0.1884 & -0.3194 & -0.1615 & -0.2727 \\ -0.3487 & 0.1396 & 0.0733 & 0.0806 & 0.2095 \\ -0.0033 & 0.2293 & -0.3364 & -0.0248 & -0.3514 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,1) = \begin{pmatrix} 0.1581 & -0.3504 & -0.4743 & 0.3964 & -0.3921 \\ 0.1340 & -0.2973 & 0.4711 & -0.3110 & -0.3211 \\ -0.2707 & 0.4550 & -0.2024 & 0.1607 & 0.2466 \\ -0.3178 & -0.4841 & 0.0251 & 0.4412 & -0.4505 \\ -0.3336 & 0.4575 & 0.3623 & 0.4757 & -0.4287 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,1) = \begin{pmatrix} -0.0109 & 0.3613 & -0.3825 & -0.3305 & 0.3679 \\ 0.3499 & 0.4091 & 0.0090 & 0.3837 & 0.2415 \\ 0.4970 & 0.3454 & -0.3312 & -0.1121 & -0.0521 \\ -0.4956 & 0.3789 & 0.3311 & -0.1174 & 0.2096 \\ 0.0426 & 0.2462 & 0.4280 & -0.2285 & 0.4443 \end{pmatrix},$$

$$\mathcal{A}(:,:,4,1) = \begin{pmatrix} -0.3259 & -0.1019 & -0.2854 & 0.4226 & 0.4257 \\ -0.2554 & -0.3845 & 0.2910 & -0.0077 & 0.3327 \\ 0.1409 & -0.4197 & 0.1547 & 0.3340 & -0.2406 \\ 0.3086 & -0.1395 & -0.4739 & -0.3686 & -0.2870 \\ 0.3534 & 0.3289 & 0.2858 & 0.2598 & 0.0223 \end{pmatrix},$$

$$\mathcal{A}(:,:,5,1) = \begin{pmatrix} -0.1026 & -0.0096 & -0.4470 & 0.0571 & -0.3613 \\ -0.0209 & -0.0621 & -0.4122 & 0.2198 & 0.3819 \\ 0.4939 & 0.2727 & 0.2980 & -0.3896 & 0.4236 \\ 0.1045 & 0.2441 & 0.1556 & -0.2834 & -0.4872 \\ 0.4449 & -0.0571 & -0.4677 & 0.3110 & -0.1228 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,2) = \begin{pmatrix} -0.3322 & 0.4716 & -0.4604 & 0.4554 & -0.4950 \\ 0.0402 & -0.1391 & -0.0306 & 0.2242 & 0.2825 \\ -0.3983 & 0.1442 & -0.3499 & 0.0809 & 0.4269 \\ -0.4607 & -0.4321 & 0.4913 & 0.0403 & -0.4917 \\ 0.4332 & -0.2921 & -0.0729 & 0.2054 & 0.3246 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,2) = \begin{pmatrix} 0.2673 & -0.3947 & -0.0302 & -0.2402 & 0.1423 \\ 0.4971 & -0.2318 & -0.2809 & 0.3781 & 0.0669 \\ -0.2723 & 0.2638 & 0.4227 & -0.3117 & -0.1236 \\ 0.4195 & 0.3055 & -0.1797 & 0.2592 & -0.2875 \\ 0.1420 & -0.3957 & 0.3575 & -0.4683 & 0.2922 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,2) = \begin{pmatrix} -0.3546 & 0.3382 & -0.0111 & 0.0942 & -0.2094 \\ -0.0109 & -0.3589 & 0.4714 & -0.0014 & 0.0613 \\ -0.4872 & 0.2322 & -0.3875 & 0.0679 & 0.1333 \\ -0.3134 & 0.1911 & 0.2432 & -0.0735 & 0.4308 \\ -0.0148 & -0.4655 & 0.1385 & -0.4238 & 0.4778 \end{pmatrix},$$

$$\mathcal{A}(:,:,4,2) = \begin{pmatrix} -0.4064 & -0.0244 & 0.0908 & 0.0041 & -0.0467 \\ 0.1617 & 0.1710 & 0.4122 & 0.2684 & 0.2374 \\ 0.1028 & 0.4596 & -0.3989 & -0.2170 & 0.0099 \\ -0.0262 & -0.4109 & -0.2067 & -0.2746 & -0.1175 \\ -0.1437 & 0.2977 & -0.4484 & -0.1687 & 0.4055 \end{pmatrix},$$

$$\mathcal{A}(:,:,5,2) = \begin{pmatrix} 0.4653 & 0.4912 & 0.1894 & 0.3398 & 0.4494 \\ 0.1283 & -0.2132 & -0.4495 & -0.3818 & -0.2436 \\ -0.3680 & 0.2062 & -0.3156 & -0.0896 & 0.4899 \\ 0.1183 & 0.0352 & -0.4543 & -0.3798 & -0.1502 \\ -0.1170 & -0.3068 & 0.3850 & 0.0721 & -0.2915 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,3) = \begin{pmatrix} 0.1658 & -0.3337 & -0.1270 & -0.1407 & 0.1455 \\ 0.4733 & -0.2687 & 0.3321 & -0.4111 & 0.0135 \\ 0.1227 & -0.4478 & 0.2538 & -0.1583 & 0.3144 \\ -0.4365 & 0.4018 & 0.1219 & 0.0487 & -0.4028 \\ -0.1265 & 0.2933 & -0.1059 & -0.0395 & -0.0363 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,3) = \begin{pmatrix} 0.0898 & 0.3423 & 0.2567 & 0.3289 & -0.3873 \\ -0.3128 & -0.0003 & 0.2961 & 0.3418 & 0.1483 \\ 0.1113 & -0.0610 & -0.2064 & 0.1652 & -0.0192 \\ -0.4481 & -0.3509 & -0.3848 & 0.4601 & -0.4335 \\ 0.0757 & -0.4717 & -0.1249 & 0.4431 & 0.3978 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,3) = \begin{pmatrix} -0.0028 & -0.3664 & -0.1185 & -0.0575 & -0.4801 \\ 0.2713 & 0.1385 & -0.2000 & -0.0458 & -0.1582 \\ -0.4396 & -0.1151 & -0.1599 & 0.4453 & 0.2660 \\ -0.2375 & 0.2657 & 0.4189 & -0.2809 & -0.1572 \\ 0.1511 & 0.1529 & -0.0437 & 0.3824 & 0.1188 \end{pmatrix},$$

$$\mathcal{A}(:,:,4,3) = \begin{pmatrix} -0.0470 & -0.1573 & -0.4016 & -0.1355 & -0.3665 \\ -0.4898 & -0.0067 & 0.1498 & 0.1762 & 0.1727 \\ 0.0991 & 0.2018 & 0.2641 & -0.1242 & -0.2974 \\ 0.1016 & 0.3878 & 0.4880 & 0.3635 & 0.3685 \\ 0.1494 & -0.4449 & -0.3747 & -0.2080 & 0.2512 \end{pmatrix},$$

$$\mathcal{A}(:,:,5,3) = \begin{pmatrix} -0.0806 & 0.1013 & 0.4251 & 0.0749 & -0.4349 \\ -0.4998 & -0.1788 & 0.0053 & -0.4535 & 0.4240 \\ -0.3505 & -0.2157 & 0.1276 & -0.0775 & 0.0341 \\ -0.2262 & -0.0647 & 0.2193 & -0.0323 & -0.1332 \\ 0.3724 & 0.4038 & -0.4761 & -0.4774 & -0.1361 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,4) = \begin{pmatrix} -0.3486 & -0.0133 & 0.3352 & 0.1246 & -0.3759 \\ -0.3504 & -0.0352 & 0.1565 & 0.2282 & -0.4972 \\ -0.1492 & -0.3687 & 0.4839 & -0.0018 & -0.3470 \\ -0.1640 & 0.3864 & 0.4798 & 0.3498 & 0.0342 \\ 0.2840 & 0.1746 & -0.2498 & -0.3091 & 0.0106 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,4) = \begin{pmatrix} -0.1148 & -0.0517 & -0.0327 & 0.3176 & 0.2653 \\ -0.1894 & -0.2559 & 0.4707 & -0.0942 & 0.0745 \\ -0.4964 & 0.3034 & 0.3412 & -0.0337 & 0.4159 \\ 0.3152 & 0.3240 & -0.4215 & 0.4515 & -0.0046 \\ 0.1384 & 0.3522 & -0.2624 & 0.4650 & -0.3340 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,4) = \begin{pmatrix} -0.1740 & -0.3332 & 0.4984 & -0.4259 & 0.1402 \\ -0.2036 & 0.4474 & 0.4875 & -0.1882 & 0.3032 \\ 0.0583 & 0.3111 & -0.3499 & 0.3952 & -0.2549 \\ -0.4325 & 0.2105 & 0.4585 & 0.3348 & -0.4359 \\ -0.4310 & 0.4702 & 0.0305 & -0.4977 & -0.2369 \end{pmatrix},$$

$$\mathcal{A}(:,:,4,4) = \begin{pmatrix} -0.3973 & -0.0794 & -0.4707 & -0.2511 & -0.3351 \\ -0.0163 & -0.2162 & 0.2023 & 0.1525 & 0.3834 \\ -0.0811 & -0.4518 & -0.4924 & -0.1797 & 0.1665 \\ -0.1187 & -0.2808 & 0.1109 & -0.3963 & 0.3477 \\ 0.3868 & -0.2608 & -0.0919 & 0.0356 & 0.2627 \end{pmatrix},$$

$$\mathcal{A}(:,:,5,4) = \begin{pmatrix} 0.3070 & 0.0316 & 0.4043 & -0.0312 & 0.0340 \\ 0.1330 & 0.3732 & 0.1302 & 0.0452 & -0.0204 \\ 0.2104 & -0.4455 & 0.4830 & -0.3209 & 0.2937 \\ 0.1887 & 0.0004 & 0.0852 & 0.1345 & -0.4073 \\ -0.1791 & -0.0672 & 0.3406 & 0.4630 & 0.3808 \end{pmatrix},$$

$$\mathcal{A}(:,:,1,5) = \begin{pmatrix} -0.4961 & -0.1795 & -0.4009 & -0.3526 & -0.4389 \\ 0.0115 & 0.1016 & 0.0110 & 0.2776 & -0.2805 \\ 0.1785 & 0.4132 & -0.3899 & -0.1009 & -0.4172 \\ 0.0657 & 0.1825 & 0.0453 & 0.3983 & 0.4504 \\ -0.0215 & 0.4467 & 0.1888 & -0.1930 & -0.4836 \end{pmatrix},$$

$$\mathcal{A}(:,:,2,5) = \begin{pmatrix} -0.3853 & 0.0170 & -0.4922 & 0.3393 & 0.3391 \\ -0.4876 & -0.2545 & 0.1027 & -0.2376 & 0.4825 \\ -0.2838 & -0.3063 & -0.0211 & 0.0142 & 0.1265 \\ -0.4886 & -0.4091 & -0.1919 & -0.0532 & -0.3187 \\ 0.1424 & -0.1316 & 0.2444 & -0.1588 & -0.3770 \end{pmatrix},$$

$$\mathcal{A}(:,:,3,5) = \begin{pmatrix} 0.0800 & 0.1785 & 0.4006 & -0.0838 & 0.4514 \\ -0.1715 & -0.4443 & 0.3466 & 0.2288 & -0.1540 \\ -0.2318 & -0.4659 & -0.1043 & -0.0935 & -0.2098 \\ 0.0502 & -0.2135 & -0.3308 & 0.4518 & 0.3867 \\ -0.3195 & -0.4226 & -0.0695 & 0.4120 & -0.2900 \end{pmatrix},$$

$$\mathcal{A}(:,:,4,5) = \begin{pmatrix} -0.3691 & -0.4213 & 0.2922 & 0.2118 & 0.0293 \\ 0.0205 & 0.4331 & -0.1665 & -0.3331 & 0.1265 \\ 0.4055 & 0.1029 & 0.1927 & -0.0572 & 0.1808 \\ -0.0975 & -0.1225 & -0.2962 & 0.1330 & 0.4232 \\ -0.2842 & 0.1649 & 0.4587 & 0.4300 & -0.3472 \end{pmatrix},$$

$$\mathcal{A}(:,:,5,5) = \begin{pmatrix} -0.0943 & 0.3058 & 0.1761 & -0.2839 & 0.0697 \\ -0.1875 & -0.1736 & 0.3284 & -0.4659 & -0.1404 \\ 0.1939 & 0.0499 & -0.3899 & -0.0634 & -0.4732 \\ 0.3907 & -0.1112 & -0.2208 & 0.4369 & 0.0004 \\ -0.0093 & 0.3968 & 0.2676 & -0.2379 & 0.3270 \end{pmatrix},$$

and again by applying the proposed methods, the factor $\mathcal{U}$ is obtained as follows:

$$\mathcal{U}(:,:,1,1) = \begin{pmatrix} 0.2294 & 0.0844 & 0.1454 & 0.3714 & 0.1783 \\ -0.0090 & 0.2134 & 0.1311 & -0.2788 & 0.0631 \\ -0.1289 & 0.1232 & -0.2312 & -0.3269 & -0.4362 \\ -0.2052 & 0.1485 & 0.1053 & -0.0469 & 0.0934 \\ 0.1994 & 0.0110 & -0.1894 & 0.1346 & -0.2060 \end{pmatrix},$$

$$\mathcal{U}(:,:,2,1) = \begin{pmatrix} 0.2831 & -0.2572 & -0.2128 & 0.1545 & -0.2174 \\ -0.0313 & -0.2889 & 0.2900 & -0.0665 & -0.2317 \\ -0.0449 & 0.3195 & -0.0141 & -0.0331 & 0.0796 \\ -0.1717 & -0.3445 & 0.1411 & 0.1043 & -0.1287 \\ -0.1254 & 0.2391 & 0.2706 & 0.1899 & -0.1458 \end{pmatrix},$$

$$\mathcal{U}(:,:,3,1) = \begin{pmatrix} 0.1329 & 0.2304 & -0.1701 & -0.1639 & 0.3100 \\ 0.0234 & 0.0633 & -0.0533 & 0.3159 & 0.1369 \\ 0.2100 & 0.1724 & -0.0714 & -0.0699 & 0.0737 \\ -0.4184 & 0.1839 & 0.1386 & 0.0075 & 0.1965 \\ 0.0981 & 0.3369 & 0.3017 & -0.0070 & 0.285 \end{pmatrix},$$

$$\mathcal{U}(:,:,4,1) = \begin{pmatrix} -0.1267 & -0.0987 & -0.0837 & 0.4187 & 0.1559 \\ -0.1063 & -0.2833 & 0.1728 & 0.1295 & 0.2680 \\ 0.1530 & -0.2273 & -0.0130 & 0.2686 & -0.2693 \\ 0.2633 & 0.1197 & -0.1228 & -0.2717 & -0.1426 \\ 0.1236 & 0.2367 & 0.2398 & 0.1160 & 0.0502 \end{pmatrix},$$

$$\mathcal{U}(:,:,5,1)=\begin{pmatrix}-0.1597 & 0.0645 & -0.0431 & -0.1642 & -0.1460\\ -0.0895 & -0.2363 & -0.1480 & 0.0285 & 0.1385\\ 0.3695 & 0.3159 & 0.1403 & -0.3026 & 0.1005\\ 0.0059 & 0.1941 & 0.1094 & -0.2380 & -0.3460\\ 0.2171 & -0.1072 & -0.1881 & 0.3113 & -0.1996\end{pmatrix},$$

$$\mathcal{U}(:,:,1,2)=\begin{pmatrix}-0.2684 & 0.2930 & -0.1414 & 0.3750 & -0.0591\\ 0.2486 & -0.0122 & -0.1685 & 0.0801 & 0.1969\\ -0.1538 & -0.0282 & -0.2241 & 0.0718 & 0.1782\\ 0.0146 & -0.2310 & 0.4298 & 0.3155 & -0.2154\\ 0.1531 & -0.0871 & -0.0474 & 0.0573 & 0.1007\end{pmatrix},$$

$$\mathcal{U}(:,:,2,2)=\begin{pmatrix}0.1286 & -0.3764 & 0.0956 & -0.0423 & 0.0800\\ 0.3112 & -0.1617 & -0.1615 & 0.1959 & 0.1596\\ -0.2735 & 0.2042 & 0.1329 & -0.2767 & -0.2008\\ 0.1108 & 0.1030 & -0.1239 & 0.2103 & -0.1982\\ 0.2319 & -0.2015 & 0.2585 & -0.2419 & 0.1380\end{pmatrix},$$

$$\mathcal{U}(:,:,3,2)=\begin{pmatrix}-0.2788 & 0.2637 & 0.1050 & 0.0348 & -0.2234\\ 0.0955 & -0.2627 & 0.4090 & -0.1302 & 0.0599\\ -0.2175 & 0.2402 & 0.0355 & 0.0384 & 0.0105\\ -0.1493 & 0.2447 & 0.0676 & -0.2437 & 0.2228\\ -0.1280 & -0.3358 & 0.2193 & -0.1305 & 0.1638\end{pmatrix},$$

$$\mathcal{U}(:,:,4,2)=\begin{pmatrix}-0.2930 & -0.1586 & 0.0750 & 0.1202 & 0.0563\\ 0.0360 & 0.0733 & 0.1291 & 0.2268 & 0.0134\\ -0.0434 & 0.2310 & -0.2069 & -0.3173 & 0.0224\\ 0.0068 & -0.2252 & -0.1935 & -0.2233 & -0.1099\\ -0.3093 & 0.2481 & -0.3776 & -0.1008 & 0.3627\end{pmatrix},$$

$$\mathcal{U}(:,:,5,2)=\begin{pmatrix}0.1918 & 0.1189 & 0.1188 & 0.2538 & 0.1535\\ -0.2004 & -0.1531 & -0.3061 & -0.0508 & -0.2156\\ -0.0821 & 0.0367 & -0.4258 & -0.1296 & 0.4410\\ 0.1216 & 0.2326 & -0.1759 & -0.1475 & -0.1294\\ -0.1364 & -0.0876 & 0.2264 & -0.1319 & -0.0819\end{pmatrix},$$

$$\mathcal{U}(:,:,1,3)=\begin{pmatrix}0.0322 & -0.1522 & -0.1520 & 0.0481 & 0.0502\\ 0.4251 & -0.2281 & 0.1599 & -0.2902 & 0.0146\\ 0.1935 & -0.3988 & 0.0127 & -0.1278 & 0.1801\\ -0.2817 & 0.3503 & -0.0404 & 0.1529 & -0.1807\\ -0.1772 & 0.1045 & -0.2053 & -0.1220 & 0.0271\end{pmatrix},$$

$$\mathcal{U}(:,:,2,3)=\begin{pmatrix}0.1562 & 0.1080 & 0.1547 & 0.1178 & -0.3201\\ -0.1666 & 0.0734 & 0.1226 & 0.2062 & 0.0591\\ 0.1697 & -0.0316 & -0.1439 & 0.1693 & -0.0413\\ -0.3584 & 0.0058 & -0.4912 & 0.2834 & -0.2288\\ 0.1103 & -0.2560 & -0.0501 & 0.1579 & 0.2024\end{pmatrix},$$

$$\mathcal{U}(:,:,3,3)=\begin{pmatrix}-0.0674&-0.2790&-0.0267&-0.1661&-0.1857\\0.0183&0.0670&-0.2810&-0.0760&-0.0875\\-0.4529&-0.0632&-0.1648&0.2337&-0.0585\\-0.1785&0.2988&0.1005&-0.1982&-0.0989\\-0.0033&0.1420&-0.0201&0.4507&0.2434\end{pmatrix},$$

$$\mathcal{U}(:,:,4,3)=\begin{pmatrix}0.1265&-0.2466&-0.3476&0.1443&-0.0930\\-0.4858&-0.0347&-0.0241&0.0110&0.1526\\0.0868&0.0583&0.0057&-0.0807&-0.0479\\0.1614&0.2710&0.2689&0.2609&0.1998\\-0.2155&-0.2204&-0.1813&-0.0740&0.2807\end{pmatrix},$$

$$\mathcal{U}(:,:,5,3)=\begin{pmatrix}0.1456&0.1018&0.3033&-0.0190&-0.3663\\-0.2515&-0.2428&-0.1656&-0.1132&0.4384\\-0.2031&-0.1261&0.1224&-0.0903&0.0940\\-0.1117&-0.0784&0.0283&-0.0009&0.0693\\0.0768&0.4281&-0.0971&-0.2734&-0.0243\end{pmatrix},$$

$$\mathcal{U}(:,:,1,4)=\begin{pmatrix}-0.2886&-0.0859&0.2511&0.1798&-0.0266\\-0.2510&-0.0649&0.0516&0.2295&-0.5907\\0.0075&-0.1743&0.2089&-0.1055&-0.0345\\-0.1728&0.1524&0.2410&0.1066&-0.0511\\0.2875&0.1107&-0.0150&-0.1831&0.0402\end{pmatrix},$$

$$\mathcal{U}(:,:,2,4)=\begin{pmatrix}-0.0880&0.0428&-0.0612&0.0188&0.2082\\-0.0734&0.0514&0.3234&0.0151&0.0640\\-0.3318&0.2065&0.1811&0.0518&0.4055\\0.2067&0.2502&-0.2629&0.3134&0.0903\\0.2347&0.2164&-0.1833&0.2153&-0.1033\end{pmatrix},$$

$$\mathcal{U}(:,:,3,4)=\begin{pmatrix}-0.0992&-0.2307&0.3292&-0.2394&0.1550\\-0.1517&0.1921&0.2620&0.0068&0.2256\\0.0703&0.1474&-0.2785&0.2915&0.0298\\-0.0507&0.1344&0.2990&0.1263&-0.3586\\-0.1411&0.0205&0.0623&-0.2138&-0.2304\end{pmatrix},$$

$$\mathcal{U}(:,:,4,4)=\begin{pmatrix}-0.0231&-0.2869&-0.1020&-0.2088&-0.1000\\-0.0472&-0.0094&0.2767&0.0533&0.0942\\0.0169&-0.3039&-0.4344&-0.2224&0.2470\\0.0233&-0.1741&0.0445&-0.1701&0.2391\\0.4754&-0.1552&0.0410&0.0024&0.0089\end{pmatrix},$$

$$\mathcal{U}(:,:,5,4)=\begin{pmatrix}0.1250&0.1455&0.1505&-0.0071&-0.1267\\-0.0370&0.3733&0.2343&0.0490&0.0815\\-0.0568&-0.3091&0.2199&-0.4150&0.1070\\0.2138&0.0512&0.1496&-0.0192&-0.2468\\-0.2192&0.0149&0.3360&0.2651&0.1600\end{pmatrix},$$

$$\mathcal{U}(:,:,1,5) = \begin{pmatrix} -0.2684 & 0.0880 & -0.2561 & -0.0673 & -0.4121 \\ 0.0858 & 0.1141 & -0.0442 & 0.2404 & -0.0370 \\ -0.0130 & -0.0125 & -0.2816 & -0.1739 & -0.2377 \\ 0.1100 & 0.2594 & -0.1603 & 0.2019 & 0.1085 \\ -0.1449 & 0.2297 & 0.1558 & -0.0937 & -0.4098 \end{pmatrix},$$

$$\mathcal{U}(:,:,2,5) = \begin{pmatrix} -0.3102 & 0.0085 & -0.3908 & -0.0540 & 0.2507 \\ -0.3707 & 0.0653 & -0.0552 & -0.1624 & 0.1307 \\ -0.2765 & -0.1392 & 0.1464 & -0.1519 & -0.0456 \\ -0.3783 & -0.1808 & -0.1917 & -0.0381 & -0.2439 \\ -0.0443 & -0.1363 & 0.1611 & -0.1097 & -0.1501 \end{pmatrix},$$

$$\mathcal{U}(:,:,3,5) = \begin{pmatrix} -0.0428 & 0.0655 & 0.2593 & -0.1831 & 0.2835 \\ -0.0679 & -0.4829 & 0.0032 & 0.2861 & 0.0487 \\ -0.1260 & -0.2217 & -0.1121 & -0.1395 & -0.1341 \\ -0.0571 & -0.0958 & 0.0534 & 0.2305 & 0.1773 \\ -0.3006 & -0.1268 & -0.0768 & 0.3748 & -0.1518 \end{pmatrix},$$

$$\mathcal{U}(:,:,4,5) = \begin{pmatrix} -0.3705 & -0.3514 & 0.2829 & 0.2440 & -0.0095 \\ 0.0337 & 0.1465 & -0.2274 & -0.2460 & 0.1606 \\ 0.2760 & 0.0902 & 0.0789 & -0.0728 & 0.1728 \\ -0.1305 & -0.0542 & -0.1005 & 0.1710 & 0.3435 \\ -0.0739 & 0.0023 & 0.2860 & 0.2186 & -0.0303 \end{pmatrix},$$

$$\mathcal{U}(:,:,4,5) = \begin{pmatrix} -0.1659 & 0.1949 & 0.0391 & -0.2956 & 0.0553 \\ -0.1278 & -0.1719 & 0.0209 & -0.5045 & -0.1588 \\ 0.1243 & 0.0618 & -0.2336 & -0.1003 & -0.2327 \\ 0.2434 & -0.0658 & -0.0855 & 0.3020 & -0.1194 \\ 0.1350 & 0.1785 & 0.1359 & 0.0297 & 0.3642 \end{pmatrix}.$$

Fig. 3 shows the results of Example 5.1.

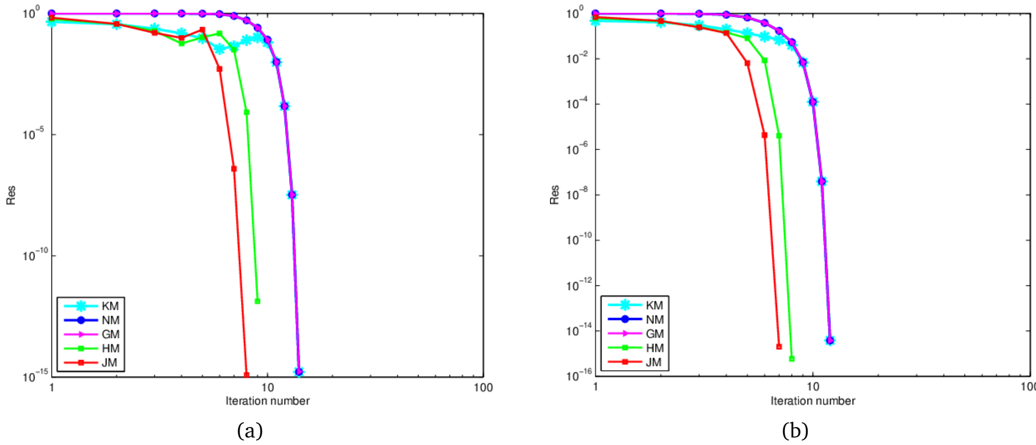
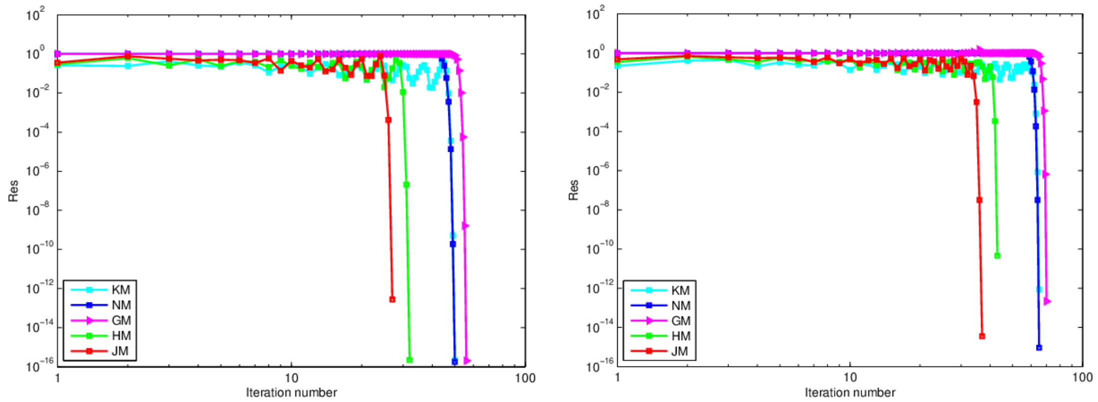


Figure 3: The residual of the results of the iterative methods for (a) $\mathcal{A} \in \mathbb{R}^{3,3,3,3}$ and (b) $\mathcal{A} \in \mathbb{R}^{5,5,5,5}$ of Example 5.1, respectively.

Example 5.2. In this example, consider square matrices with sizes $n = 10, 20$ as follows:

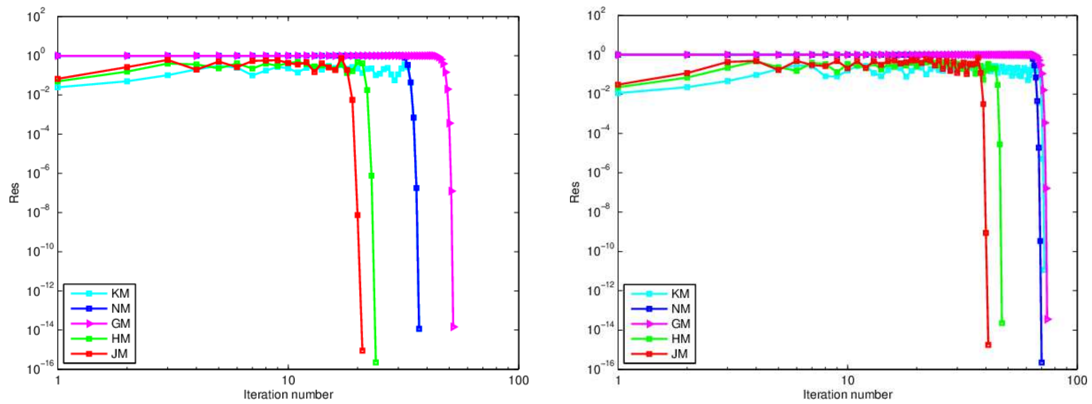
- Hilbert matrix with entries $a_{ij} = 1/(i + j - 1)$.
- Pascal matrix with entries $a_{i1} = a_{1j} = 1$ and $a_{ij} = a_{i-1,j} + a_{i,j-1}$.

Figs. 4 and 5 display the results of Example 5.2.



(a) (b)

Figure 4: The residual of the results of the iterative methods for the Hilbert matrix with (a) $n = 10$ and (b) $n = 20$, respectively.



(a) (b)

Figure 5: The residual of the results of the iterative methods for the Pascal matrix with (a) $n = 10$ and (b) $n = 20$, respectively.

Example 5.3. As the last example, we consider the random tensor $\mathcal{A} \in \mathbb{R}^{n,n,n,n}$ generated in Matlab code by

$$\mathcal{A} = \texttt{tenrand}(n, n, n, n).$$

Fig. 6 represents the results of Example 5.3.

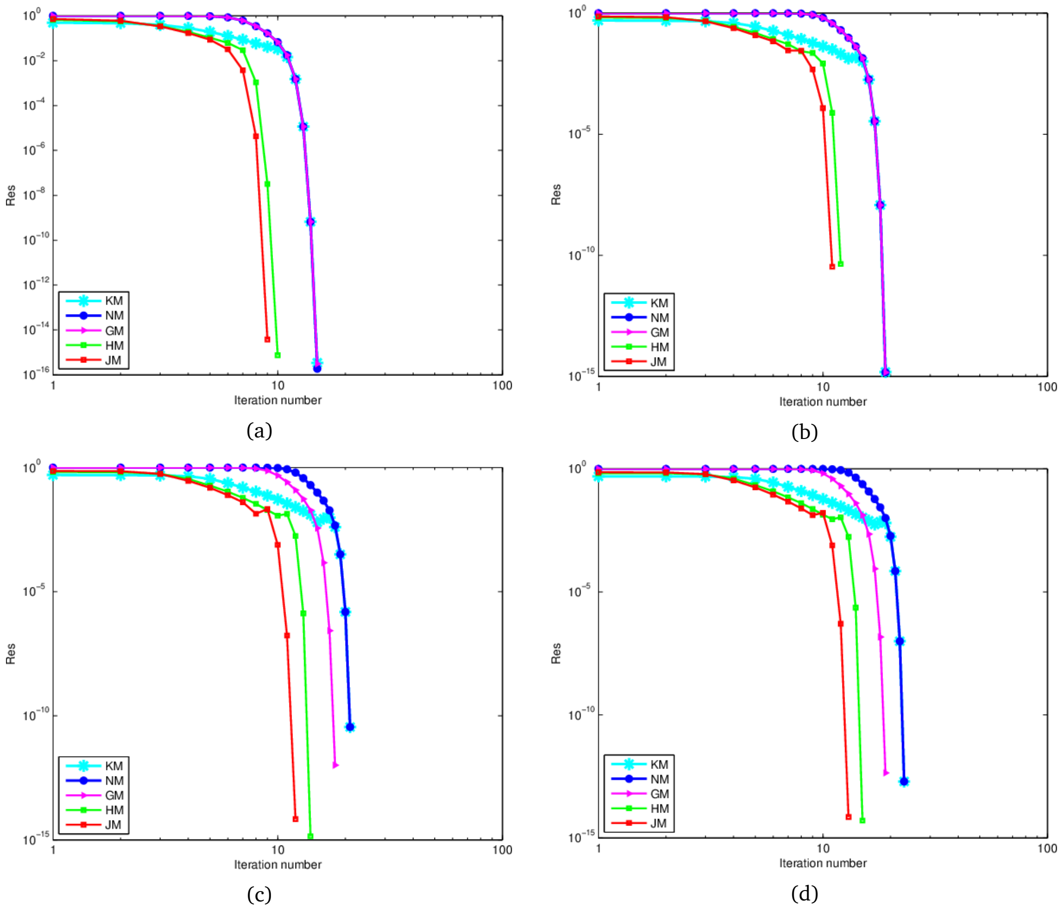


Figure 6: The residual of the results of the iterative methods for random tensors with (a) $n = 10$, (b) $n = 20$, (c) $n = 30$ and (d) $n = 40$ for Example 5.3, respectively.

6. Conclusions

In this paper, we have shown that the polar decomposition of the tensor can be computed using the singular value decomposition of the tensor with the Einstein product. Then several iterative methods for finding the polar decomposition of the matrices have been developed into iterative methods to compute the polar decomposition of tensors. Therefore, a novel iterative method for computation of the polar decomposition of the tensors was thoroughly proposed and investigated. It has been shown that the proposed method is globally convergent of order four. Also, the new parametric method was free from calculating the tensor inversion, while the other mentioned iterative methods needed to calculate the tensor inversion in each iteration. Thus, the new method was superior in terms of efficiency index. Finally, we tested the presented methods for calculating the $\mathcal{U}$ coefficient of the polar decomposition of a wide range of random tensors with the Einstein product, and the new parametric method has excellent superiority in comparing the other well-known methods such as Kovarik method, Newton method, Gander method, and Halley method.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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