NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **1**, No. 1, pp. 57-77 (2008)

Spectral Analysis for HSS Preconditioners

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Received 3 July, 2007; Accepted (in revised version) 30 November, 2007

Abstract. In this paper, we are interested in HSS preconditioners for saddle point linear systems with a nonzero $(2,2)$-th block. We study an approximation of the spectra of HSS preconditioned matrices and use these results to illustrate and explain the spectra obtained from numerical examples, where the previous spectral analysis of HSS preconditioned matrices does not cover.

AMS subject classifications: 65F10
Key words: Saddle point problems, iterative methods, preconditioning, eigenvalues.

1. Introduction

Saddle point linear problems arise in many different practical applications [4]. In this paper we study iterative methods for solving saddle point problems $\mathscr{A}'x = b$ where

$$\mathscr{A}' = \begin{bmatrix} W & K \\ K^T & -\mu I \end{bmatrix}, \tag{1.1}$$

$W \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite and $K \in \mathbb{R}^{m\times n}$ (so $I \in \mathbb{R}^{n\times n}$). We confine our discussion that $m \geq n$ (or even $m \gg n$) and K is assumed to have full (column) rank. Examples include discrete Stokes equations [6], and weighted Toeplitz least squares problems [5] where applications arise from image reconstruction [1, 10] and nonlinear image restoration [9]. In imaging applications, μ refers to the regularization parameter.

In the literature, there are many solution methods for solving saddle point linear systems; see [4] for references. In this paper, we are interested in HSS preconditioning techniques [3]. The idea is first to transform $\mathscr{A}'$ in (1.1) to a nonsymmetric matrix

$$\mathscr{A} = \begin{bmatrix} W & K \\ -K^T & \mu I \end{bmatrix}. \tag{1.2}$$

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Then we split $\mathscr{A}$ into $\mathscr{H}+\mathscr{S}$ where

$$\mathscr{H}=\begin{bmatrix} W & 0 \\ 0 & \mu I \end{bmatrix} \quad \text{and} \quad \mathscr{S}=\begin{bmatrix} 0 & K \\ -K^T & 0 \end{bmatrix}. \tag{1.3}$$

Here $\mathscr{H}$ is the symmetric part of $\mathscr{A}$ and $\mathscr{S}$ is the skew-symmetric part of $\mathscr{A}$. The HSS preconditioner is in the form

$$\mathscr{P}=\frac{1}{2\alpha}(\mathscr{H}+\alpha I)(\mathscr{S}+\alpha I), \tag{1.4}$$

where $\alpha>0$ is referred to as the preconditioning parameter for the HSS preconditioner. The idea of HSS preconditioner is motivated from the Hermitian and skew-Hermitian splitting (HSS) method [2]. It was shown in [3] that the eigenvalues of $I-\mathscr{P}^{-1}\mathscr{A}$ lie in $\overline{\mathbf{D}(1)}=\{z\in\mathbb{C} \mid |z|\le 1\}$ and in particular $\mathbf{D}(1)=\{z\in\mathbb{C} \mid |z|<1\}$ if W is positive definite.

In [11] a more thorough spectral analysis for the case of $\mu=0$ is presented where a more refined inclusion region is given, and sufficient conditions with respect to α for a real or a clustered spectrum are provided.

When $\mu\neq 0$, an analysis under the special case when $\alpha=\mu$ can be found in [5]. However, when $\alpha\neq\mu$ it remains a rather difficult task to make any conclusion on the spectrum.

The main aim of this paper is to explain the spectra of HSS preconditioned matrices when $\mu\neq 0$. The argument involved is to approximate the transformed preconditioned matrices for which the eigenvalues of the approximations can be determined and analyzed. When α is smaller than the eigenvalues of W our approximations are quite accurate. Our analysis can be used to study the spectra of HSS preconditioned matrices when $\mu=0$.

The outline of this paper is as follows. In Section 2, we give some preparatory work. In Section 3, we study an approximation of spectra of HSS preconditioned matrices. In Section 4, numerical examples are presented to illustrate the approximation scheme. In Section 5, we study the spectra when $\mu=0$. Finally, we give some concluding remarks in Section 6.

2. Decomposition of matrices

For simplification, throughout this paper we shall denote by $\mathbf{R}(r)$, $\mathbf{D}(r)$ and $\overline{\mathbf{D}(r)}$ the sets $\{z\in\mathbb{C} \mid |z|=r\}$, $\{z\in\mathbb{C} \mid |z|<r\}$ and $\{z\in\mathbb{C} \mid |z|\le r\}$ respectively. We shall also denote by I_k the $k\times k$ identity matrix, whenever ambiguities may arise from the size of the identity matrix.

We first note that the spectrum of the matrix $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ is the same as the spectrum of $I-\mathscr{P}^{-1}\mathscr{A}$.

Lemma 2.1. ([11]) *Consider the HSS preconditioner $\mathscr{P}$ to a linear system with coefficient matrix $\mathscr{A}=\mathscr{H}+\mathscr{S}$, where $\mathscr{H}$ and $\mathscr{S}$ are respectively the Hermitian part and skew-Hermitian part of $\mathscr{A}$. Then the matrices $I-\mathscr{P}^{-1}\mathscr{A}$ and $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ have the same spectrum.*