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A Parameter-Uniform Finite Difference Method for a Coupled System of Convection-Diffusion Two-Point Boundary Value Problems

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Abstract. A system of m (≥ 2) linear convection-diffusion two-point boundary value problems is examined, where the diffusion term in each equation is multiplied by a small parameter ε and the equations are coupled through their convective and reactive terms via matrices B and A respectively. This system is in general singularly perturbed. Unlike the case of a single equation, it does not satisfy a conventional maximum principle. Certain hypotheses are placed on the coupling matrices B and A that ensure existence and uniqueness of a solution to the system and also permit boundary layers in the components of this solution at only one endpoint of the domain; these hypotheses can be regarded as a strong form of diagonal dominance of B. This solution is decomposed into a sum of regular and layer components. Bounds are established on these components and their derivatives to show explicitly their dependence on the small parameter ε. Finally, numerical methods consisting of upwinding on piecewise-uniform Shishkin meshes are proved to yield numerical solutions that are essentially first-order convergent, uniformly in ε, to the true solution in the discrete maximum norm. Numerical results on Shishkin meshes are presented to support these theoretical bounds.

AMS subject classifications: 65L10, 65L12, 65L20, 65L70

Key words: Singularly perturbed, convection-diffusion, coupled system, piecewise-uniform mesh.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

While the numerical analysis of singularly perturbed convection-diffusion problems has received much attention in recent years [6, 12, 14], the main focus has been on single equations of various types—systems of equations appear relatively rarely. Nevertheless

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coupled systems of convection-diffusion equations do appear in many applications, notably optimal control problems and in certain resistance-capacitor electrical circuits; see [7].

In this paper we consider a system of $m \geq 2$ convection-diffusion equations in the unknown vector function $\boldsymbol{u} = (u_1, u_2, \cdots, u_m)^T$. This system is coupled through its convective and reactive terms:

$$L\boldsymbol{u} := (-\varepsilon \boldsymbol{u}'' - B\boldsymbol{u}' + A\boldsymbol{u})(x) = \boldsymbol{f}(x), \quad x \in (0,1) \tag{1.1}$$

and it satisfies boundary conditions $\boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}$. Since the problem is linear there is no loss in generality in assuming homogeneous boundary conditions. Here $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times m$ matrices whose entries are assumed to lie in $C^3[0,1]$, and $\varepsilon > 0$ is a small diffusion parameter whose presence makes the problem singularly perturbed. We assume that $\boldsymbol{f} = (f_1, \cdots, f_m)^T \in (C^3[0,1])^m$.

Systems of this type from optimal control problems often have a different diffusion coefficient ε_i associated with the i^{th} equation for $i = 1, \cdots, m$, but with all ratios $\varepsilon_i/\varepsilon_j$ bounded by a fixed constant [7, p.503]; one can then rescale all equations to the form (1.1) with affecting the analysis and conclusions of this paper, so our assumption of a single value ε is not a restriction in this case.

Assumption 1.1. *In the matrices $B = (b_{ij})$ and $A = (a_{ij})$, for $i = 1, \cdots, m$ one has*

$$\beta_i := \min_{x \in [0,1]} b_{ii}(x) > 0 \tag{1.2a}$$

and

$$a_{ii}(x) \geq 0 \quad \textit{for } x \in [0,1]. \tag{1.2b}$$

Similar assumptions are often made in scalar convection-diffusion equations, where in particular any sign change or vanishing of the coefficient of the first-derivative term alters significantly the nature of the solution; see, e.g., [12]. Each component u_i of our solution $\boldsymbol{u}$ will exhibit a boundary layer and (1.2a) enables us to predict that the layer in $u_i(x)$ will be at $x = 0$.

Further hypotheses will be placed on B in Section 2, but our collective hypotheses are not strong enough to guarantee that the differential operator of (1.1) satisfies a standard maximum principle; see, e.g., [11, Example 2.1]. This excludes the most commonly-used tool in finite difference analysis of singularly perturbed differential equations and forces us to develop an alternative methodology.

Notation. Throughout the paper C denotes a generic constant that is independent of ε and any mesh, and can take on different values at different points in the argument. Write $\|\cdot\|_\infty$ for the norm on $L_\infty[0,1]$. Set

$$\|\mathbf{g}\|_\infty = \max\{\|g_1\|_\infty, \cdots, \|g_m\|_\infty\}$$

for any vector-valued function $\boldsymbol{g} = (g_1, \cdots, g_m)^T$ having $g_i \in L_\infty(0,1)$ for all i. For each $w \in W^{-1,\infty}$ define the norm

$$\|w\|_{-1,\infty} = \inf\{\|W\|_\infty : W' = w\}.$$

We shall also use the usual $L_1[0,1]$ norm $\|\cdot\|_{L_1}$.