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Meshfree First-order System Least Squares

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Abstract. We prove convergence for a meshfree first-order system least squares (FOSLS) partition of unity finite element method (PUFEM). Essentially, by virtue of the partition of unity, local approximation gives rise to global approximation in $H(div) \cap H(curl)$. The FOSLS formulation yields local *a posteriori* error estimates to guide the judicious allotment of new degrees of freedom to enrich the initial point set in a meshfree discretization. Preliminary numerical results are provided and remaining challenges are discussed.

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1. Introduction

1.1. Summary

Interest remains in avoiding the proper tessellation of a computational domain used to solve partial differential equations, especially in the context of moving meshfree, or meshless, particle methods. However, as will be made clear, the flexibility inherent to using merely the cover of a domain does not come without cost. A number of mostly-related meshfree approaches have been proposed, yielding a variety of approximation spaces from which to choose. For example, consider the diffuse element method (DEM), element free Galerkin (EFG), finite point method (FPM), HP clouds, meshfree local Petrov Galerkin (MLPG), smooth particle hydrodynamics (SPH), moving least squares SPH (ML-SPH), material-point method (MPM), partition of unity finite element method (PUFEM), reproducing kernel particle method (RKPM); see [1,2] for a classification and review. Below, we employ the partition unity (PU) approach, given its flexibility and local nature, to discretize the prototypical first-order system least-squares (FOSLS) formulation for Poisson's equation. This synthesis can be generalized to existing FOSLS formulations of more

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complicated PDE systems. Since the FOSLS formulation presents local a posteriori error estimates to guide adaptive enrichment of an initially-sparse point set, the synthesis may enhance the utility of meshfree methods.

Below, after a brief discussion about meshfree methods, we introduce FOSLS and a vector PUFEM discretization. Then, convergence is proved from assumptions about the local approximation spaces defined on each patch used to cover the domain. Finally, several numerical examples are provided.

1.2. Meshfreedom

The notion of altogether avoiding a mesh requires some clarification. First, a set of points in a domain, along with an associated covering of the domain, must be chosen. Given generic specifications for this covering, such as the density used in point set selection and the degree of overlap, a provision of points and their associated patches can be supplied probabilistically, without use of a background mesh [3]. However, enrichment of an initially sparse (or coarse) point set nevertheless entails relaxation of the new point set, in concert with updating the associated cover. While explicit retriangulation is thus avoided, significant refinement costs persist to ensure that both point placement and patch size are suitable.

Integration during assembly of the discrete problem leads to a second unavoidable computational cost. In lieu of a tessellation, integration over each intersecting pair of patches entails defining quadrature points either locally, in a consistent and efficient manner, or globally, appealing to some background mesh. Below we simply use circular patches Ω_i to cover a domain $\Omega \subset \mathbb{R}^2$. Rather than perform quadrature on each individual lens, $\Omega_i \cap \Omega_j$, which would yield a symmetric linear system, it is more efficient to simply set a quadrature rule on each Ω_i. Of course, this leads to an asymmetric system due to inexact integration, i.e., A_{ij} is computed using quadrature on Ω_i while A_{ji} is computed using quadrature on Ω_j.

Our approach is truly meshfree in the sense that integration is performed according to neighbor connectivity, point locations, and neighboring support radii, not according to the elements of a tessellation. No background mesh is utilized. As a result, adding and/or moving individual points is unencumbered by the need to re-tessellate the domain. This flexibility comes with less-efficient assembly of the discrete problem, primarily because there are many more regions of overlap, $\Omega_i \cap \Omega_j$, than there are elements of a comparable tessellation. This cost is compounded by the partition of unity construction, which yields conforming discretizations at the expense of pointwise conditions on the degree of overlap; e.g., requiring that each point be covered by at least three elements of the cover, $\#\{j|\Omega_i \cap \Omega_j \neq \emptyset\}$ approaches 30 in Fig. 1.

2. A FOSLS partition of unity method

FOSLS has been applied to far more difficult PDE systems than the simplistic elliptic problem considered below [4]. As a methodology, it is only distinct from least-squares (LS) in that a residual on vorticity, or the curl of velocity, is introduced in lieu of solving for