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New Estimates for the Rate of Convergence of the Method of Subspace Corrections

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Abstract. We discuss estimates for the rate of convergence of the method of successive subspace corrections in terms of condition number estimate for the method of parallel subspace corrections. We provide upper bounds and in a special case, a lower bound for preconditioners defined via the method of successive subspace corrections.

AMS subject classifications: 65F10, 65J05, 65N12, 65N55
Key words: Method of subspace corrections, preconditioning convergence rate of linear iterative method.

1. Introduction

The method of subspace corrections is a general iterative method for solving the linear system of equations arising from the variational formulation in a Hilbert space. The modern theory of the subspace correction methods has showed that the multigrid method and the domain decomposition method are systematically equivalent. In this paper, we study the method of subspace corrections. We refer readers to von Neumann [7], Bramble [1], Bramble and Zhang [2], Hackbusch [4], Griebel and Oswald [3], Trottenberg, Oosterlee and Schüller [5], Xu [8, 9] and Xu and Zikatanov [10] for the method of subspace corrections.

One main focus in this paper is to provide an estimate for the rate of convergence of the method of successive subspace corrections (MSSC) in terms of the method of parallel subspace corrections (MPSC). This work can be considered as an extension and application of the convergence theory by Xu and Zikatanov [10]. Based on this framework, we obtain a formula for the convergence rate, which can be employed to derive many other estimates related to the method of subspace corrections. We then show how the convergence rate of the MSSC can be estimated in terms of the MPSC. Similar results and other approaches on deriving estimates relating MSSC and MPSC are also found in earlier works [2, 3, 8, 9].

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The remainder of this paper is organized as follows. In Section 2, we present the variational problem in a Hilbert space and recall some notation and algorithms in [10]. In Section 3, we relate the convergence rate to the condition number of multiplicative preconditioner and derive a formula for this condition number. In Section 4, we present estimates for the convergence rate, using the formula for the condition number. We discuss the special cases of the subspace correction methods in Section 5.

2. Notation and preliminaries

Let V be a Hilbert space with an inner product $(\cdot,\cdot)$ and its induced norm $\|\cdot\|$ and let V^* be the dual space of V. We consider the variational problem:

Find $u \in V$ such that for any given $f \in V^*$

$$a(u,v) = \langle f, v\rangle, \qquad \forall\ v \in V. \tag{2.1}$$

Here, $a(\cdot,\cdot): V \times V \mapsto \mathbb{R}$ is a continuous symmetric positive definite (SPD) bilinear form. Since $a(\cdot,\cdot)$ is a SPD, it introduces an inner product and a norm which we denote with $(\cdot,\cdot)_a$ and $\|\cdot\|_a$. In more classical notation, we define an operator $A: V \mapsto V$ by

$$(Au, v) = a(u,v) \quad \forall u \in V, \forall v \in V.$$

Following [10], we introduce some notation and the parallel and successive subspace correction algorithms. We first consider a collection of closed subspaces

$$V_k \subset V, \quad k = 1, \cdots, J,$$

such that

$$(\mathbf{A0}) \quad V = \sum_{k=1}^{J} V_k.$$

Associated with each subspace V_k, we define a continuous positive definite bilinear form $a_k(\cdot,\cdot)$ to be an approximation of $a(\cdot,\cdot)$ on V_k. We point out that in general $a_k(\cdot,\cdot)$ may not be symmetric. To assure the well-posedness of the subspace problems, we assume that the bilinear forms $a_k(\cdot,\cdot)$ satisfy appropriate inf-sup conditions.

The method of parallel subspace corrections (MPSC) is an iterative algorithm that correct the residual equations in parallel in each subspace. MPSC is described as follows.

Algorithm 2.1 (MPSC). *Let $u^0 \in V$ be given.*

for $\ell = 1, 2, \cdots$

  for $i = 1 : J$

    Let $e_i \in V_i$ solve

$$a_i(e_i, v_i) = f(v_i) - a(u^{\ell-1}, v_i) \quad \forall v_i \in V_i,$$

  endfor

  $u^\ell = u^{\ell-1} + \sum_{i=1}^{J} e_i,$

endfor